Signal recognition by finite automata
– work in progress –

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Outline

1. Pre-signal and approximation
2. Automata and signals
3. Exercises
4. Cardinality of the set of signals
5. Open questions
Pre-signals

Σ : finite alphabet

**Pre-signal** \( f : [0, 1] \rightarrow \Sigma \)

Represented by its graph

![Graph of Pre-signal](image)
\( \mathcal{E} \):-approximation

\[ \tilde{\Sigma} = \Sigma \times (\mathcal{P}(\Sigma) \setminus \{\emptyset\}) \]

\[ w_1 \ldots w_n \in \tilde{\Sigma} \quad f : [0, 1] \rightarrow \Sigma \]

\[ w_1 \ldots w_n \ \varepsilon \text{-approximates } f \]

\[ \iff \exists x_1, x_2, \ldots, x_{n+1} \]

\[ \begin{cases} x_1 = 0 \\ x_i < x_{i+1} \\ x_{n+1} = 1 \end{cases} \]

\[ w_i = \left( f(x_i), f\left(\lfloor x_i, x_{i+1}\rfloor\right) \right) \]

\[ \left| f\left(\lfloor x_i, x_{i+1}\rfloor\right) \right| > 1 \Rightarrow |x_i - x_{i+1}| < \epsilon \]
$\varepsilon$-approximation

\[ \frac{1}{n+1} \quad \frac{n}{n+1} \]

Diagram showing points labeled a, b, and c.
ε-approximation

\[
\frac{1}{n+1} \quad \frac{n}{n+1}
\]

\(x_1\) \(x_2\) \(x_3\) \(x_4\) \(x_5\) \(x_6\) \(x_7\) \(x_8\) \(x_9\) \(x_{10}\) \(x_{11}\) \(x_{12}\) \(x_{13}\)
\( \varepsilon \)-approximation

\[
\frac{1}{n+1} \quad \frac{n}{n+1}
\]

\[ |x_1 - x_2| < \varepsilon \text{ and } |x_{12} - x_{13}| < \varepsilon \]

\[
w = (a, \{a, b\}) (b, \{a\}) (b, \{a\}) \ldots (b, \{a\}) (b, \{a, b\})
\]
Finite automaton for pre-signals

\[ \mathcal{A} = \left( \tilde{\Sigma}, Q, \delta, I, F \right) \]

\( \mathcal{L}(\mathcal{A}) \) language (on \( \tilde{\Sigma} \)) recognized

\( f \) in \( \Sigma^{[0,1]} \)

\( \mathcal{A} \) signal-recognizes \( f \)

\( \iff \)

\( f \in \mathcal{S}(\mathcal{A}) \)

\( \iff \)

\( \forall \varepsilon > 0, \exists w \in \mathcal{L}(\mathcal{A}), w \varepsilon \)-approximates \( f \)
Equivalence and signals

\[ f, g \text{ in } \Sigma^{[0,1]} \]

\[ f \approx g \]

\[ \iff \forall A, f \in S(A) \iff g \in S(A) \]

\[ [f] \text{ is a signal} \]

\[ \iff [f] \text{ is an equivalence class for } \approx \]

Open question  Characterize these classes (links to scattered linear orders)
Example of an acceptance

\[ f \in S(\mathcal{A}) \]

\[ g \in S(\mathcal{A}) \iff \exists z_1, z_2, \ldots \]

\[ [f] = S(\mathcal{A}) \]

\[
\begin{cases}
  z_1 = 0 \\
  z_i < z_{i+1} \\
  \lim_{i \to \infty} z_i = 1 \\
  g(x) = b \iff \exists i, x = z_i
\end{cases}
\]
Exercises

Find the signal languages for:

\[ b, \{a\} \]

\[ b, \{a, b\} \]

\[ c, \{a\} \]

\[ b, \{a, b\} \]

\[ b, \{a\} \]

\[ b, \{a, b\} \]

\[ c, \{a\} \]

\[ c, \{a, b, c\} \]

\[ b, \{a, b\} \]

\[ b, \{a\} \]
Exercises

Find the signal languages for:

- $b, \{a\}$
- $b, \{a, b\}$
- $b, \{a\}$
- $b, \{a, b\}$
- $c, \{a\}$
- $b, \{a, b\}$
- $b, \{a\}$
- $b, \{a\}$
- $c, \{a\}$
- $b, \{a, b\}$
- $c, \{a, b, c\}$
Exercises

Find the signal languages for:

- $b, \{a\}$
- $c, \{a\}$
- $b, \{a, b\}$
- $\omega$
Exercises

Find the signal languages for:

- $b, \{a\}$
- $b, \{a,b\}$
- $c, \{a\}$
- $b, \{a,b\}$
- $b, \{a\}$
- $b, \{a,b\}$
- $\omega^2$
- $c, \{a\}$
- $c, \{a,b,c\}$
- $b, \{a,b\}$
- $b, \{a\}$
- $\omega_2$
Exercises

Find the signal languages for:

- $b, \{a\}$
- $b, \{a,b\}$
- $c, \{a\}$
- $b, \{a,b\}$
- $\omega$

- $b, \{a\}$
- $b, \{a\}$
- $b, \{a\}$
- $\zeta$

- $b, \{a\}$
- $b, \{a\}$
- $b, \{a\}$
- $\eta$

- $c, \{a\}$
- $c, \{a,b,c\}$
- $\omega^2$
- $b, \{a,b\}$

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Cardinality of the set of signals

Notation: \( \mathbb{U}_0 = \omega \) et \( \mathbb{U}_{i+1} = 2^{\mathbb{U}_i} \), \( (\text{e.g. } |\mathbb{R}| = \mathbb{U}_1) \)

\[ \left| \{ \mathcal{A} \text{ automate} \} \right| = \mathbb{U}_0 \]

\[ \left| \{ \text{ signaux} \} \right| = ??? \]

\[ \left| \sum^{[0,1]} \right| = \mathbb{U}_2 \]
Cardinality of the set of signals

Notation: $\mathbb{N}_0 = \omega$ et $\mathbb{N}_{i+1} = 2^{\mathbb{N}_i}$, (e.g. $|\mathbb{R}| = \mathbb{N}_1$)

$|\{ A \text{ automate } \}| = \mathbb{N}_0$

$|\{ \text{ signaux } \}| = \mathbb{N}_1$

$|\Sigma^{[0,1]}| = \mathbb{N}_2$

Almost no signal is characterized by an automaton

Open question Look at companion signals
Let $\{A_i\}_{i \in \mathbb{N}}$ be an enumeration of the automata

$$\varphi [f] \mapsto w \in \{0, 1\}^\omega$$

$$w_i = 1 \iff [f] \subseteq S(A_i)$$

$\varphi$ is one-to-one because

$$[f] = \bigcap_{x_i=1} S(A_i) \cap \bigcap_{x_i=0} \overline{S(A_i)}$$
\[ \mathbb{I}_1 \leq \left\{ \text{signaux} \right\} \]

\[
\begin{align*}
    w \in \{0, 1\}^\omega & \quad \mapsto \quad f \in \{0, 1, \#\}^{[0,1]} \\
    f \left( \frac{n-1}{n} \right) = w_i \\
    f \left( \left[ \frac{n-1}{n}, \frac{n}{n+1} \right] \right) = \{\#\}
\end{align*}
\]

\[ w \neq w' \Rightarrow \exists i_0, w_{i_0} \neq w'_{i_0} \]

Accepts \( \psi(w) \) but not \( \psi(w') \) thus \( [\psi(w)] \neq [\psi(w')] \) \( [\psi(.)] \) is one-to-one

Possible with two letters
Open questions

Classical operations on automata
  - Union OK
  - Concatenation... (I am missing an inclusion or a counterexample)
  - Star...

Closure
  - Complements... (conj. no)
  - Intersection...

Extra operations
  - $\omega$, $-\omega$ and $\zeta$-iterations ($\Diamond$ and $\#$)

Identification of signal languages
  - Regular expressions
  - KLEENE like theorem