New Spline Quasi-Interpolant for Fitting 3-D Data on the Sphere: Applications to Medical Imaging

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***Abstract*—In this paper, a new local spline quasi-interpolant is constructed for fitting 3-D data defined on the sphere-like surface S. After mapping the surface S onto a rectangular domain, we use the tensor product of cubic polynomial B-splines and 2π-periodic uniform algebraic trigonometric B-splines (UAT B-splines) of order four to introduce a new expression of the associated quasi-interpolant $\mathcal{Q}$. The use of UAT B-splines is necessary to enforce some boundary conditions which are useful to ensure the $\mathcal{C}^1$ continuity of the associated surface. The new method is particularly well designed to render 3-D closed surfaces. It has been successfully applied to reconstruct human organs such as the lung and left ventricle of the heart.**

***Index Terms*—B-spline, medical data, quasi-interpolant, sphere-like surface reconstruction, uniform algebraic trigonometric B-splines (UAT B-splines).**

I. INTRODUCTION

SPLINES have been widely used in medical imaging for surface reconstruction and visualization of human organs [1], [2]. Usually, medical representations are obtained from a few scattered noisy data [3]. However, physicians require realistic organ representations to refine their diagnoses. Depending on the availability or the nature of the original data, interpolated-based and/or smooth- based surface reconstruction methods are required. Various methods are developed in the literature for fitting 3-D data on the sphere-like surface, in particular see [4]–[11]. In this paper, we propose a new method for fitting 3-D data by using a spline quasi-interpolant. It is based on the tensor product method of polynomial B-splines and periodic uniform algebraic trigonometric B-splines (UAT B-splines) recently developed in [12]. This new method can be successfully applied to sphere-like surface of various organs (heart, lung, bladder, kidney, etc.).

II. PROBLEM STATEMENT

Let S be a closed and bounded surface in $\mathbb{R}^3$ which is topologically equivalent to a sphere, i.e., there exists a one to one mapping of S onto the unit sphere. In many applications, one needs to construct a function F, defined on S, and which satisfies $F(P_i) \approx r_i, i = 1, \ldots, d$, where $r_1, \ldots, r_d$ are given real numbers and $P_1, \ldots, P_d$ are points on S. The construction of F is done so that its associated surface $S_F = \{F(s)s,\ s \in S\}$ has at every point a tangent plane that varies continuously over the surface S_F. Without loss of generality, we assume that S is the unit sphere. Then it can be identified with the rectangular domain $D = I \times J$, where $I = [-(\pi/2), \pi/2]$ and $J = [0, 2\pi]$, by the mapping $\chi : D \longrightarrow S$ such that $\chi(\theta, \phi) = (z_1, z_2, z_3)$, where $z_1 = \cos\theta\cos\phi$, $z_2 = \cos\theta\sin\phi$, and $z_3 = \sin\theta$.

The associated surface of the polar coordinates' representation f of F, defined on D by $f = F \circ \chi$, is identical to that of F, i.e., $S_F = S_f = \{f(\theta, \phi)\chi(\theta, \phi),\ (\theta, \phi) \in D\}$. However, the smoothness properties of f are not equivalent to those of its corresponding closed surface S_f. According to [4], S_f is of class $\mathcal{C}^1$ if it has at every point a tangent plane that varies continuously over the surface S_f. More specifically, S_f is of class $\mathcal{C}^1$ if $f \in \mathcal{C}^1(D)$ and satisfies the following boundary conditions:

$$f(\theta, 0) = f(\theta, 2\pi), \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \qquad (C1)$$

$$f\left(\pm\frac{\pi}{2}, \phi\right) = c_\pm, \quad 0 \le \phi \le 2\pi \qquad (C2)$$

$$\frac{\partial f}{\partial \phi}(\theta, 0) = \frac{\partial f}{\partial \phi}(\theta, 2\pi), \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \qquad (C3)$$

$$\frac{\partial f}{\partial \theta}\left(\pm\frac{\pi}{2}, \phi\right) = a_\pm\ \cos\phi + b_\pm\ \sin\phi, \quad 0 \le \phi \le 2\pi \quad (C4)$$

where $a_\pm, b_\pm$ and $c_\pm$ are constants.

Now, if we set $\mathcal{F} := \{f \in \mathcal{C}^1(D) :$ conditions $(C1)$-$(C4)$ hold$\}$, then the problem of finding F such that S_F is of class $\mathcal{C}^1$ and satisfies $F(P_i) \approx r_i$, $1 \le i \le d$, becomes equivalent to finding f in $\mathcal{F}$ that satisfies $f(\theta_i, \phi_i) \approx r_i$ where (θ_i, ϕ_i) are the polar coordinates of P_i, i.e., $\chi(\theta_i, \phi_i) = P_i$. Since the problem is now posed on a rectangular domain, it is natural to use tensor-products for the construction of an approximating function $\widetilde{f}$ of the form

$$\widetilde{f}(\theta, \phi) = \sum_{i=1}^{n}\sum_{j=1}^{m} c(i, j) v_i(\theta)\, \tilde{v}_j(\phi) \qquad (1)$$

where $\{v_1(\theta), \ldots, v_n(\theta)\}$ (respectively, $\{\tilde{v}_1(\phi), \ldots, \tilde{v}_m(\phi)\}$) is a linearly independent set of functions on $[-(\pi/2), \pi/2]$ (respectively, on $[0, 2\pi]$).

Various choices of v_i and $\tilde{v}_j$ have been introduced in the literature (see [4] and [5]). The obvious one for both sets would be polynomial B-splines. However, since the trigonometric functions cosine and sine cannot be expressed in terms of polynomial splines, enforcing condition $(C4)$ is impossible using this choice, especially for $\tilde{v}_j$. Thus, condition $(C4)$ can only be approximately satisfied. To overcome this problem, the authors in [6]–[8] have chosen the periodic trigonometric B-splines of

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order three for $\tilde{v}_j$ and the quadratic polynomial B-splines for v_i. This work is generalized in [9] and [11] by using B-splines of high order, but in this case the trigonometric B-splines must be of odd order, in particular the cubic case cannot be used. Here, we use the cubic polynomial B-splines and 2π-periodic UAT B-splines of order four to introduce a new expression of the associated quasi-interpolant $\mathcal{Q}$. The use of UAT B-splines is necessary to enforce boundary conditions which are useful to ensure the $\mathcal{C}^1$ continuity of the associated surface.

III. CUBIC POLYNOMIAL B-SPLINES

For $I = [a, b] = [-(\pi/2), \pi/2]$ and given positive integers n and k, let $\Theta_n = \{\theta_i\}_{i=1}^{n+k}$ with mesh length $h_\theta = h = (b-a)/(n-k)$ be a uniform partition of the interval I defined by

$$\begin{cases} \theta_1 = \theta_2 = \cdots = \theta_k = a \\ \theta_i = a + (i-k)h, \text{ for } i = k+1, \ldots, n \\ \theta_{n+1} = \theta_{n+2} = \cdots = \theta_{n+k} = b \end{cases}.$$

The associated polynomial spline space of order k is defined by

$$\mathcal{S}_k(I, \Theta_n) = \left\{ s \in \mathcal{C}^{k-2}(I) : \ s_{|[\theta_i, \theta_{i+1}]} \in \mathbb{P}_{k-1} \right\}$$

where $\mathbb{P}_k$ is a polynomial space of degree $\leq\ k$. The classical normalized B-splines N_i^k of order k satisfy *supp* $N_i^k = [\theta_i, \theta_{i+k}]$ and $N_i^k(\theta) > 0$, for $\theta_i < \theta < \theta_{i+k}$. They form a partition of unity, i.e., $\sum_{i=1}^n N_i^k(\theta) = 1$ and the family $\left\{N_i^k,\ i = 1, 2, \ldots, n\right\}$ forms a basis of $\mathcal{S}_k(I, \Theta_n)$. Here, we denote by $N_i = N_i^4$ the cubic B-splines on the interval I endowed with the partition Θ_n.

We now give a local linear operator $\mathcal{Q}_1$ which maps a given function f onto a cubic spline space $\mathcal{S}_4(I, \Theta_n)$ and which has an optimal approximation order. This operator is the discrete $\mathcal{C}^2$ cubic spline quasi-interpolant defined by

$$\mathcal{Q}_1 f = \sum_{i=1}^{n} \lambda_i(f) N_i \tag{2}$$

where the coefficients $\lambda_i(f)$ are defined as linear combinations of some values of f on the set Θ_n in order to have the exactness of the quasi-interpolant $\mathcal{Q}_1$ on $\mathbb{P}_3$, i.e., $\mathcal{Q}_1 p = p$, for all $p \in \mathbb{P}_3$. More specifically, these coefficients are defined as follows:

$$\begin{aligned} \lambda_1(f) &= f(\theta_4) = f(a) \\ \lambda_2(f) &= \frac{1}{18}\left(7f(\theta_4) + 18f(\theta_5) - 9f(\theta_6)\right. \\ &\quad \left. + 2f(\theta_7)\right) \\ \lambda_j(f) &= \frac{1}{6}\left(-f(\theta_{j+1}) + 8f(\theta_{j+2}) - f(\theta_{j+3})\right) \\ &\quad \text{for } j = 3, \ldots, n-2 \\ \lambda_{n-1}(f) &= \frac{1}{18}\left(2f(\theta_{n-2}) - 9f(\theta_{n-1}) + 18f(\theta_n)\right. \\ &\quad \left. + 7f(\theta_{n+1})\right) \\ \lambda_n(f) &= f(\theta_{n+1}) = f(b). \end{aligned}$$

This quasi-interpolant has an order four approximation, i.e., $\|f - \mathcal{Q}_1 f\|_\infty = \bigcirc(h_\theta^4)$

IV. PERIODIC UNIFORM ALGEBRAIC TRIGONOMETRIC B-SPLINES

Let $J = [c, d] = [0, 2\pi]$ and for a given positive integer $m > 2$ and $0 < \alpha = (2\pi)/m < \pi$, let

$$\Phi_m = \{\phi_j = (j-1)\alpha, j = 1, \ldots, m+1\}$$

be a set of knots that subdivide the interval J uniformly. In order to define a trigonometric B-spline basis of order l, we add knots to the left of ϕ_1 and to the right of ϕ_{m+1} namely $\phi_{2-l} < \cdots < \phi_{-1} < \phi_0 < \phi_1$ and $\phi_{m+1} < \phi_{m+2} < \phi_{m+3} < \phi_{m+l}$. Using the partition Φ_m, we define the algebraic trigonometric spline space of order l by

$$\widetilde{\mathcal{S}}_l(J, \Phi_m) = \left\{ s \in \mathcal{C}^{1-2}(\mathbb{R}) : \ s_{|[\phi_j, \phi_{j+1}]} \in \Gamma_l \right\}$$

where $\Gamma_l = \left\{1, \phi, \ldots, \phi^{l-3}, \cos(\phi), \sin(\phi)\right\}$. A basis of a linear space $\widetilde{\mathcal{S}}_l(\Phi)$ is called a uniform algebraic trigonometric B-spline (UAT B-spline) basis of order l if the basis functions are nonnegative, form a partition of unity and have a minimal support. To construct a UAT B-spline basis of $\widetilde{\mathcal{S}}_l(J, \Phi_m)$, we first define a set of functions over $\widetilde{\mathcal{S}}_2(\Phi)$.

For $l = 2$, we put

$$M_0^2(\phi) = \frac{-1}{2[\cos(\alpha) - 1]} \begin{cases} \alpha \sin(\phi), & 0 \leq \phi \leq \alpha \\ \alpha \sin(2\alpha - \phi), & \alpha \leq \phi \leq 2\alpha \\ 0, & \text{elsewhere} \end{cases} \tag{3}$$

$$M_j^2(\phi) = M_0^2(\phi - j\alpha),\ j \in \mathbb{Z} \tag{4}$$

and for $l \geq 3$, we put

$$M_i^l(\phi) = \frac{1}{\alpha} \int_{\phi-\alpha}^{\phi} M_i^{l-1}(t)\, dt. \tag{5}$$

Some basic properties of the UAT B-spline basis of order l are listed as follows:

- $M_j^l \in \mathcal{C}^{l-2}(\mathbb{R})$, $M^k_{j\,|[\phi_j, \phi_{j+1}]} \in \Gamma_l$;
- $M_j^l(\phi) = M_0^l(\phi - \phi_j)$;
- $DM_j^l(\phi) = 1/\alpha \left[M_j^{l-1}(\phi) - M_{j+1}^{l-1}(\phi) \right]$;
- *supp* $M_j^l = [\phi_j, \phi_{j+l}]$ and $M_j^l(\phi) > 0$, $\phi_j < \phi < \phi_{j+l}$;
- they form a partition of unity: $\sum_{j=2-l}^{m} M_j^l(\theta) = 1$;
- the family $\left\{M_j^l,\ j = 2-l, \ldots, m\right\}$ forms a basis of the spline space $\widetilde{\mathcal{S}}_l(J, \Phi_m)$.

Similarly to those of the polynomial B-spline basis, the UAT B-spline basis has many optimal properties, such as the subdivision property, the variation diminishing property and the convexity preserving property.

In order to define the periodic UAT B-splines, we choose the additional knots such that

$$\phi_{-j} = \phi_{m-j} - 2\pi, \text{ and } \phi_{m+2+j} = \phi_{2+j} + 2\pi, j = 0, \ldots, l-2.$$

The associated periodic UAT B-splines are hence defined by

$$T_j^l(\phi) = \begin{cases} M_j^l(\phi), & j = 1, \ldots, m-l+1 \\ M_j^l(\phi) + M_j^l(\phi + 2\pi), & j = m-l+2, \ldots, m. \end{cases} \tag{6}$$

When $l = 4$, we denote by $T_j = T_j^4$ the periodic UAT B-splines of order four associated with the periodic knots Φ_m. These B-splines are of class $\mathcal{C}^2$ on the interval J and their restrictions to $[\phi_j, \phi_{j+1}]$ are in the space $\Gamma_l = \{1, \phi, \cos(\phi), \sin(\phi)\}$. We can verify that

$$1 = \sum_{j=1}^{m} T_j(\phi),\ \cos(\phi) = \sum_{j=1}^{m} \frac{\alpha \cos(\phi_{j+2})}{\sin(\alpha)}\, T_j(\phi)$$
$$\phi = \sum_{j=1}^{m} \phi_{j+2}\, T_j(\phi),\ \sin(\phi) = \sum_{j=1}^{m} \frac{\alpha \sin(\phi_{j+2})}{\sin(\alpha)}\, T_j(\phi).$$

The algebraic trigonometric quasi-interpolant which maps a given function f into $\widetilde{\mathcal{S}}_4(J, \Phi_m)$ is defined by

$$\widetilde{\mathcal{Q}}_2 f = \sum_{j=1}^{m} \widetilde{\lambda}_j(f)\, T_j \tag{7}$$

where $\widetilde{\lambda}_j$ is the linear functional defined by $\widetilde{\lambda}_j(f) = \sum_{s=1}^{4} \beta_{j,s}\ f(\tau_{j,s})$. In order to have the exactness of the quasi-interpolant $\widetilde{\mathcal{Q}}_2$ on Γ_4, i.e., $\widetilde{\mathcal{Q}}_2 q = q$, for all $q \in \Gamma_4$, the coefficients $\beta_{j,s}$ will satisfy the following equations:

$$\sum_{s=1}^{4} \beta_{j,s} = 1,\ \sum_{s=1}^{4} \beta_{j,s}\ \cos(\tau_{j,s}) = \frac{\alpha \cos(\phi_{j+2})}{\sin(\alpha)}$$
$$\sum_{s=1}^{4} \beta_{j,s}\ \tau_{j,s} = \phi_{j+2},\ \sum_{s=1}^{4} \beta_{j,s}\ \sin(\tau_{j,s}) = \frac{\alpha \sin(\phi_{j+2})}{\sin(\alpha)}.$$

If we choose $\tau_{j,s} = \phi_{j+s-1}$, then the above linear system has the unique solution $\{\beta_{j,s}, s = 1, \ldots, 4\}$.

By using the trigonometric Taylor's expansion of f about the point t, we have $f = T + R$ with $T \in \Gamma_3$ and $R(\phi) = \int_t^{\phi} \sin((\phi - u)/2)^2 D(D^2 + 1) f(u)\, du$, and by using the usual Taylor's expansion of R about the point t, we have $R = P + S$ with $P \in \mathbb{P}_1$ and $S(\phi) = \int_t^{\phi} (u - t) D^2 R(u)\, du$. Since $\widetilde{\mathcal{Q}}_2 P = P$ and $\widetilde{\mathcal{Q}}_2 T = T$, we obtain $|f - \widetilde{\mathcal{Q}}_2 f| = |S - \widetilde{\mathcal{Q}}_2 S| \leq K \max_{1 \leq j \leq m} |\widetilde{\lambda}_j(S)|$. After computation, we obtain $|\widetilde{\lambda}_j(S)| \leq \sum_{s=1}^{4} |\beta_{j,s}| |S(\tau_{j,s})| \leq C_j\ h_\phi^4$ where $h_\phi = \max_{1 \leq j \leq m} (\phi_{j+1} - \phi_j) = \alpha$. Consequently, we have:

Lemma 4.1: The quasi-interpolant $\widetilde{\mathcal{Q}}_2$ based on the UAT B-splines of order four has an order four approximation, i.e., $\|f - \widetilde{\mathcal{Q}}_2 f\|_\infty = \bigcirc(h_\phi^4)$.

V. QUASI-INTERPOLANT ON THE SPHERE

In this section, we construct a local linear operator $\mathcal{Q}$ which maps a given function in the space $\mathcal{F} := \{f \in \mathcal{C}^2(D) : \text{conditions } (C1) \text{ - } (C4) \text{ hold}\}$ into splines of the form (1) which also lie in $\mathcal{F}$. Using the linear functionals λ_i and $\widetilde{\lambda}_j$ described above, we define for $f \in \mathcal{C}^2(D)$

$$\mathcal{Q}f(\theta, \phi) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\lambda_i \widetilde{\lambda}_j f\right)\ N_i(\theta)\ T_j(\phi). \tag{8}$$

According to the preceding properties of N_i and T_j, the approximate function $\mathcal{Q}f$ is of class $\mathcal{C}^2$ on D. On the other hand, the conditions $(C1)$ – $(C4)$ can be satisfied if the coefficients $\lambda_i \widetilde{\lambda}_j f$ verify some additional conditions.

For these choices of $\{v_i\}$ and $\{\tilde{v}_j\}$ the approximating function $\widetilde{f}$ in (1) can be written in the form

$$\widetilde{f}(\theta, \phi) = \sum_{i=1}^{n} \sum_{j=1}^{m} c(i, j)\ N_i(\theta)\ T_j(\phi). \tag{9}$$

Using the fact that $\{T_j\}$ are 2π-periodic, we easily verify that the function $\widetilde{f}$ defined in (9) satisfies conditions $(C1)$ and $(C3)$. In order to satisfy the remainder conditions, some coefficients $c(i, j)$ of $\widetilde{f}$ are imposed as follows.

Lemma 5.1: The function $\widetilde{f}$ given in (9) satisfies the condition (C2) if and only if

$$c(1, j) = c_-, \text{ and } c(n, j) = c_+,\ j = 1, \ldots, m.$$

Lemma 5.2: The function $\widetilde{f}$ given in (9) satisfies the condition (C4) if and only if

$$c(2, j) = c(1, j) + \frac{h}{3}\left[a_- \frac{\alpha \cos(\phi_{j+2})}{\sin(\alpha)} + b_- \frac{\alpha \sin(\phi_{j+2})}{\sin(\alpha)}\right]$$
$$c(n-1, j) = c(n, j) - \frac{h}{3}\left[a_+ \frac{\alpha \cos(\phi_{j+2})}{\sin(\alpha)} + b_+ \frac{\alpha \sin(\phi_{j+2})}{\sin(\alpha)}\right]$$

for all $j = 1, \ldots, m$.

Theorem 5.1: If the function f lies in $\mathcal{F}$, then the associated quasi-interpolant $\mathcal{Q}f$ given in (8) lies also in $\mathcal{F}$. Moreover, we have $\mathcal{Q}p(\theta, \phi) = p(\theta, \phi)$, for all $p \in \mathbb{P}_3 \otimes \Gamma_4$, where $p \in \mathbb{P}_3 \otimes \Gamma_4$ is the tensor product of $\mathbb{P}_3$ and Γ_4, and $\mathcal{Q}$ has an order four approximation, i.e., $\|f - \mathcal{Q}f\|_\infty = \bigcirc(h^4)$ where $h = \max(h_\theta, h_\phi)$.

VI. NUMERICAL RESULTS AND APPLICATIONS TO MEDICAL IMAGING

The implementation of the proposed method can be summarized in four steps.

- The first step consists in transforming the cartesian coordinates of given 3-D scattered data to spherical coordinates.
- The second step includes the subdivision of the rectangular domain D into subrectangles $D_{\mu,\nu}$, and the application of the least-square method to the given scattered data in order to construct the initial matrix $f_0(i, j)$.
- The third step consists in constructing the matrix $c(i, j) = \lambda_i \widetilde{\lambda}_j f_0$ and consequently computing the associated spline quasi-interpolant $\mathcal{Q}f_0$ of the form defined in (8).
- The last step reconstructs the associated approximating sphere-like surface $S_{\mathcal{Q}f_0}$, by retransforming the spherical coordinates of $\mathcal{Q}f_0$ on D to Cartesian coordinates.

To test the method, let f_0 be the function defined explicitly on the rectangular domain D by $f(\theta, \phi) = \sum_{i=1}^{3} (g_i(\theta, \phi))^{-(1/2)}$, where

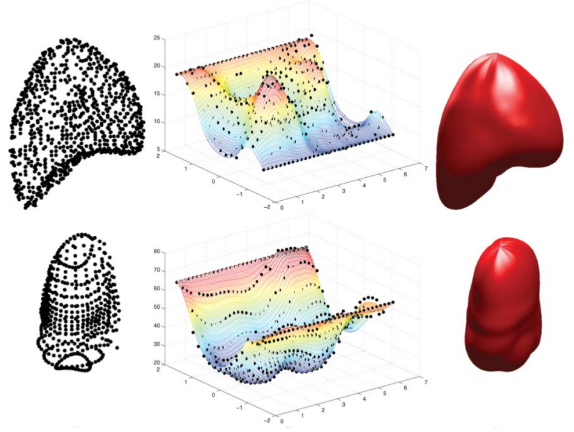
$$g_i(\theta, \phi) = \left(\frac{\cos\theta\ \cos\phi}{\alpha_i}\right)^2 + \left(\frac{\cos\theta\ \sin\phi}{\alpha_{i+1}}\right)^2 + \left(\frac{\sin\theta}{\alpha_{i+2}}\right)^2$$

with $(\alpha_1, \ldots, \alpha_5) = (5, 1, 2, 5, 1)$ [5]. It is straightforward to verify that $f \in \mathcal{F}$. Table I gives the maximum error and the time of execution corresponding to different values of n and m.

Two different sets of 3-D medical data have been used to evaluate the proposed method. The first experiment considers a set of 922 surface points of real data of the human left lung, provided from perfusion scintigraphy images [Fig. 1(a), top]. The second experiment deals with real data of the left ventricle (LV) of a human heart acquired with a new multidimensional

TABLE I

n	m	Error	Time (s)
50	50	0.1454	22.37
100	100	0.0192	45.39
150	150	0.0047	102.01
200	200	0.0016	134.95
250	250	$0.6969\ 10^{-3}$	171.29
300	300	$0.1888\ 10^{-3}$	210.06
350	350	$0.1117\ 10^{-3}$	253.04
400	400	$0.7024\ 10^{-4}$	299.01
450	450	$0.4631\ 10^{-4}$	350.73



(a) (b) (c)

Fig. 1. (a) 3-D given data of the lung (top) and the LV (bottom). (b) Surface meshes on the rectangle D. (c) Quasi-interpolant closed surfaces.

imaging ultrasound system [14]. It consists of 1024 scattered points [Fig. 1(a), bottom].

3-D closed surfaces are first developed in 2-D periodic surfaces by moving from Cartesian to Spherical coordinates [black dots on Fig. 1(b)]. Then, spline quasi-interpolated surfaces are reconstructed with $n = 150$ and $m = 150$ [continuous surfaces on Fig. 1(b)]. Finally, Fig. 1(c) shows the corresponding 3-D closed $\mathcal{C}^1$ surfaces. Numerical comparisons with other interpolated- based or smooth-based surface reconstruction methods are not straightforward since such methods usually depend on different adjustable parameters. Complementary detailed studies on the comparison of the new method with quasi-interpolant ones will be carried out in the future.

VII. CONCLUSION

The quasi-interpolation method proposed in this letter is based on the tensor product of cubic polynomial B-splines and periodic UAT B-splines of order four. In contrast to least-square methods, it has two major advantages. First, the local reconstruction of a disturbed subset of a given surface is possible without affecting the whole surface data set. Second, the reconstruction process is achieved without the need to solve a large linear system and is hence easy to compute. In addition, compared to previously developed spline quasi-interpolant methods, our proposed algorithm is based on a spline quasi-interpolant with an order four approximation, exact on $\{1, \theta, \theta^2, \theta^3\} \otimes \{1, \phi, \cos(\phi), \sin(\phi)\}$, and the reconstructed closed surface is of class $\mathcal{C}^2$ except on the two poles where it is of class $\mathcal{C}^1$. Promising results have been obtained using numerical and real medical data. The method is also suitable for other applications related to the problem of 3-D reconstruction. It will be evaluated shortly in aerial and satellite imaging domains.

APPENDIX
PROOF OF THEOREM 5.1

In order to prove that $\mathcal{Q}f$ lies in $\mathcal{F}$, it suffices to show that $\mathcal{Q}f$ satisfies conditions ($C1$)-($C4$). Indeed, from the fact that $\{T_j\}$ are 2π-periodic, $\mathcal{Q}f$ defined in (8) satisfies conditions ($C1$) and ($C3$). Now, let us show that $\mathcal{Q}f$ satisfies conditions ($C2$) and ($C4$) which is equivalent to prove that the coefficients $\lambda_i\widetilde{\lambda}_j f$ satisfy the conditions given in the lemmas 5.1 and 5.2 respectively. According to the definitions of the linear functionals λ_i and $\widetilde{\lambda}_j$, we have $\lambda_1\widetilde{\lambda}_j f = \widetilde{\lambda}_j f(-(\pi/2), .) = \sum_{s=1}^{4} \beta_{j,s}\ f(-(\pi/2), \tau_{j,s})$. On the other hand, since $f \in \mathcal{F}$, we have $f(-(\pi/2), \tau_{j,s}) = c_-$. Consequently, we obtain $\lambda_1\widetilde{\lambda}_j f = c_- \sum_{s=1}^{4} \beta_{j,s} = c_-$. In a similar way, we get $\lambda_n\widetilde{\lambda}_j f = c_+$. Thus, $\mathcal{Q}f$ satisfies condition ($C2$). For the condition ($C4$), using the expressions of λ_1, λ_2 and $\widetilde{\lambda}_j$, we have

$$\lambda_2\widetilde{\lambda}_j f - \lambda_1\widetilde{\lambda}_j f$$
$$= \frac{1}{18}\left[7\widetilde{\lambda}_j f\left(-\frac{\pi}{2}, .\right) + 18\widetilde{\lambda}_j f(\theta_5, .) - 9\widetilde{\lambda}_j f(\theta_6, .) + 2\widetilde{\lambda}_j f(\theta_7, .)\right]$$
$$-\widetilde{\lambda}_j f\left(-\frac{\pi}{2}, .\right) \simeq \frac{h}{3}\frac{\partial f}{\partial \theta}\left(-\frac{\pi}{2}, \tau_{j,s}\right).$$

Since $f \in \mathcal{F}$, we have $\partial f/\partial\theta(-\pi/2, \tau_{j,s}) = a_- \cos\tau_{j,s} + b_- \sin\tau_{j,s}$, and consequently we obtain

$$\lambda_2\widetilde{\lambda}_j f - \lambda_1\widetilde{\lambda}_j f = \frac{h}{3}\left[a_-\frac{\alpha\cos(\phi_{j+2})}{\sin(\alpha)} + b_-\frac{\alpha\sin(\phi_{j+2})}{\sin(\alpha)}\right].$$

By using the same technique, we obtain the result for the case $\theta = \pi/2$.

Finally, using the fact that the quasi-interpolants $\mathcal{Q}_1$ and $\widetilde{\mathcal{Q}}_2$ have an order four approximation, we obtain $\|f - \mathcal{Q}f\|_\infty = \bigcirc(h^4)$.

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