4D-Variational Data Assimilation using POD Reduced-Order Model

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Model based flow control:

Need for a dynamical model:
- Representative of the flow \textit{(at least input/output behavior)}.
- Fast $\Rightarrow$ Low order model.
**Context**

*Model based flow control:*
- Huge number of DoF $\Rightarrow$ POD Reduced-Order Model
- POD ROM coming from Galerkin projection imperfect
  $\Rightarrow$ **Data assimilation** to improve the POD ROM

**Principle:** To combine different sources of information to estimate at best (*in an optimal way*) the state of the system.

- Imperfect observations (*incomplete, noised*)
- Imperfect model (*simplified*)
- *A priori* knowledge of the state of the system

**Approaches:**
- Stochastic method (*ex: Kalman Filter*)
- Variational method: Minimisation of a cost functional $J$
Outline

1. What is 4D-Var?
2. Formalism
   - Strong dynamical constraint
   - Weak dynamical constraint
3. Results
   - DNS Data
   - DNS Data - Twin experiments
   - PIV Data
We have some observations $\mathcal{Y}$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{data_assimilation_graph.png}
\end{figure}
Data Assimilation
What is Data Assimilation?

We know a model $\mathbb{M}$. $\quad u$ model parameters

$$\frac{\partial X(t)}{\partial t} + \mathbb{M}(X(t), u) = 0 \quad ; \quad X(0) = X_0$$
Data Assimilation
What is Data Assimilation?

We know a background solution \((X_0^b, u^b)\). \((\eta, u)\) control parameters.

\[
\frac{\partial X(t)}{\partial t} + M(X(t), u) = 0 \quad ; \quad X(0) = X_0^b + \eta
\]
Data Assimilation

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Data Assimilation
What is Data Assimilation?

4D-Var: search for \((\eta^a, u^a) = \text{argmin}(J(\eta, u))\)

\[
J(\eta, u) = \frac{1}{2} \int_0^T \| \mathcal{Y} - \mathbb{H}(X(t; \eta, u)) \|^2_{R^{-1}} dt + \frac{1}{2} \| \eta \|^2_{B^{-1}} + \frac{1}{2} \| u - u^b \|^2_{C^{-1}}
\]
Data Assimilation
What is Data Assimilation?

4D-Var: search for \((\eta^a, u^a) = \text{argmin}(J(\eta, u))\)

\[
J(\eta, u) = \frac{1}{2} \int_0^T \left\| Y - H(X(t; \eta, u)) \right\|_R^2 dt + \frac{1}{2} \left\| \eta \right\|_B^2 + \frac{1}{2} \left\| u - u^b \right\|_C^2
\]
**Objectives of Data Assimilation:** Estimate

\[ X^a(t) = X(t; X_0^a, u^a) \]
Objectives of Data Assimilation: Forecast

\[ X^a(t) = X(t; X_0^a, u^a) \]
Formalism

Strong dynamical constraint
Variational Data Assimilation
Formalism

Strong dynamical constraint (Papadakis, 2007)

Model:

\[
\frac{\partial X(t)}{\partial t} + \mathbb{M}(X(t), u) = 0
\]

\[
X(0) = X_0 + \eta
\]

Cost functional:

\[
J(\eta, u) = \frac{1}{2} \int_0^T \| Y - \mathbb{H}(X(t; \eta, u)) \|^2_{R^{-1}} dt + \frac{1}{2} \| \eta \|^2_{B^{-1}} + \frac{1}{2} \| u - u^b \|^2_{C^{-1}}
\]

- \( \mathbb{H} \) observation operator
- \( R, B \) and \( C \) correlation matrix which represent how we trust in the observations and the background solutions.
Formalism

Strong dynamical constraint

Find $\nabla J$ for minimisation:

- Adjoint equation

$$-\frac{\partial \lambda}{\partial t}(t) + \left(\frac{\partial M}{\partial X}\right)^* \lambda(t) = \left(\frac{\partial H}{\partial X}\right)^* R^{-1} (H(X(t)) - Y)$$

$$\lambda(T) = 0$$

- Optimality condition

$$\frac{\partial J}{\partial \eta} = \lambda(0) + B^{-1} \eta$$

$$\frac{\partial J}{\partial u} = -\int_0^T \left(\frac{\partial M}{\partial u}\right)^* \lambda(t) dt + C^{-1}(u - u^b)$$
Formalism

Weak dynamical constraint

Variational Data Assimilation
Formalism

Weak dynamical constraint (Papadakis, 2007)

Model:

\[
\frac{\partial X(t)}{\partial t} + \mathbb{M}(X(t)) = w(t) \quad ; \quad X(0) = X_0 + \eta
\]

Cost functional:

\[
J(\eta, w(t)) = \frac{1}{2} \int_{0}^{T} \| \mathcal{Y} - \mathbb{H}(X(t)) \|^2_{\mathcal{R}^{-1}} dt + \frac{1}{2} \| \eta \|^2_{\mathcal{B}^{-1}} + \frac{1}{2} \int_{0}^{T} \| w(t) \|^2_{\mathcal{W}^{-1}} dt
\]

Adjoint equation: Unchanged

Optimality condition:

\[
\frac{\partial J}{\partial \eta} = \lambda(0) + \mathcal{B}^{-1} \quad ; \quad \frac{\partial J}{\partial w(t)} = \lambda(t) + \mathcal{W}^{-1} w(t)
\]
### DNS Data

DNS data from a cylinder wake (DNS code lcare - IMFT)

- \( Re = 200 \)
- \( N_t = 200 \)
- 2 periods of vortex shedding
- 6 POD modes used and represent 99% of the flow energy

**POD Decomposition:**

\[
\mathbf{v}(x, t) = \mathbf{v}_m(x) + \sum_{i=1}^{N_t} a_i^p(t) \Phi_i(x)
\]
Results

DNS Data

Observations \( \mathcal{Y} : a_i^P(t) \) (D’Adamo, 2007)

Model \( \mathcal{M}_{gal} \):

\[
\begin{aligned}
\frac{d a_i^R(t)}{dt} &= C_i + \sum_{j=1}^{N_{gal}} L_{ij} a_j^R(t) + \sum_{j=1}^{N_{gal}} \sum_{k=1}^{N_{gal}} Q_{ijk} a_j^R(t)a_k^R(t) \\
 a^R(0) &= a^P(0) + \eta
\end{aligned}
\]

Background solution: *taken as initial control parameters*

- \( \eta^0 = 0 \)
- \( C_i^0 = L_{ij}^0 = Q_{ijk}^0 = 0 \)

Covariance matrices:

- \( \mathcal{R}^{-1} = \mathbb{I}, \mathcal{B}^{-1} = \mathcal{C}^{-1} = \sigma^2 \mathbb{I} \), \( \sigma = 10^{-3} \)
Results

DNS Data

Figure: 4D-Var for the DNS dataset.
Results

DNS Data - Twin experiments

Figure: 4D-Var twin experiments for the DNS dataset.
Results

DNS Data - Twin experiments

**Equation:**

\[
e(t) = \sqrt{\sum_{i=1}^{N_{\text{gal}}} (a_i(t) - a_i^{\text{True}}(t))^2} / \frac{1}{T} \int_0^T \sum_{i=1}^{N_{\text{gal}}} a_i^{\text{True}}(t)^2 dt
\]

**Figure:** Time evolutions of errors. Comparison between the case of perfect observations and the case of twin experiments.
PIV Data of a cylinder wake (N. Benard, Pprime)

- \( Re = 40000 \)
- \( N_t = 128 \)
- 10 periods of vortex shedding
- 16 POD modes used and represent 31\% of the flow energy

**Background solution:** *taken as initial control parameters*

<table>
<thead>
<tr>
<th>Strong 4D-Var</th>
<th>Weak 4D-Var</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta^0 = 0 )</td>
<td>Solution of the strong 4D-Var.</td>
</tr>
<tr>
<td>( C_i^0, L_{ij}^0, Q_{ijk}^0 ) found by Galerkin projection.</td>
<td></td>
</tr>
</tbody>
</table>

**Covariance matrices:**

- Strong: \( \sigma = 1 \)
- Weak: \( \sigma = 10^{-3} \)
Results

PIV Data

Strong and Weak constraint

Figure: 4D-Var for the PIV dataset. Strong and weak constraint.
Weak 4D-Var can relax the dynamical constraints and fit better the observations.

Figure: Time evolution of the errors between observations and estimation.

But no forecast is possible! \((w(t) \text{ undefined})\)
Conclusion

Method coming from meteorology/oceanography, applied on flow control.

4D-Var is a well established tool and the results depend on the quality of the inputs (*model, observations, background*).

The belief we have in each information has to be consistent with its imperfections.

Summary

Two methods of 4D-Var were presented.

POD ROM improved by 4D-Var.

Tested by twin experiments, then on PIV data.
QUESTIONS???
Formalism
Perfect model

Descent algorithm, we need $\nabla J$:

Finite differences

$$\left( \frac{\partial J}{\partial \eta}, \delta \eta \right) = \frac{J(\eta + \epsilon \delta \eta, u) - J(\eta, u)}{\epsilon}$$

$$\left( \frac{\partial J}{\partial u}, \delta u \right) = \frac{J(\eta, u + \epsilon \delta u) - J(\eta, u)}{\epsilon}$$

$N + 1$ temporal integrations

Adjoint method

$\nabla J$ is a function of $\lambda(t)$

2 temporal integrations
Twin experiments

Define a true state:
Analysed solution of the previous 4D-Var

Create artificial observations:
Noising the original observations by adding $\mathcal{N}/\sigma_i$ where
$\mathcal{N} \sim \mathcal{N}(0, \sigma_t^2)$, $\sigma_t = 0.2$ and $\sigma_i = 2\left[\frac{i+1}{2}\right]$.

Compare the true state and the new analysed solution:

$$e(t) = \sqrt{\sum_{i=1}^{N_{gal}} (a_i(t) - a_i^{\text{true}}(t))^2 / \frac{1}{T} \int_0^T \sum_{i=1}^{N_{gal}} a_i^{\text{true}}(t)^2 dt}$$
\( \eta \) is small (\( \approx 0.1\% \) mode amplitude).
Is it a sensitive control parameter?

Now we run the same simulation with \( \eta = 0 \) ...
Figure: 4D-Var with $\eta = 0$.
Results

DNS Data - Initial condition sensitivity

Perfect Model

Figure: Time evolutions of errors. Comparison between the case of perfect observations with $\eta \neq 0$ and $\eta = 0$. 

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