Reduced-Order Modeling in the Frequency Domain for Closed-Loop Flow Control

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Background

**Motivation:** MURI for Airfoil Control by Synthetic-Jets

Closed-Loop Control of Synthetic-Jets over an Airfoil Pitch Requires Modeling the Flow as a Dynamic System

\[ \downarrow \]

Reduced-Order Modeling of the Governing Flow Equations

*Figures by Brzozowski and Glezer (Georgia Tech)*
1-D Reduced-Order Modeling: Flow Between Two Parallel Plates

\[ u = \cos(\omega_0 t) \leftrightarrow \]

\[ \frac{\partial u}{\partial t} = \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial y^2} \]

Exact solution:

\[ u = \text{Real} \left\{ \frac{\sinh\left(\sqrt{i\text{Re} \omega_0} y\right)}{\sinh\left(\sqrt{i\text{Re} \omega_0}\right)} \exp(i\omega_0 t) \right\} \]
Impulse-Response: Flow Between Two Parallel Plates

\[ u = \delta(t) \]

Fourier transform of Eq (I) and B.C,

\[ \hat{u} = \int_{0}^{\infty} u \exp(i\omega t) dt \]

yields

\[ \hat{u} = \frac{\sinh(\sqrt{i\text{Re} \omega} y)}{\sinh(\sqrt{i\text{Re} \omega})} \]

which is sampled for a range of frequencies.
Reduction of Impulse-Response Problem to an ODE System

Proper-Orthogonal-Decomposition (POD) in Frequency Domain:

\[ \hat{u}(y, \omega) = \sum_{j=1}^{N} \hat{c}_j(\omega)u_j(y) \]

where the velocity modes \( u_j(y) \) satisfy the orthogonality relation

\[
\int_{0}^{1} u_j(y)u_k^*(y)dy = \delta_{jk}
\]

* is the complex conjugate
Reduction of Impulse-Response Problem to an ODE System (cont.)

ODE System Derived from Galerkin Projection of Eq (I) onto the POD modes:

\[
\frac{dc_n(t)}{dt} = \sum_{j=1}^{N-1} H_{jn} c_j(t) + F(t) \quad 1 \leq n \leq N - 1
\]

and the coefficient \( c_N(t) \) is determined by imposing \( u(1,t) = F(t) \) on the upper wall:

\[
c_N(t)u_N(1) = -\sum_{j=1}^{N-1} c_j(t)u_j(1) + F(t)
\]
Validation of the ODE System: Velocity Reconstructed from 3 POD Modes for Arbitrarily Chosen Forcing Frequency ($\omega_0=4$)

Re=5

Analytical Solution

Reconstruction from ODE Solution
Validation of the ODE System: Velocity Reconstructed from 3 POD Modes for Arbitrarily Chosen Forcing Frequency ($\omega_0=4$) $Re=5$

Analytical Solution  Reconstruction from ODE Solution
Validation of the ODE System for Higher Re: \( \omega_0=4 \) and \( Re=50 \)

Analytical Solution
Reconstruction from ODE Solution

3 POD modes

5 POD modes
2-D Reduced-Order Modeling: Linearized Navier-Stokes Equations

**Objective:** Develop low-order, real-time computable ODE models of the boundary-actuated flow by projecting solutions of Navier-Stokes onto a set of orthogonal modes in the frequency domain.

\[
\begin{align*}
(I) & \quad \frac{\partial \widetilde{\mathbf{v}}}{\partial t} + \widehat{\mathbf{v}} \cdot \nabla \mathbf{V}_s + \mathbf{V}_s \cdot \nabla \widetilde{\mathbf{v}} = -\nabla \widetilde{p} + \frac{1}{\text{Re}} \nabla^2 \widetilde{\mathbf{v}}, \\
(II) & \quad \nabla \cdot \widetilde{\mathbf{v}} = 0,
\end{align*}
\]

where \( \mathbf{V}_s = \mathbf{V}_s(x, y) \) is a given base-flow and \( \widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}(x, y, t) \) is the deviation to be controlled.
Impulse-Response: 2-D Linearized Actuated Flow

With B.C. for Equations (I)-(II),

\( v_\perp(y) \) is a given distribution (normal to wall component) based on the type of actuator

by Fourier transforming, Eqs (I)-(II) are solved in the frequency domain for a range of frequencies \( \omega \).
Reduction of Impulse-Response Problem to an ODE System

Proper-Orthogonal-Decomposition (POD) in Frequency Domain:

\[
\hat{v}(x, y, \omega) = \sum_{j=1}^{N} \sum_{k=1}^{M} \hat{c}_{jk}(\omega)v_{jk}(x, y)
\]

where the “partially separable” velocity modes \( v_{jk}(x, y) = \xi_{jk}(x)\eta_j(y) \)
satisfy the orthogonality relation

\[
\int_{x_0}^{x_1} \int_{y_0}^{y_1} v_{jk}(x, y) \cdot v^*_{nm}(x, y) dx dy = \delta_{jn} \delta_{km}
\]

\[
\int_{x_0}^{x_1} \int_{y_0}^{y_1} \xi_{jk}(x)\xi^*_{nm}(x) dx \int_{x_0}^{x_1} \eta_j(y)\eta^*_n(y) dy = \delta_{jn} \delta_{km}
\]
Reduction of Impulse-Response Problem to an ODE System (cont.)

ODE System Derived from Navier-Stokes Equations

\[
\frac{dc_{nm}(t)}{dt} = \sum_{j=1}^{N} G_{jn} \sum_{k=1}^{M-1} H_{km} c_{jk}(t) + F_{n}(t) \quad 1 \leq n \leq N
\]

\[
1 \leq m \leq M - 1
\]

and the coefficients \( c_{jM}(t) \) are determined by projection onto the POD modes \( \eta_j(y) \), and imposing \( \mathbf{v}_{in}(x_0, y, t) = \mathbf{v}_\perp(y, t) \) (wall-normal component) at inlet:

\[
c_{jM}(t) \xi_{jM}(x_0) = - \sum_{k=1}^{M-1} c_{jk}(t) \xi_{jk}(x_0) + \int_{y_0}^{y_1} \mathbf{v}_\perp(y, t) \, \eta^*_j(y) \, dy
\]

\[F_j(t)\]
Reduction of Impulse-Response Problem to an ODE System (cont.)

Pressure Contribution Computed for Each Mode using Poisson Equation for Pressure:

\[
\nabla^2 p_{jk} = -\nabla \cdot \left( v_{jk} \cdot \nabla V_S + V_S \cdot \nabla v_{jk} \right)
\]
Model Problem: Open-Cavity Flow

Steady Base-Flow for Re=5 (Based on Cavity Height):
Open-Cavity Flow: Coordinate Transformation

\[(x, y) \rightarrow (\sigma, \tau)\]

Flow-fields computed in \((x, y)\) are transformed to \((\sigma, \tau)\), where POD and projection of the N-S Eqs are done in \((\sigma, \tau)\) plane.
Open-Cavity Flow: Coordinate Transformation (cont.)

The modified N-S Eqs in $(\sigma, \tau)$ plane

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + \tilde{\mathbf{v}} \cdot \nabla \mathbf{V}_S + \begin{pmatrix} \sigma_x & \sigma_y \\ \sigma_y & -\sigma_x \end{pmatrix} \mathbf{V}_S \cdot \nabla_{\sigma\tau} \tilde{\mathbf{v}} = -\begin{pmatrix} \sigma_x & \sigma_y \\ \sigma_y & -\sigma_x \end{pmatrix} \nabla_{\sigma\tau} \tilde{p} + \frac{1}{\text{Re}} \left( \sigma_x^2 + \sigma_y^2 \right) \nabla^2_{\sigma\tau} \tilde{\mathbf{v}}$$

Project onto the POD modes $\mathbf{v}_{jk}(\sigma, \tau) = \xi_{jk}(\sigma)\eta_j(\tau)$,

where

$$\int_{\sigma_0}^{\sigma_1} \int_{\tau_0}^{\tau_1} \mathbf{v}_{jk}(\sigma, \tau) \cdot \mathbf{v}_{nm}^*(\sigma, \tau) d\sigma d\tau =$$

$$\int_{\sigma_0}^{\sigma_1} \xi_{jk}(\sigma) \xi_{nm}^*(\sigma) d\sigma \int_{\tau_0}^{\tau_1} \eta_j(\tau) \eta_n^*(\tau) d\tau = \delta_{jn} \delta_{km}$$
Open-Cavity Flow:
Actuated Flow in the Frequency Domain

Injection/Suction Actuation on Upper Part of Cavity Wall

Flow Fields for $\omega=0$:

$(x, y)$ plane

$(\sigma, \tau)$ plane
Validation of the ODE System:
Velocity Reconstructed from 9 POD Modes for Arbitrarily Chosen Forcing Frequency ($\omega_0=1$)

$x$-velocity ($\text{Re}=5$)

Unsteady Numerical Simulation

Reconstruction from ODE Solution
Validation of the ODE System:
Velocity Reconstructed from 9 POD Modes for Arbitrarily Chosen Forcing Frequency ($\omega_0=1$)

$y$-velocity ($\text{Re}=5$)

Unsteady Numerical Simulation

Reconstruction from ODE Solution
Validation of the ODE System: Velocity Reconstructed from 15 POD Modes for Arbitrarily Chosen Forcing Frequency ($\omega_0=1$)

$x$-velocity ($Re=50$)

Unsteady Numerical Simulation

Reconstruction from ODE Solution
Validation of the ODE System:
Velocity Reconstructed from 15 POD Modes for Arbitrarily Chosen Forcing Frequency ($\omega_0=1$)

$y$-velocity ($Re=50$)

Unsteady Numerical Simulation  Reconstruction from ODE Solution
Application to Pitching Airfoil

Motivation: MURI for Airfoil Control by Synthetic-Jets

Closed-Loop Control of Synthetic-Jets over an Airfoil
Pitch Requires Modeling the Flow as a Dynamic System

\[ \downarrow \]

Reduced-Order Modeling of the Governing Flow Equations

*Figures by Brzozowski and Glezer (Georgia-Tech)
Experimental Approach
(Brzozowski and Glezer at Georgia Tech)

- Wind tunnel experiments (by Dan P. Brzozowski and Ari Glezer at Georgia Tech) use an airfoil mounted on a programmable, 1-DOF pitch traverse.

- The airfoil executes commanded maneuvers driven by synthetic-jet actuators on the upper and lower surfaces in the airfoil, close to the trailing edge.
PIV Measurements (Phase-Averaged Vorticity) by Brzozowski and Glezer at Georgia Tech

$U_\infty = 30 \text{ m/s}$

Inset area investigated, in frame moving with pitching airfoil
Reduced-Order Modeling Based on Inviscid Vorticity Transport Equation

Objective: Develop low-order, real-time computable ODE models of the boundary-actuated flow by projecting solutions of the inviscid vorticity transport equation

$$\frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta = \mathbf{\theta} \cdot \nabla \mathbf{v}$$

onto a set of orthogonal vorticity modes in the frequency domain, and relate the dynamical variables to measured quantities (surface pressure, forces, moments) to obtain an ODE system.
Linearizing the Actuated Flow about the Time-Averaged (Airfoil Wake) 2D-Flow

\[ \theta = \left[ \Theta(x, y) + \theta(x, y, t) \right] e_z \]

\[ \mathbf{v} = \left[ U(x, y) + u(x, y, t) \right] e_x + \left[ V(x, y) + v(x, y, t) \right] e_y \]
Impulse Response for 2-D Linearized Actuated Flow

Vorticity Transport Equation

\[
\begin{aligned}
\frac{\partial \theta}{\partial t} + U \frac{\partial \theta}{\partial x} + V \frac{\partial \theta}{\partial y} + \frac{\partial \Theta}{\partial x} u + \frac{\partial \Theta}{\partial y} v &= 0 \\
\end{aligned}
\]

where \( U, V, \) and derivatives of \( \theta \) are provided experimentally.

2-D Continuity

\[
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\end{aligned}
\]
Impulse-Response 2-D Linearized Actuated Flow (cont.)

B.C. for Equations (I)-(II):

\[ u_{in}(y) \text{ and } \theta_{in}(y) \text{ are instantaneous inlet distributions from experiment} \]

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial x} &= 0
\end{align*}
\]

\[
\begin{align*}
u &= 0 \\
\frac{\partial \theta}{\partial x} &= 0
\end{align*}
\]

By Fourier transforming, Eqs. (I)-(II) are solved in the frequency domain for a range of frequencies \( \omega \).
Reduction of Impulse-Response Problem to an ODE System

Vorticity Proper-Orthogonal-Decomposition (POD) in Frequency Domain:

\[
\hat{\theta}(x, y, \omega) = \sum_{j=1}^{N} \sum_{k=1}^{M} \hat{c}_{jk}(\omega) \theta_{jk}(x, y)
\]

where the “partially separable” vorticity modes \( \theta_{jk}(x, y) = \xi_{jk}(x) \eta_j(y) \) satisfy the orthogonality relations

\[
\int_{y_0}^{y_1} \int_{x_0}^{x_1} \theta_{jk}(x, y) \theta_{nm}^*(x, y) dx dy = \\
\int_{x_0}^{x_1} \xi_{jk}(x) \xi_{nm}^*(x) dx \int_{y_0}^{y_1} \eta_j(y) \eta_n^*(y) dy = \delta_{jn} \delta_{km}
\]
Reduction of Impulse-Response Problem to an ODE System (cont.)

ODE System Derived from Vorticity Transport Equation

\[
\frac{dc_{nm}(t)}{dt} = \sum_{j=1}^{N} G_{jn} \sum_{k=1}^{M-1} H_{km} c_{jk}(t) + F_n(t) \quad 1 \leq n \leq N
\]

\[
1 \leq m \leq M - 1
\]

and the coefficients \( c_{jM}(t) \) are determined by projection onto the POD modes \( \eta_j(y) \), and imposing \( \theta(x_0,y,t) = \theta_{in}(y,t) \) at the inlet:

\[
c_{jM}(t) \xi_{jM}(x_0) = - \sum_{k=1}^{M-1} c_{jk}(t) \xi_{jk}(x_0) + \int_{y_0}^{y_1} \theta_{in}(y,t) \eta_j^*(y) dy
\]

\[
F_j(t)
\]
POD Modes from Impulse-Response PDE Solution

First three modes $\theta_{1k}(x, y) = \xi_{1k}(x)\eta_1(y)$ ($k = 1, 2, 3$):

Spanwise Vorticity

Streamwise Velocity
Validation of the ODE System:
Vorticity Reconstructed from Six POD Modes using
Inlet Conditions at Two Different Phases

Experimental Data

Reconstruction from ODE Solution
Relating Reduced-Order Model of Flow to Forces and Moments

PIV data (and resulting POD modes) are available in only part of the airfoil wake and not in a region surrounding it. Thus, force and moment cannot be directly predicted by the reduced-order model.

Vorticity from CFD (by Moser’s group), obtained by time-averaging lower and upper actuation levels for one cycle of the experiment.

CFD simulations by Jee & Moser at the University of Texas provide flow fields around the whole airfoil, and are used for relating lift and moment to the flow predicted by the reduced-order model.
Summary and Conclusions

- A reduced-order-modeling approach has been proposed for a 1-D model problem governed by a linear (diffusion) equation. The resulting ODE system has shown to be valid for time-oscillating (harmonic) forcing with different frequencies on one of the boundaries.

- A more generalized approach of the reduced-order-modeling has been proposed to construct a forced ODE system from the 2-D linearized Navier-Stokes equations, accounting for general time-dependent forcing. This approach may be useful in applications where a given flow is to be controlled by actuators producing relatively small changes in the flow-fields.

- The proposed approach has been validated on a model problem of open-cavity flow, controlled by an injection/suction actuator. The approach can be easily applied to 3-D flows (since it uses streamlines, not streamfunction) and for different types of actuators as well.

- The proposed model has been used for the linearized Vorticity Transport Equation on data from experimental two-dimensional flow past an airfoil trailing-edge, controlled by synthetic-jet actuators.