LIMITS OF THE TOOLS FOR DETECTION OF CHAOS IN ECONOMY APPLICATION TO THE STOCK RETURNS

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Abstract

Many recent developments about chaos ‘s presence in economy are opening new perspectives and reviving debate between endogenous or exogenous shocks in financial and economical theories. It seems difficult to discuss the impact of chaos's theory in economy before studying and validating tools for its detection. We focus our work on these tools, especially on Lyapunov exponents and correlation dimension, in order to establish the determinist or stochastic nature of stock returns. The results are divergent and conflicting, and don't allow to conclude about chaos ‘s presence in stock returns. Our reservations are not about algorithms but about the lack of significativity test. In order estimate the "validity" of the tools, we applied them on to chaotic series (logistic equation) in random series. The results of these simulations show the weakness of the tools: the first Lyapunov exponent estimated on a random series is positive.

Keywords: Stock returns, chaotic system, attractor, Lyapunov exponent, correlation dimension.

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Introduction.

Why being interested in chaos in economy ? The first reason lies in the capacity of dynamic chaotic to reproduce apparently random trajectories, property observed on a number of economic lot series and especially financial. The second reason emanates from the appearance of chaos in certain macroeconomic models whose most famous illustration is the model of Day (1982). The third reason takes its rise in the study of the business cycle. Indeed, the various phases expansion - recession - crisis - recovery are recurring, not periodicals, and differ by their amplitudes and their lengths. Thus, the assumption usually selected of linearity which admittedly presents the major asset to simplify estimations, cannot answer this asymmetry of the business cycle. The linearity leaves place to non-linearity thus generating the possibility of observing the dynamic chaotic ones, since it is necessary to have a certain degree of non-linearity to meet chaotic behaviours. The last reason for the search of the theory of chaos in economy is due to its universal range put forward by the studies undertaken on the point of Feigenbaum[1], where it appears a regularity and a topological constancy of geometrical convergence for various functions. Moreover, this universality was increased by the undertaken studies over the twenty last years on the deterministic dynamic systems, particularly on the appearance of chaos in simple dynamic systems: chaos is born neither from a dynamic complexity nor of an ignorance but it can emerge through the dynamic simple ones which control us every day. The object of this work consists in the study of the tools for detection of the chaos which are the correlation dimension and the Lyapunov exponents. These two tools make it possible in theory to distinguish a stochastic process from a chaotic process. The correlation dimension measures the degree of instability of a dynamic system while being interested in the long term evolution of the system and the Lyapunov exponents allow to estimate the rate of separation between two initially close trajectories. We propose here to apply these tools to the series of American stock returns (Dow-Jones and SP 500) and German (DAX). The search for chaos in the financial series has significant economic implications. Indeed, the traditional financial theory is based on the assumption of informational efficiency of the markets. According to this assumption, the price observed reflects all information available. Consequently, the fluctuations of market price can be allotted only to the appearance of new information not anticipated concerning the fundamental ones. In other words it is of the shocks of exogenous nature which disturb the market, but the assumption of chaos, by its deterministic character, generates an endogenous nature of the shocks calling into question the theory of the efficiency of the markets. In a first section, we will present the properties of dynamic chaotic, then in a second section the tools

[1] In addition let us note the importance of technical progress in the emergence of theory of chaos. The computers indeed made it possible to have a great capacity of calculation and to carry out many simulations. Thus, the topological properties and the instability of the numerical calculation of the attractor of Lorenz could " be visualised " convincing the more sceptics scientists.

for detection of chaos and in a last section we will judge relevance of these tools for the detection of chaos in the time series, particularly, for the financial series.

1. Concepts and formalisations of the properties of dynamic chaotic.

The chaotic dynamics show two fundamental characteristics: a local instability and a global stability. All the trajectories diverge on the attractor[1] (local instability), thus two points as close as possible will have two different trajectories. This property of sensitive dependence in the initial conditions is measured using the first Lyapunov exponent (cf infra). Conversely, all the trajectories converge towards the attractor (global stability), this topological characteristic is measured using the dimension of attractor (cf infra).

Thus, a chaotic dynamics is characterised by four properties: non-linearity, determinism, sensitivity to the initial conditions and presence of on attractor. A system is known as determinist[2] if its law of evolution is defined by an application F of X in X This law describes the evolution of the system per unit of time. If the system is represented by the initial condition x_0 at time T = 0, then it will be represented at time T (T integer) by the point x_t :

$x_t = f^t(x_0) = f(f(\ldots.f(x_0))$ the made up one is carried out t time

Explicitly, according to the definition of Brock (1986), a series $\{y_t\}_{t = 1,\ldots,T}$, has a deterministic explanation if there is a system $\{ h, f, x_0 \}$ such as:

- $y_t = h(x_t)$
- $x_t = f(x_{t-1})$
- X_0 condition initial

Where: $\{y_t\}$ is the series observed, h is an unknown function of $\mathbb{R}^n$ in $\mathbb{R}$ and F is the deterministic law of the definite system of $\mathbb{R}^n$ in $\mathbb{R}$.

Within this deterministic framework, an endogenous explanation of economic fluctuations is privileged: it is the system itself which "fluctuates". This approach is radically opposed to the theories of exogenous shocks, where in the absence of shocks (exogenous) the economy tightens in a monotonous way towards steady state. With the chaotic dynamics to introducing shocks is not useful to observe fluctuations. The debate between exogenous and endogenous shocks is started again by the emergence of chaos and the introduction of the assumption of non-linearity, making it possible to reveal series with the behaviours much closer to series observed in macroeconomics and finance. To observe a chaotic behaviour it is necessary for the system that governs dynamics to be deterministic, but any deterministic system is not chaotic, it must also have an attractor that is there is an attraction of all the trajectories in

[1] One speaks about attractor when all the points of a dynamic system, at the end of a certain time, converge towards a limited space: the attractor.
[2] To simplify, let us restrict with the models in discrete time.

under space. The attractor of a dynamic system corresponds to the unit on which is driven any X_t point representing the state of the deterministic dynamic system when one waits a time rather long (so that there is disappearance of the phenomena transitional[1]). Hence, it represents the long term behaviour of the dynamics system (it is defined in asymptotic term of limit). Formally, to define the concept of attractor, it is necessary to be placed within a deterministic framework:
That is to say the application f such as: $x_{t+1} = f(x_t)$.

A compact unit A is attractor if there is a open unit U in the nearness of A in the space phases such as: $f^t(x)$ tends towards A when T tends towards the infinite for $x \in U$.

The union of all the vicinities of U constitutes the space attraction of A, i.e. the set of initial conditions which at the end of a certain time will be attracted by the attractor. Most frequently met attractors are the fixed points and the cycles. The tores and the strange attractors are different attractors which are less current in economy. A tore of dimension r is noted T^r and corresponds to quasi-periodicals attractor having r independent periods, their particular form is in " doughnut ". Strange attractors, Lorenz's one is the which form are not always precise, are objects of not-integer dimension, defined by B Mandelbrot like fractals. This property is necessary for an attractor to be called a chaotic attractor but not sufficient; indeed, fractal dimension does not make it possible to conclude with chaotic attractor. Moreover, it is necessary that the attractor presents a sensitivity to the initial conditions.

An application F on a space X checks the property of sensitivity to the initial conditions if there is a B positive such as:

- $x \in X$ et $\varepsilon > 0$
- $\exists\, y \in X$ such as: $d(x,y) < \varepsilon$ et $d(f^t(x),f^t(y)) > \beta$

That is, there is a point y located at more at a distance ε of x for which the distance between reiterated f defined respectively in x and t is at least equal to β This definition implies that the chaotic dynamic are locally unstable, in measurement or two orbits initially close divergent. Two separate points of a " microscopic " distance lower than ε will diverge from a " macroscopic " distance higher than β at the end of a certain iteration number, until having totaly different trajectories. However the measuring accuracy of the initial conditions is finite, making unforecasting the chaotic systems. Such a tiny error will be amplified in the run of time: there is an unpredictability of long term of chaotic dynamics, only a very short term forecast is possibly. From now, it is not possible any more to conceive the unforecasting

[1] Ruelle (1991), let us note that it is advisable to consider dissipate systems, i.e. systems which dissipate energy so that attractor can exist.

like a difficulty of measurement external with the system or as being due to human ignorance, but like an intrinsic property with the system.

This short presentation of the chaotic dynamics properties leads us to the definition of chaotic dynamics[1]. An application f is known as chaotic on a unit invariant A[2], called attractor, if:

- f is transitive on A, i.e. if the orbit { X, f(x),…,f^t(x) } is dense in A. (the attractor is not decomposable).
- the periodic points, if there are some, are dense on A.
- f to show of the sensitivity to the initial conditions on A. The sensitivity to the initial conditions is of primary importance in the study of dynamic chaotic. The tools for detection of chaos are mainly based on this property.

2. Presentation of the tools for detection of chaos.

The chaotic dynamics show several characteristics: sensitivity to the initial conditions, unpredictability, an apparently randomed structure… This last property limits the use of traditional methods, based on the autocorrelation and the spectral concentration to detect the chaos presence. As deferred in table 1 values of the function of autocorrelation and the function of autocorrelation partial of the logistic function[3]:

$$X_t = 4\, X_{t-1}(1-X_{t-1})$$

and, of a series $\{Y_t\}$ presenting a phenomenon of antipersistence[4] simulated by a modelling ARFIMA (0, d, 0):

$$(1-L)^d\, Y_t = u_t$$

where - d is a real ranging between -0.5 and 0.
- u_t is a white noise

For the study we retained a parameter d equal to - 0.4.

[1] There are various definitions of chaos, we will present here most usually used, based on the sensitivity to the initial conditions cf Mignon (1998).
[2] A unit A is known as invariant by f if and only if: $f^t(A) = A$ for any t.
[3] The logistic function is one of the chaotic processes simplest. Let us note here that we are in the chaotic area insofar as the parameter of control is higher than the point of Feigenbaum (here equal to 4). For a detailed study of the logistic function, one will be able to refer to Abraham-Frois and Berrebi (1995).
[4] A antipersistent process is a particular form of process to long memory characterised by a function of autocorrelation which alternates of sign and which decreases at an hyperbolic rate

Table 1 – Functions of autocorrelation and partial autocorrelation:.

	Autocorrelation	Partial autocorrelation	Autocorrelation	Partial autocorrelation
Retards	Function logistic	Function logistic	ARFIMA	ARFIMA
1	0.0017	0.016	-0.00011	-0.00011
2	0.006	0.0059	-0.00092	-0.00092
3	0.0039	0.004	-0.0034	-0.0034
5	-0.001	-0.001	0.00047	0.00047
7	0.003	0.004	0.0011	0.0011
10	0.013	0.0013	-0.0006	-0.0006

The simulated series shows the expected characteristics a long memory process of antipersistence (autocorrelation decreasing sinusoidal function) and whose values are without significant difference. The logistic equation showing these same characteristics, one cannot differentiate these two series starting from the usual statistical tools. Hence, it is fundamental to introduce new tools to detect the presence of chaos in a time series.

2.1 Rebuilding of attractor by the m-histories.

The detection of chaos lies in the search of the properties of attractor subjacent with a dynamics. However, the estimation of the values of the various instruments of detection of chaos requires to know the equations of evolution of the dynamic system. Those are unknown in economy since time series are the only one observed. So, it is impossible to understand dynamics starting from only one series. The Takens theorem (1981) makes it possible to mitigate this insufficiency of information by rebuilding the attractor by m-histories. Let us take again in a heuristic way formalisation of Brock (1986) between the observations and the state vector:

Let : x_t the state vector dynamic system at the date t

"Nature" knows the f function making it possible to pass from x_t to x_{t+1} (i.e. equations of dynamics evolution). On the other hand the observer knows only the time series $\{a_t\}_{t=1\ T.}$

By Takens' theorem (1981), it is possible to know all the properties of the system starting from $\{a_t\}$ observations. For that it is necessary to build m-histories.

The principle of construction of the m-histories is to create (T-m+1) vectors of m dimension starting from the series $\{a_t\}$[1]:

$$\{a^m_t\}_{t=1,\ldots,T-m+1}$$

$$\text{with} \quad a^m_t = (a_t, a_{t+\tau}, a_{t+2\tau}, \ldots, a_{t+(m-1)\tau})$$

The Takens theorem establishes that on the condition of choosing m rather large, the behaviour of the m-histories will imitate that of dynamics subjacent with the system (which is unknown). Takens considers m rather large if it is higher than 2n+1, where N is the unknown dimension of " true " attractor. From the rebuilding of attractor can apply the tools for detection of chaos. We paid our attention on the two most used tools: Lyapunov exponents and the correlation dimension. The first makes it possible to quantify the sensitivity to the initial conditions and the second gives an estimation of the complexity of attractor.

2.2 Exponents of Lyapunov.

Let us suppose that a series $\{x_t\}$ is described by the function[2]:

$$x_t = f(x_{t-1}) = 10\, x_{t-1} = 10^t\, x_0$$

where x_0 is the initial value measured with a precision 10^{-n}.

x_t represents the position of the point x_0 after x_t turns on a circle. The initial value is known with a precision of 10^{-N} . Thus, after N iterations the evolution of the system becomes independent of the initial condition. It does not provide any more any information.

The theory of chaos is paradoxical, it exists a sensitivity to the initial conditions preventing any forecasibilty although we can have recourse to a negligible precision. Of the another with dimensions information provided by the initial conditions does not have any more influence at the end of a certain time on the evolution of dynamics. To characterise a chaotic process, it is necessary to determine its sensitivity to the initial conditions. This one is measured by the value of the first Lyapunov exponent whose principle of estimation is based on the evolution of the variation in the course of time between two very close points initially. More precisely it measures the rate of separation between two initially close trajectories. This variation in the course of time diverges in an exponential way, i.e. that a small initial variation dx_0 between two trajectories will increase in an exponential way in the course of time:

$$\delta x_T = \delta x_0 \exp(\lambda_1 T)$$

[1] Where: T is the number of observations of the time series a_t , m is called embedding dimension. It corresponds to the number of axes necessary to represent the attracor, τ corresponds to the shift applied to each successive value of the series observed, it is selected like the first delay allowing the absence of temporal dependence in the series (τ is fixed)

[2] This example is borrowed from Ekeland (1991).

Where - λ_1 is the largest Lyapunov exponent

- δx_T the difference between the two trajectories after T iterations.

If λ_1 is positive that means that the variation grows in the course of time, decrease if there is negative and remains constant if λ_1 is null. So that a dynamic system is chaotic it is necessary that at least one of the exponents is positive. It is a theoretically sufficient condition to show the presence from chaos. However, insofar as there is not significativity test of the positivity of Lyapunov exponents, this condition is generally regarded as necessary but not sufficient to the presence of chaos. To clarify the estimation of Lyapunov exponents, let us consider an application F:

$\mathbb{R} \to \mathbb{R}$, such as $x_{t+1} = f(x_t)_{t=1\ldots T}$ et x_0 fixed.

To measure speed to which two trajectories diverge exponentially, it is necessary to introduce the average rate of separation δ after T iterations, definite like:

$$\delta = \frac{1}{T} \ln \left[\frac{f^T(x_0+\varepsilon) - f^T(x_0)}{|\varepsilon|} \right]$$

where ε is the smallest possible difference between two initial points

One defines the Lyapunov[1] exponents by:

$$\lambda = \lim_{T\to\infty} \lim_{\varepsilon\to 0} \delta$$

One can still define the Lyapunov exponents by:

$$\lambda = \lim_{T\to\infty} \frac{1}{T} \ln \left| \frac{df^T(x_0)}{dx_0} \right| = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0\ldots T} \ln \left| \frac{df(x_t)}{dx_t} \right|$$

The sign of the first Lyapunov exponent is of primary importance for the study of the detection of chaos in a time series. There are several estimation algorithms to estimate it. We retained the algorithm of Wolf and Al (1985) which, by its rather simple formalisation, allows to give an estimation of the first Lyapunov exponent. Initially it is necessary to rebuild the attractor by the use of m-histories. The positioning of a point on the attractor is then given by:

$$\{x(t),x(t+\tau),..,x(t+(m-1)\tau\}$$

One considers an initial point on an unspecified orbit of attractor, defined by:

[1] The Lyapunov exponents note λ and by convention (Oseledec) are arranged in descending order ($\lambda_n > \lambda_{n+1}$). The number of Lyapunov exponent is equal to the dimension of dynamics: here there is thus only one of them. In a space $\mathbb{R}^n$, one carries out the analogue between the derivative and the matrix Jacobean to calculate the λ. Let us note finally that the Lyapunov exponents are calculated in asymptotic limit, one thus considers only the behaviour of long term of the system (i.e. the behaviour on the attractor).

$\{x(t_0),x(t_0+\tau),..,x(t_0+(m-1)\tau\}$ (initial value of a trajectory A of reference)

One locates the point nearest to this initial point at the date t_0 (value initial of a trajectory B), and one notes L(t0) the distance between these two points. This distance will evolve in time. On date t_1 ($t_1 > t_0$), the distance is $L'(t_1)$, two cases are to be considered:

Cas1: there is a point pertaining to a trajectory C such as:

- $L(t_1) < L'(t_1)$ ($L(t_1)$ must be weak 10% of extended from the series)
- Angle ϑ_1, between $L'(t_1)$ et $L(t_1)$ is small

i.e., that at the date t_1, the point of the trajectory C is closer to trajectory A than is to it the point of the trajectory B (Fig.1).

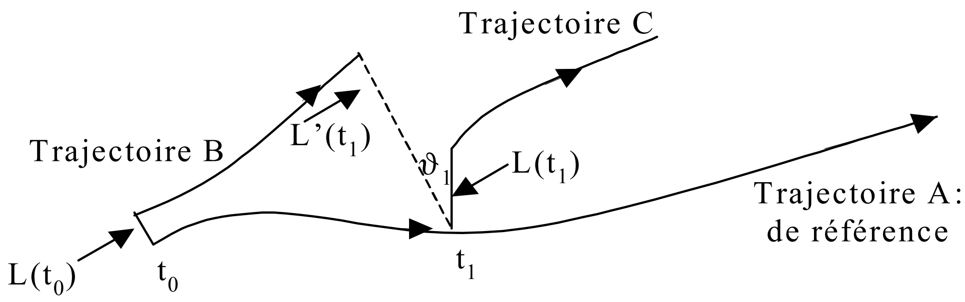

Fig.1- *Diagrammatic representation of the algorithm of Wolf and Al (1985).*

Cas 2 : there is not point answering the preceding criteria and:

- If the distance $L'(t_1)$ lower than 10% of is extended from the series, then the same initial points are kept and one looks at the evolution of the distance for a time higher than t_1
- Or if the distance $L'(t_1)$ higher than 10% of is extended from the series, then one cannot determine the value of the first Lyapunov exponent, it is necessary to choose another initial point for the trajectory of reference. One reiterates this procedure until the trajectory of reference crossed the totality of the data. The first Lyapunov exponent is estimated by:

$$\lambda_1 = \left(\frac{1}{t_M - t_0}\right) \sum_{k=1,\dots,M} \log_2\left(\frac{L'(t_k)}{L(t_{k-1})}\right)^1$$

[1] M is the number of replacements of trajectory.

The various values limit given by Wolf et al. for the distances $L'(t_1)$ and $L(t_1)$ makes it possible to obtain a stable value of the first Lyapunov exponent. These values are the same ones for each iteration of the procedure.

2.3 ***The correlation dimension of Grassberger and Procaccia (1983).***

One measures the complexity of a dynamic system by his dimension because it characterises the geometrical properties of the system. Thus, the concept of dimension provides a further information on the dynamics of the systems to the Lyapunov exponents. Indeed, the latter make it possible to highlight the instability or not of a dynamic system while being interested in the evolution of long term of the system, i.e. in dynamics on the attractor. The measurement of dimension makes it possible to detect a dynamics subjacent with a stochastic process (very large dimension) or with a deterministic process (small dimension); it thus allows, in theory, to establish a distinction between these two kinds of process. The correlation dimension is generally used to estimate the dimension of attractor in economy because it has the major advantage to be very easy to calculate compared to other dimensions whose estimate algorithms are enough difficult. For developments relating to various dimensions we return the reader to "Walked financial and modelling of stock returns" (1998) Valerie Mignon.

The existence of attractor generates a long-term convergence towards a limited space. Thus the correlation dimension measures a space and non temporal correlation between the points. It quantifies the number of points close on the attractor (i.e. at a distance lower than ε) estimated by the integral to noted correlation C(ε). Let us consider a whole of points on attractor the $\{ X_i, _{i=1 \dots N} \}$. the integral of correlation is given by:

$$C(\varepsilon) = \lim_{N\to\infty} \left(\frac{1}{N^2 - N}\right) \times [\text{ number of pair } (X_i, X_j) < \varepsilon]$$

Thus, Grassberger and Procaccia (1983) define the correlation dimension like

$$D_C = \lim_{\varepsilon\to 0} \frac{\ln C(\varepsilon)}{\log \varepsilon}$$

The economist having only time series, it is again necessary to recreate the dynamics of attractor by building the m-histories (theorem of Takens (1981)). One then carries out estimations of integral and correlation dimension on the m-histories:

$$C_m(\varepsilon) = \lim_{Nm\to\infty} \left(\frac{1}{Nm^2 - Nm}\right) \times \sum_{i,j=1,..,Nm} \times H(\varepsilon - |X_{im}, X_{jm}|)$$

Where Nm is the number of m-histories which can be created starting from a sample of N observations and H(X) is related to heaviside H=1 if X>0 (if not = 0).

The authors show that for small values of ε:

$$Dm = \lim_{\varepsilon \to 0} \frac{\ln Cm(\varepsilon)}{\log \varepsilon}$$

and if Dm is stabilised when m increases then:

$$Dc = \lim_{m \to \infty} Dm$$

For a deterministic process Dm is stabilised when m increases what is not the case for a stochastic process; indeed the correlation dimension becomes independent of the embedding dimension m (m > 2n+1) for a deterministic process. The economist having finite samples cannot make tighten the embedding dimension m towards the infinite one and even less to calculate one dimension which would tend towards the infinite one. Thus, the measurement of the correlation dimension mainly makes it possible to detect a small value of dimension, and thus to characterise the dynamic determinists. To study the process subjacent with a dynamics it is thus necessary to couple several concepts in order to try to show the presence or not of chaos. Let us recall that the Lyapunov exponents allow to characterise the structure of dynamics and the correlation dimension informs about the geometrical properties of the attractor. There are two tests of validation of the tools for detection of chaos bringing additional information on the study of the distinction between deterministic and stochastic dynamics: the residual test of Brock (1986) and the random mixing test of Scheinkman and LeBaron (1989).

2.4 Residual test of Brock[1] (1986).

Base test: The properties of a deterministic process are not modified by a linear transformation.

Presentation of the test:

- $\{a_t\}_{t=1...T}$, a time series
- $\{\varepsilon_t\}_{t=1...T}$,, the residue estimated by the following autoregressive model:

$\varphi(L)\, a_t = \varepsilon_t$ (1) Where $\varphi(L)$ is a polynomial delay.

If the process subjacent with dynamics is of deterministic type chaotic then $\{a_t\}$ and $\{\varepsilon_t\}$ have the same values for the first Lyapunov exponent and the correlation dimension .

If we were in the presence of a chaotic dynamics, we would find a first Lyapunov exponent positive and a correlation dimension positive and finite. But, not being able to have

[1] The mathematical demonstration of the residual test being rather heavy, we return the reader to the article of Brock (1986) " Distinguishing random and determinic systems "

statistics of test for the results obtained, it would be difficult to conclude with certainty. The Brock test allows a validation of the results then.

One seeks thus, using the residual test, to differentiate the dynamic determinists from the stochastic ones. The making use of the test as well as the possible results are described schematically hereafter:

1- One builds the m-histories and one applies the various tools for detection to the series original

If stochastic	If deterministic
Negative first Lyapunov exponent Dimension Large	Positive first Lyapunov exponent Dimension Small

If one shows a deterministic dynamics initially then:

2- The residual test is applied:

If the results are divergent	If the same results exactly are found
One is in the presence of a stochastic dynamics	One is in the presence of a deterministic dynamics

2.5 Random mixed test of Scheinkman and LeBaron (1989).

Base test: There is attractor in the case of dynamic chaotic.

Presentation of the test: Within the framework of a chaotic dynamics, a random mixing of the data prevents the rebuilding of attractor subjacent. Indeed, the mixing will cause to destroy the internal structure of the system. Conversely if dynamics is stochastic, the mixing will have only one very weak impact.

Let us note that insofar as these tests are based on the properties of dynamic by the estimation of the tools for detection of chaos, they are not statistical tests of significativity but only of validation.

1- One builds the m-histories and one applies the various tools for detection to the original series

If stochastic	If deterministic
Negative first Lyapunov exponent Dimension Large	Positive first Lyapunov exponent Dimension Small

2- One applies the random mixing test:

If stochastic	If deterministic
The mixing of the data should not have any impact on dynamics. One must find the same results	The mixing of the data destroyed the attractor (it exists). One then finds a negative first Lyapunov exponent, and a large correlation dimension

Let us note that insofar as these tests are based on the properties of dynamic by the estimation of the tools for detection of chaos, they are not statistical tests of significativity but only of validation.

3. Empirical application.

The focus of our study is on the SP 500 composite, Dow-Jones composite and the DAX. The choice of taking some series in composite was made after reading many articles and particularly Dufrénot and Matthieu (1994). The authors suggest the rejection of presence of chaos in the studied series by the existence of a aggregation skew which present the economic variables. By considering some of the indices in " composite ", reduce " skew " because only the industrial, pharmaceutical part … index was retained. As previously noticed tools for detection of chaos are defined in asymptotic limits. The three series were thus retained in daily[1], in order to have a maximum number of data to be able to calculate the values of the Lyapunov exponents and correlation dimension .

Table 2 – Series.

INDEX	Country	Dates	Number observations
DAX	Germany	22/10/71 - 21/07/00	7500
DJ	USA	22/10/71 - 21/07/00	7500
SP 500	USA	22/10/71 - 21/07/00	7500

The study of the time series is carried out starting from many statistical tests, of which the use is not possible that with stationary series. Moreover, for the application of the tools for detection of chaos, the series must also be stationary. The first stage of any study thus consists

[1] The choice to take data in frequency intra-day (to have a more significant sample) was not retained because the economic interpretation of the results would be then delicate.

with stationarize the series. The passage in logarithm and differentiation on the indexes of security prices made it possible to obtain series stationarized. The series of indexes of security prices in variation logarithmic curve can be then to be interpreted in terms of stock returns. We thus will work on stock returns, and will study their stochastic or deterministic or even chaotic character.

Table 3 – Descriptive statistics on the series of stock returns

Series	Number observations	Statistic of ADF test (model, statistic test*)	Skewness	Kurtosis	Mean	Standard errors
DAX	7499	(model [1], -21.26)	-3.03	69.9	0.00037	$9\ 10^{-3}$
DJ	7499	(model [1], -35.51)	-0.66	9.55	0.00031	$1.1\ 10^{-3}$
SP 500	7499	(model [1], -34.17)	-1.93	46.6	0.00036	$9.5\ 10^{-3}$

* statistics test are to be compared with the threshold from 5% to -1.95.

The three series have a negative skewness testifying to a distribution spread out towards the left, significant of an asymmetrical distribution. This asymmetry can be a conjecture in favour of the non-linearity of the series since the linearity makes it possible to give an account only of symmetrical distributions. Moreover, the kurtosis are very high, significant of a great probability for the extreme points. The two preceding observations suggest the search for chaos in the series of stock returns. Indeed, we noted previously that non linearity is a condition necessary to the appearance of chaos, and one of the principal characteristics of dynamic chaotic is to present trajectories being able to evolve/move in an apparent disorder and to take extreme values. Before the application of the tools for detection of chaos we will test the character iid (identically and identically distributed) of the various series by means of BDS test.

3.1 Presentation of BDS test (Brock, Dechert, Scheinkman 1987).

BDS test makes it possible to test the null assumption of series iid against a not specified alternative assumption. The alternative assumption is not defined, the rejection of H_0 can derive:

- Either from a non stationarity of the series
- Or from a structure of dependence resulting from a linear stochastic process
- Or from a structure of dependence resulting from a non linear process.

Let us note that under the null assumption the statistics of test follow a normal [0;1], it thus should be compared with the threshold 5% to 1.96.
Brock, Hsieh and LeBaron (1992) defined in an empirical way that the statistics of test are correctly approximated in finite sample if:

- the number of data T is higher than 500.
- ε must lie between 0.5 θ and 2 θ, where θ is the standard deviation of the series.
- the embedding dimension m must be lower than T / 200.

The following tables give the statistics of BDS test, according to the embedding dimension m and of ε / θ the statistics of test are always higher than 1.96 with the threshold of 5%, H_0 of series iid is thus rejected for the three series of studied stock returns. Consequently, stock returns depend on passed stock returns, which goes against the markets efficiency theory where the unforeseeable character of stock returns is modelled by a random walk. As noticed by Mignon (1994) and Leroy (1992), the rejection of H_0 does not imply the markets inefficiency, for example it is enough that neutrality respecting risk is not respected. Conversely, the acceptance of the assumption of iid stock returns involves the markets efficiency.

Table 4 – Results of BDS test.

embedding dimension m	DAX ε / θ				DJ ε / θ			
	0.5	1	1.5	2	0.5	1	1.5	2
2	14.38	16.9	18.09	17.9	6.75	9.01	11.7	13.6
5	31.58	32.2	31.71	30.02	15.48	15.4	17.5	19.1
10	86.3	63.4	47.7	39.25	14.7	29.8	25.2	24.6
15	45.02	149.2	72.9	47.89	6972	65.1	34.5	28.6

embedding dimension m	SP ε / θ			
	0.5	1	1.5	2
2	7.001	9.33	11.9	13.8
5	14.49	17.2	19.56	20.7
10	36.1	32.8	29.4	27.6
15	106.2	65.06	41.9	33.2

Stock returns are stationary series, the rejection of H_0 can be due only to the existence of a linear dependence or not. While using the residual test of Brock, we will eliminate the linear dependence if there is one, and consequently the rejection of H_0 of BDS test will be ascribable only with the presence of a non linear process, which one will be able to study the stochastic character or deterministic by the estimation of the Lyapunov exponents and correlation dimension . Within this framework BDS test becomes a non linearity test (cf. infra).

3.2 Estimation of the correlation dimension of stock returns.

The estimation of the correlation dimension is made starting from the estimation of the correlation integral where it is necessary to choose values for m and ε. We retained the same criteria as those of BDS test. The following table gives the results obtained for the correlation dimension for values of m going from 2 to 15 and ε between 0.5 θ and 2 θ.

Table 5 – Estimation of the correlation dimension

embedding dimension m	DAX ε / θ				DJ ε / θ			
	0.5	1	1.5	2	0.5	1	1.5	2
2	0.6	0.339	0.202	0.117	0.502	0.274	0.151	0.082
5	1.37	0.748	0.437	0.253	1.18	0.642	0.34	0.19
10	2.48	1.27	0.733	0.423	1.89	1.14	0.623	0.338
15	3.39	1.69	0.967	0.559	2.07	1.52	0.85	0.465

embedding dimension m	SP ε / θ			
	0.5	1	1.5	2
2	0.511	0.285	0.161	0.09
5	1.21	0.661	0.367	0.204
10	2.22	1.17	0.644	0.357
15	3.12	1.59	0.87	0.484

The correlation dimension grows with the embedding dimension for the three series of studied stock returns. This leads us to show the existence from a stochastic structure of stock returns. Nevertheless the correlation dimension increases less quickly than the embedding dimension what is indicating presence of a certain structure in the series. With through this estimation of the correlation dimension , one meets one of the major problems of the application of the tools for detection of chaos to the economic lot sizes. The low number of data which one lays out, in particular compared to the physical series for which are intended these various tools, prevents from calculating the correlation dimension for a high embedding dimension. For m large one would have can be a saturation of the correlation dimension , testifying this time, of a deterministic structure of stock returns. One should not draw definitive conclusions, although within sight of the only results, stock returns seem to be characterised by a stochastic structure. The value of the first Lyapunov exponent estimated on stock returns will allow us can be to obtain more distinct results.

3.3 Estimation of the first Lyapunov exponent

The estimation of the first Lyapunov exponent was carried out by means of the algorithm of Wolf[1]. It is necessary to vary time passed between two trajectories to obtain stable values of the exponent of Lyapunov. Here, one has varied 10 days, either two weeks, 20 days and 40 days or approximately a month and two months. In addition, we estimate for various values of the embedding dimension m: 2,4,8 and 16.

Table 6 – Estimation of the first exponent of Lyapunov

	Steps	m =2	m = 4	m = 8	m = 16
DAX	10	0.0258	0.042	0.0219	0.0464
	20	0.0165	0.029	0.0181	0.0254
	40	0.0073	0.011	0.0053	0.003
DJ	10	-0.0001	0.0111	0.0128	0.00251
	20	0.0071	0.0064	0.0096	0.00143
	40	-0.0002	0.0042	0.0041	0.00102
SP	10	-0.0038	0.0076	0.0156	0.0025
	20	-0.0028	0.0059	0.0081	0.0018
	40	-0.0013	0.0051	0.0047	0.0005

All the Lyapunov exponents are positive, except for m = 2 for stock returns DJ and SP, because the low embedding dimension limits the rebuilding of attractor by the m-histories. The estimation of the first Lyapunov exponent poses the problem of the positivity of the exponent: to consider a value of about 10^{-3} significantly different from zero seems to be a

[1] This method requires a calibration we return the reader to work of Wolf and Al (1985) and Médio (1992) for more details.

little " abusive ". However the positivity of the first exponent implies a structure of nature deterministic to stock returns. Let us note all the same that the value of the first Lyapunov exponent increases with the embedding dimension. This could be explained by a dimension of attractor subjacent with very high dynamics not being able to be rebuilt by the too low number of our data. The various estimations of the correlation dimension and the first Lyapunov exponent do not enable us to conclude as for the stochastic or deterministic character from the studied series. Thus let us apply the validation tests.

3.4 Residual test of Brock.

We used the method of Box-Jenkins in order to model our series by models ARMA. To validate the absence of autocorrelation in the residue we based ourselves on the Durbin test and the Ljung-Box test. The tables below present the estimation of the correlation dimension and the value of the Lyapunov exponent on the residues of modelling ARMA. Let us recall that if the structure of stock returns is of deterministic nature us should find the same characteristics for the first Lyapunov exponent and the correlation dimension estimated on stock returns.

Table 7 - Estimation of correlation dimension on the residues.

embedding dimension m	RES DAX ε / θ				RES DJ ε / θ			
	0.5	1	1.5	2	0.5	1	1.5	2
2	0.594	0.34	0.195	0.859	0.506	0.277	0.153	0.084
5	1.368	0.75	0.423	2.04	1.19	0.649	0.353	0.192
10	2.48	1.28	0.71	3.75	1.91	1.15	0.621	0.342
15	3.37	1.71	0.935	*	2.074	1.52	0.85	0.47

Embedding dimension m	RES SP ε / θ			
	0.5	1	1.5	2
2	0.112	0.298	0.171	0.098
5	1.249	0.69	0.39	0.22
10	2.28	1.22	0.682	0.385
15	3.18	1.67	0.92	0.525

For the three series studied, the estimation of the correlation dimension on the residues is identical to that found for the series of stock returns. Moreover, the first Lyapunov exponent estimated on the residues is very close to that obtained on the series of stock returns.

The results of the residual test testify in favour of a chaotic structure of stock returns. Let us note all the same that the correlation dimension does not arrive at saturation although it increases less quickly than the embedding dimension and than the Lyapunov exponents are very close to zero.

Table 8 - Estimation of the first Lyapunov exponent on the residues.

	Steps	m =2	m = 4	m = 8	m = 16
DAX	10	0.0202	0.044	0.024	0.0048
	20	0.071	0.022	0.0109	0.0012
	40	0.007	0.012	0.0047	0.0011
DJ	10	-0.0022	0.11	0.0124	0.00215
	20	-0.0011	0.0087	0.0077	0.0013
	40	-0.0005	0.005	0.0047	0.0007
SP	10	-0.0087	0.0133	0.019	0.0026
	20	-0.0015	0.0063	0.0101	0.0014
	40	0.0026	0.0054	0.005	0.0051

Before applying the second validation test let us re-estimate statistics BDS on the residues in order to carry out a non-linearity test on stock returns. By modelling the three series by a process ARMA, we removed the linear dependence, the rejection of the assumption H0 (iid) could be ascribable only with one non-linear structure of stock returns. The statistics of test, deferred in the table below, are higher than the threshold of 5 % to 1.96.

Table 9 – Results of BDS test on the residues.

embedding dimension m	RES DAX ε / θ				RES DJ ε / θ			
	0.5	1	1.5	2	0.5	1	1.5	2
2	14.19	16.4	17.7	13.79	6.68	8.54	10.81	12.8
5	30.13	31.28	30.97	28.8	15.9	15.2	17.2	18.8
10	78.7	61.13	46.19	132.2	152.2	30.2	25.4	24.6
15	40.58	143.6	69.2	*	76.3	68.28	34.9	28.72

embedding dimension m	RES SP ε / θ			
	0.5	1	1.5	2
2	7.5	9.37	11.27	12.7
5	15.07	17.45	19.45	20.05
10	39.3	33.8	29.9	27.86
15	130.5	68.8	43.5	33.9

Thus, stock returns have a structure of dependence non linear, but BDS test does not give any information on the stochastic or deterministic character of non linearity.

3.5 Random mixing test

After having carried out a random mixing of the data of the series of stock returns we estimated the correlation dimension and the first Lyapunov exponent on the transformed series whose results are summarised in the following tables:

Table 10 – Estimation of the correlation dimension on the mixed

embedding dimension m	Mixed DAX ε / θ				Mixed DJ ε / θ			
	0.5	1	1.5	2	0.5	1	1.5	2
2	0.616	0.35	0.212	0.124	0.51	0.282	0.157	0.09
5	1.53	0.882	0.255	0.309	1.276	0.707	0.394	0.217
10	3.04	1.75	1.05	0.617	2.57	1.423	0.796	0.44
15	*	2.62	1.57	0.923	3.71	2.149	1.2	0.662

embedding dimension m	Mixed SP ε / θ			
	0.5	1	1.5	2
2	0.521	0.293	0.167	0.094
5	1.304	0.737	0.419	0.235
10	2.6	1.487	0.846	0.474
15	3.93	2.23	1.27	0.713

For the transformed series the correlation dimension increases more quickly than the embedding dimension contrary to the results obtained for the series of stock returns. The random mixing thus destroyed the structure testifying rather to a deterministic structure of stock returns. On the other hand the estimation of the first Lyapunov exponent is always positive whereas the mixing would have to destroy the deterministic structure and to thus return negative Lyapunov exponent.

Table 11 - Estimation of the first Lyapunov exponent on the mixed series.

	Steps	m =2	m = 4	m = 8	m = 16
DAX	10	0.093	0.059	0.0233	0.0039
	20	0.0174	0.032	0.0135	0.0023
	40	0.0115	0.015	0.0052	0.0015
DJ	10	0.0018	0.181	0.029	0.0025
	20	-0.002	0.008	0.0142	0.0015
	40	-0.0015	0.0041	0.0063	0.0012
SP	10	-0.0029	0.0139	0.0238	0.0032
	20	0.018	0.0096	0.0119	0.0018
	40	0.002	0.0061	0.0064	0.001

3.6 Conclusion.

Although the embedding dimension does not arrive at saturation and the Lyapunov exponents are slightly higher than zero, the application of the tools on stock returns and the residues estimated by modelling ARMA, had led us to conclude rather in favour of the existence of a chaotic structure of the studied series. The random mixing test, where the Lyapunov exponents remain positive, invalid this conclusion but also invalids the estimation of the first Lyapunov exponent by the algorithm of Wolf. Indeed, these original stock returns are chaotic or stochastic, we should have found, after mixing, the first Lyapunov exponent is negative. Within sight of these results, it seems useful to apply the tools for detection of chaos to the logistic equation as presented in the first part and to check that the theoretically expected results appear empirically. Particularly, we will be interested in the values of the Lyapunov exponents .

4. Application of the tools for detection of chaos on simulated series.

We retained the logistic equation in order to work on a series whose chaotic character is certain. For that, we will take a parameter of control equal to 3.9. Let us point out what the chaotic area appears for a value of w higher than 3.569. We moreover will put in parallel the various estimations of the tools for detection of chaos for a random series simulated starting from a random draw of 7500 observations in a uniform law [0,1]. The table below recapitulates the estimations of the correlation dimension for various values of the embedding dimension m and ε.

Table 12 – Estimation of the correlation dimension on the simulated series.

embedding dimension m	Equation logistic ε / θ				Random series ε / θ			
	0.5	1	1.5	2	0.5	1	1.5	2
2	0.948	1.004	1.093	1.07	1.2	1.14	0.718	0.928
5	1.65	2.012	2.439	3.073	2.99	2.835	1.79	2.325
10	2.67	3.631	4.531	6.361	5.98	5.672	3.59	4.748
15	3.58	5.132	6.562	9.556	9.038	8.44	5.38	6.97

The correlation dimension of the logistic equation increases definitely less quickly than the embedding dimension. We find here the characteristics of our financial series, where within sight of the estimation of the correlation dimension we had concluded in favour of the existence of a structure of stock returns. Nevertheless, we do not arrive at saturation of the latter. Let us notice that the correlation dimension of the random series grows very quickly with the embedding dimension.

The obtained results on the correlation dimension of the simulated series are in agreement with the theory. Let us interest in the estimation of the first Lyapunov exponent, which results are presented below.

Table 13 – Estimation of the first Lyapunov exponent on the simulated series

	Steps	m =2	m = 4	m = 8	m = 16
Equation logistique	10	0.1866	0.228	0.3055	0.2651
	20	0.0865	0.118	0.1584	0.1593
	40	0.0455	0.0662	0.0788	0.0824
Série aléatoire	10	0.296	0.178	0.157	*
	20	0.121	0.092	0.079	*
	40	0.0733	0.049	0.041	*

The first Lyapunov exponent is positive for the logistic equation, expected result since dynamics is chaotic; on the other hand the positivity obtained for the random series is in disagreement with the theory. The validity of the algorithm of Wolf thus seems here to be called into question. Let us return to the results found on stock returns: we had rejected the chaotic assumption of dynamics starting from the random mixing test, and in particular on the positivity of the Lyapunov exponents. The study undertaken previously shows that the estimation of the Lyapunov exponents by the algorithm of Wolf " is skewed ", we can suppose that it over-estimates the value of the first Lyapunov exponent since it is positive whereas it should be negative. Consequently, the results resulting from the estimation of the first Lyapunov exponent on the series of stock returns are also skewed, and their positivity is called in question. The absence of significativity test of the tools for detection of chaos then seems one of the main difficulties of their application. Indeed, one could find the largest Lyapunov exponent is positive for a random series, but to have a test allowing to identify his non statistical significativity.

Conclusion .

The object of this work lies in the search of chaotic behaviours in the series of stock returns. In this focus, the two principal tools for detection of chaos, the correlation dimension and the Lyapunov exponents were applied. We showed that the correlation dimension did not arrive to saturation and that the first Lyapunov exponent was positive. Nevertheless the values estimated for the first Lyapunov exponents are very low, that leads us to wonder about their significant difference with zero. To answer, we implemented the residual test and the random mixing test. The obtained results proved to be contradictory and simulations implemented called into question the reliability of the algorithm of Wolf (1985). Thus, the current tools did not enable us to accept or to reject the assumption of chaos, it is impossible to show the presence of chaos in the economic series. Morever, starting from the results of BDS test, we can maintain that the financial series studied are non-linear, which confirms many current search of non-linear modelling in economy. Notions conveyed by the theory of chaos: determinism, irreversibility, arrow of time, which have already given place to a revolution in the physical science, open in economy some new prospect, as the search for state unsteady state, the modelling unify determinism and probabilistic[1] or still the use of multifractal[2] the forecast of financial market.

[1] Ilya Prigogine and Isabelle Stengers were the first to be been interested in the unification of these two terms in their rebuilding of nature. We return the reader to the works of Prigogine and Stengers (1979,1992).

[2] We return the reader to the article of Mandelbrot, Fisher and Calvet (1997).

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