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Grain Sorting in the One-dimensional Sand Pile Model

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Abstract

We study the evolution of a one-dimensional pile, empty at first, which receives a grain in its first stack at each iteration. The final position of grains is singular: grains are sorted according to their parity. They are sorted on trapezoidal areas alternating on both sides of a diagonal line of slope $\sqrt{2}$. This is explained and proved by means of a local study. Each generated pile, encoded in height differences, is the concatenation of four patterns: 22, 1313, 0202, and 11. The relative length of the first two patterns and the last two patterns converges to $\sqrt{2}$. We make asymptotic expansions and prove that all the lengths of the pile are increasing proportionally to the square root of the number of iterations.

1 Introduction

We consider an infinite sequence of stacks. Each stack can hold any finite number of grains, this number is called the *height* of the stack.

The Sand pile model (SPM) and Chip firing games (CFG) are based on local interactions. Both models conserve the total number of grains. In SPM, a grain goes to a neighbor stack if the height difference between stacks is more than a given threshold; whereas in CFG a stack gives a chip to each of its neighbors if its number of chips is above a certain value [3, 4]. In the one-dimensional case, both models are one-dimensional cellular automata and they are equivalent via some simple encoding.

Both SPM and CFG, like Petri nets [11], are used to model in parallel computing the flows of information in systems. SPM is used to model gradian-driven dynamic load balancing. Grains model data or tasks, and stacks, a processor network [12]. The aim is to find a simple, fast, and relatively inexpensive local rearrangement which ensures that all processors have almost the same amount of work.

SPM is important for granular flows in physics. It admits invariants, entropy like functions, and verifies the so-called "Self-organized criticality" and is related to the "$1/f$ phenomenon" [2, 9, 10, 13].

The sequential one-dimensional SPM and the related CFG are studied in [6, 7, 8]. They have proved the uniqueness of the final pile whatever the order of the iterations as well as described the dynamics in various sequential cases.

The problem studied in this paper is the parallel evolution of a one-dimensional SPM, empty at first, which receives a grain onto its first stack at each iteration. It can also be seen as sand dripping in a thin but long hourglass.

In section 2, we define the SPM and CFG models and recall that they are equivalent in dimension one. We also define the dripping process studied.

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In section 3, we prove that the generated piles, in height differences, are made of four patterns: 22, 1313, 0202 and 11. The frontiers between patterns act like signals. The silhouette of each pile is made of two parts of different slopes: 2 then 1.

In section 4, grains are marked depending on their parity. Even and odd grains are arranged in a very special way: they are located in trapezoidal areas alternating on both sides of a diagonal line of slope $\sqrt{2}$. We explain this by looking locally at the interactions between moving grains and signals.

In section 5, we give asymptotic approximations of the different parameters. We do this by making a continuous approximation of the pile and use a differential resolution as in [1]. We prove that the length of the part of slope 1 is $\sqrt{2}$ times the length of the part of slope 2 and that the lengths of all the piles are increasing proportionally to the square root of the number of iterations.

2 Definitions

The one-dimensional sand pile is an infinite sequence of stacks. Each stack can hold any finite number of grains. We use the notation from [8], the difference is that our model is parallel.

A pile is encoded by the sequence of the number of grains, or *height*, of the stacks. It is then denoted with square brackets: $\nu = [[\, \nu_0\, \nu_1 \, \dots \, \nu_k \,]]$. We call *slope* the difference of height, $\nu_i - \nu_{i-1}$, between two consecutive stacks. If more stacks are considered, the slope is the average slope.

If a stack is higher by at least two grains than the next stack, then one grain "tumbles down." This is depicted by the movement of the grains `a` to `f` in Figure 1.

The starting pile is empty. At each iteration, a grain falls onto stack number 1. Grains `c` to `f` in Figure 1 are the newly arrived grains. The number of grains is finite. Except for the grain added to the pile at each iteration, the number of grains is constant. The number of grains is then equal to the iteration number.

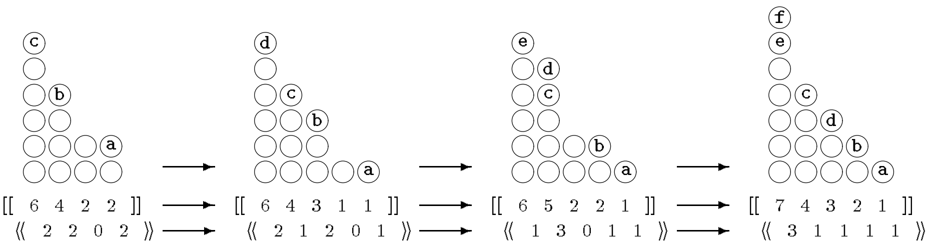

Figure 1: Iterations 14 to 17.

Since grains are only moved to smaller stacks, a direct induction proves that only decreasing sequences are generated from the initial pile. A *pile* is now an element of $\mathbb{N}^{\mathbb{N}}$ decreasing to zero. This ensures that the height difference between any two consecutive stacks is always positive.

Definition 1 Let $\mathbb{1}(\, n \,)$ be the following threshold function: $\forall n \in \mathbb{Z}$, $\mathbb{1}(\, n \,) = 1$ if $0 \le n$, otherwise 0. Let ν be a pile. The dynamics of SPM with dripping is driven by the following transition function F:

$$\begin{aligned} F(\nu)_0 &= \nu_0 - \mathbb{1}(\, \nu_0 - \nu_1 - 2\,) + 1 \ , \\ 0 < i, \quad F(\nu)_i &= \nu_i - \mathbb{1}(\, \nu_i - \nu_{i+1} - 2\,) + \mathbb{1}(\, \nu_{i-1} - \nu_i - 2\,) \ . \end{aligned}$$

The negative terms correspond to the possibility of giving a grain to the next stack, while the positive term corresponds to the possibility of getting a grain from the previous stack. All the stacks are updated at the same time, making this is a parallel process.

Definition 2 A pile can be encoded by the list of the height differences between stacks: for any pile ν $\varphi(\nu) = \langle\!\langle\, (\nu_0 - \nu_1)\, (\nu_1 - \nu_2)\, (\nu_2 - \nu_3) \, \dots \rangle\!\rangle$. With this encoding, the transition function becomes:

$$\begin{aligned} \Theta(x)_0 &= x_0 - 2\,\mathbb{1}(\, x_0 - 2\,) + \mathbb{1}(\, x_{i+1} - 2\,) + 1 \ , \\ \forall i,\ 0 < i, \quad \Theta(x)_i &= x_i + \mathbb{1}(\, x_{i-1} - 2\,) - 2\,\mathbb{1}(\, x_i - 2\,) + \mathbb{1}(\, x_{i+1} - 2\,) \ . \end{aligned}$$

We call the difference of height of one between a stack and the next one *chip*. Definition 2 is equivalent to a stack having more than two chips and firing one to both neighbors. This is the *chip firing game* (CFG). In a one-dimensional lattice, SPM and CFG are equivalent with this encoding.

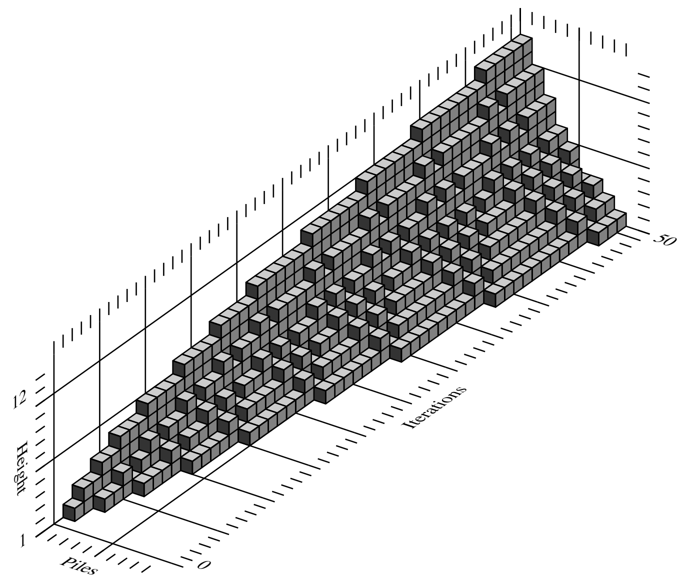

Figure 2: Iterations 0 to 50.

Figure 2 illustrates the first 50 steps of this dynamic. The lengths and heights, as well as the slopes, exhibit some regularity. After some iterations, there are two straps of triangles drawn on the surface as depicted in Figure 3 for iteration 100 to 150.

3 Triangles and signals

Piles are encoded in height differences in Figure 4 (steps 1 to 120). Triangles appear with patterns 22, 1313, 0202, and 11. Those patterns are stable. It should be noted that for the second and third patterns, digits are alternating, like in a chessboard and the frontier between them is either 12 or 30.

Let ε be the empty word. The Kleene operator is denoted *; that is, $(13)^*$ is ε or 13 or 1313 or 131313 .... We use the theory of languages in the next proposition in order to get a synthetic expression.

Proposition 1 *The pile, encoded in height differences, is a word of the following language:*

$$2^* \ (\varepsilon\,|\,3)\ (1\,3)^*\ (\varepsilon\,|\,1\,2)\ (0\,2)^*\ (\varepsilon\,|\,0)\ 1^* \ .$$

Proof. We prove proposition 1 by induction. It is true for the first 120 iterations, as can be seen in Figure 4. Interaction, as expressed in definitions 1 and 2, only depends on the two closest neighbors. It is enough to look locally at the interactions of the frontiers in Figure 4.

Suppose that the nth pile is the concatenation of four parts with the patterns 22, 1313, 0202, and 11 respectively. We call *frontier* the limit between two patterns and *border* the limits of the pile. We denote L (left), M (middle), and R (right) the positions of the frontiers between respectively; first and second, second and third, third and fourth patterns. They are represented in Figure 4 where L and R behave like signals moving on both sides of M. Geometric definitions are given in Figure 5.

First we investigate each signal alone, from left to right: L is going left (right) if it is equal to $2|1$ ($2|3$) (lines 107 to 117); M is not moving (lines 96 to 102) and R is going left (right) if it is equal to $0|1$ ($2|1$) (lines 94 to 104). While the proposition is true, only the following encounters are possible, from left to right: on

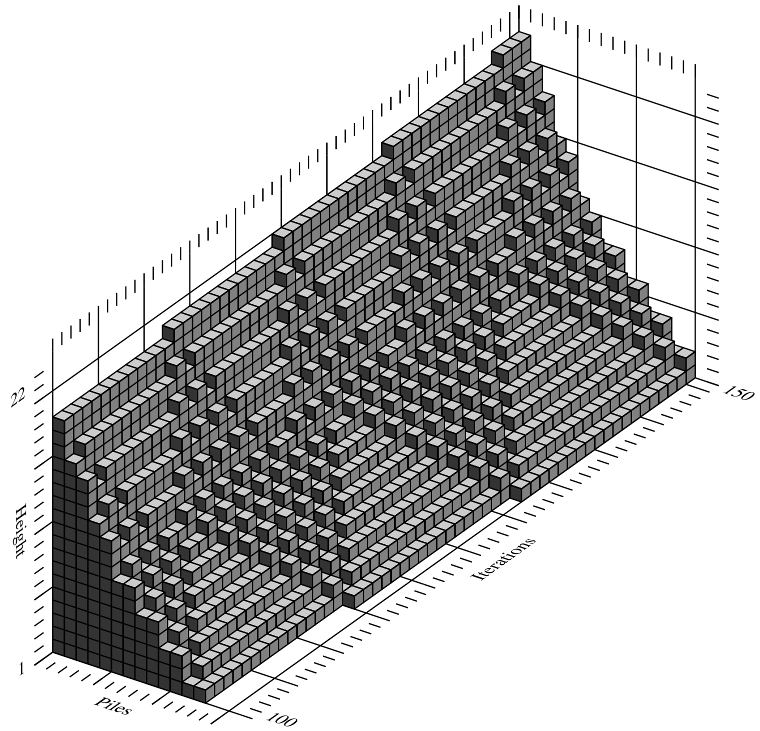

Figure 3: Iterations 100 to 150.

the left border, L bounces (lines 59 to 65); when L meets M, it bounces and M is moved one step to the right (lines 81 to 87); when R meets M, it bounces and M is moved one step to the left (lines 50 to 57).

The order of the signals is kept and the only possible encounter with more than two frontiers is L-M-R. The meeting can be exactly synchronous (lines 40 to 44) or not (lines 62 to 67 and 103 to 109). In all cases the order is kept and no other case arises. ■

The dynamics are very simple except when signal L or R reaches one of its limits; the rest are only linear displacements. When L reaches the left border, it only bounces back. When R reaches the right border, it bounces back and the total length is increased by one. When R comes back to the center, the total length has been increased by one.

In height differences, piles are the concatenation of four parts of patterns 22, 13, 02, and 11 respectively. The two first parts have a slope of 2 while the two last parts have a slope of 1. This explains the shape of piles (in heights) as depicted in Figure 5.

4 Labeling grains

Grains are labeled according to the iteration during which they enter the pile. In Figure 6, at iteration 5000, all odd grains are spotted in black. Their localization is singular.

The odd grains, like the even ones, are located on trapezoidal areas delimited by the axis, lines of slope 1 and 2, and a diagonal line. These areas are alternating like in a chessboard. The diagonal separation seems to be a straight line.

There is also some kind of relation between the intersections of the line of slope 1 and 2 with the axis and the edges in the middle as depicted in Figure 6. We do not have any explanation nor proof for this phenomenon yet. Nevertheless, if the diagonal separation is a line, because of such coincidences, its slope would be: $\mathtt{a}/\mathtt{b} = (\mathtt{b}+2\mathtt{a})/(\mathtt{a}+\mathtt{b})$ which leads to $\mathtt{b}/\mathtt{a} = \sqrt{2}$. We prove in section 5 that indeed it is a straight line with slope $\sqrt{2}$.

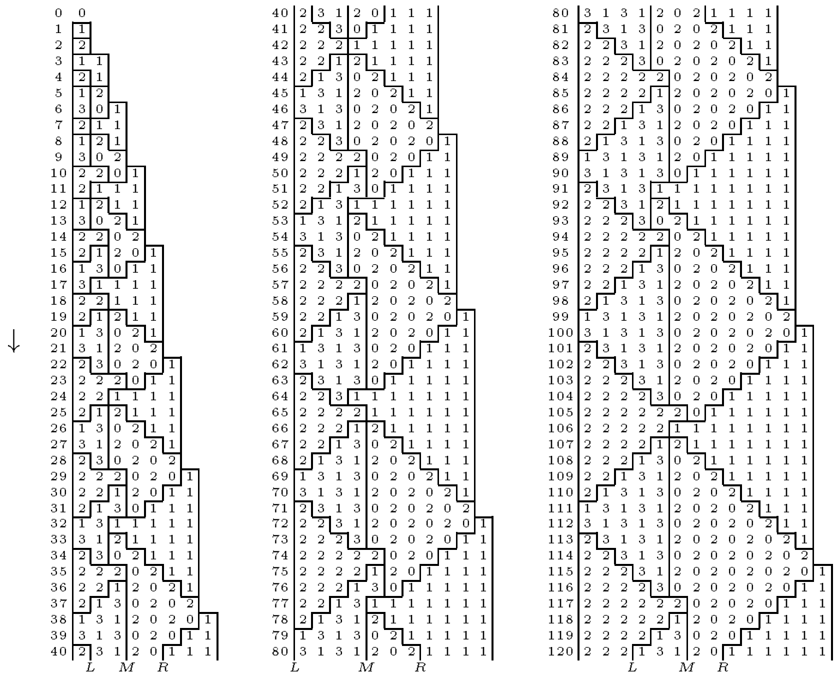

Figure 4: Representation with height differences.

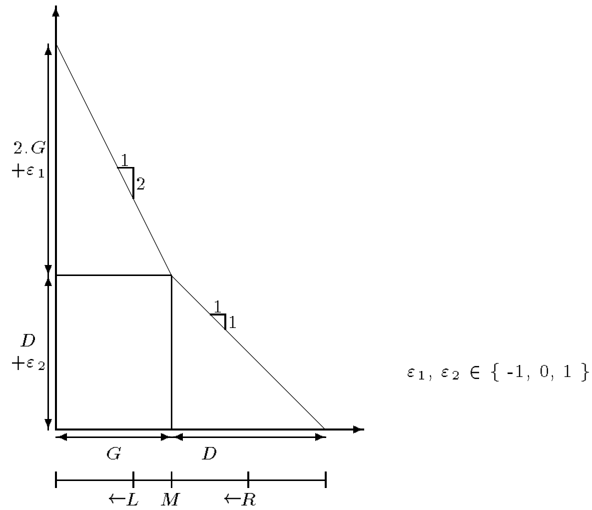

Figure 5: Geometric definitions of G, D, L, and M.

Theorem 1 *Odd and even grains are always sorted in trapezoid areas delimited by a diagonal, lines of slope 1 and 2, and the axes.*

With figures 7 through 11, we prove that the grains are always on either side of the frontier, depending on their parity. In all these figures, grains are either black or white depending on their parity. Grains for which parity is unknown are drawn with a little circle. The grains which do not move any more are represented by their silhouette.

Let us first consider that signal L is away from the left border. Even and odd grains come alternatly and go down the pile one after the other as depicted in Figure 7. Grains behave like a wave of marbles on stairs.

From this, a direct induction proves that the pattern 22 corresponds to an even-odd wave of grains. Let

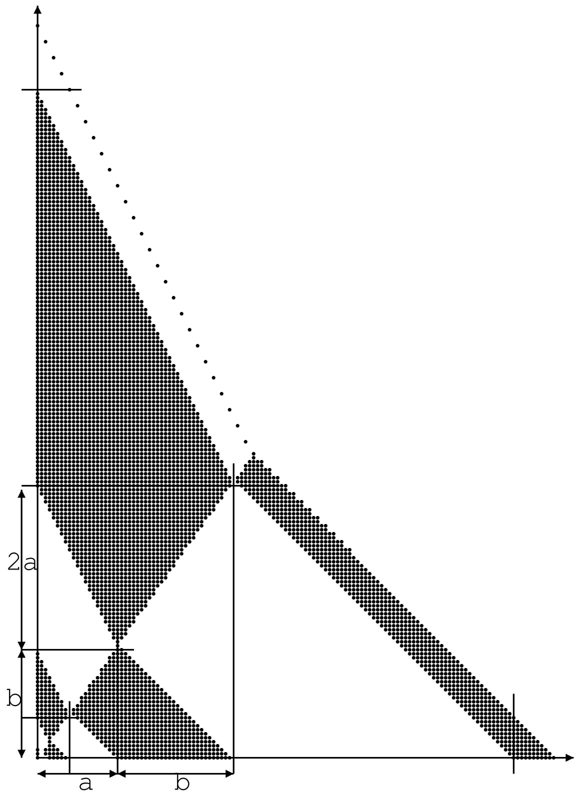

Figure 6: Position of the odd grains (in black).

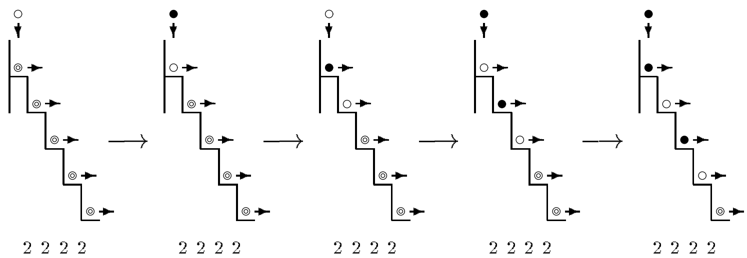

Figure 7: Arrival of new grains.

us consider that signal L is going right. As depicted in Figure 8, the wave is just going down with scarce grains running in front of it.

Going right, the signal L encounters the middle border M as depicted in Figure 9. The first grain crosses the border and because of the height difference 1, the second gets locked. The third passes over the second and restrains the fourth from passing, and so on.

The phenomena of one grain getting locked and the next passing over it, one layer up, is the way the signal L goes right as depicted in Figure 10. When L reaches the left border, it ends building a layer and goes back to the middle on the new layer as depicted in Figure 10. In comparison to Figure 7, we know now that the grains that are running in front of the wave are all of the same parity. The first grain of the wave is of the same parity as the first grain of the previous wave, which is also the parity of the grains the running scarce grains, so that the phenomena starts again and loops.

We now consider signal R. When R is away from the middle M, it has no action whatsoever since the selection of grains is made before. Signal R only orders the grains on layers in the right part. When L or R meets M it only moves it and that does not change the dynamic of Figure 9. But, when all three signals

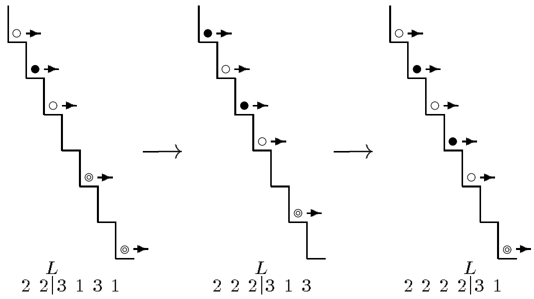

Figure 8: Signal L goes right.

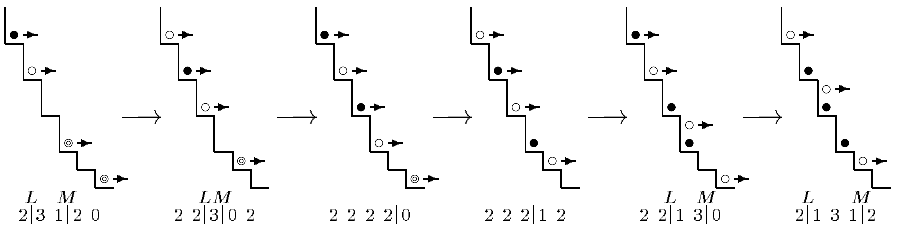

Figure 9: Signal L reaches the middle border alone.

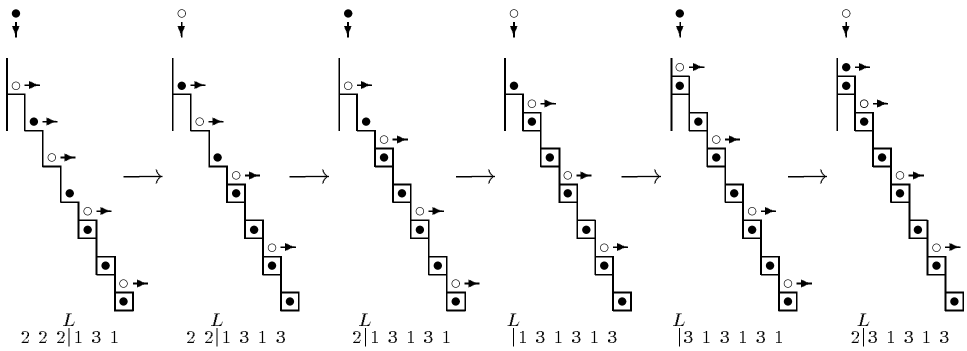

Figure 10: Signal L goes left and reaches the left border.

L, M, and R meet, things are different as depicted in Figure 11. This time, the fate of odd and even grains are switched. The changes in the destination of odd and even grains in Figure 6 are directly linked to the synchronous encounter of L and R detailed in section 3.

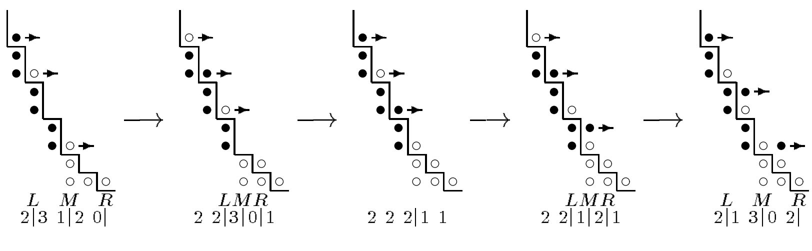

Figure 11: Signals L and R exactly synchronized.

In Figure 6, the separation lines represent the silhouettes of piles at some iterations and the diagonal separation is the trace of the middle border M.

Since there are as even grains as odd grains, the two symmetric areas in Figure 6 have the same surface, that is, they correspond to the same number of grains.

5 Asymptotic behavior

All the results in this section can also be found in [5]. The proof of [5] is too long to fit here, we give a shorter one that we feel is more like an *a posteriori* verification.

Theorem 2 *The diagonal separation is a line of slope* $\sqrt{2}$*.*

The value of G increases (decreases) by one for each round trip of L (R). The value of D decreases (increases) by one (two) for each loop of L (R). The round trip delay for a signal is twice the length of the part it evolves in, up to a constant. Since every quantity can go to infinity, when G and D are very big, the equations can be extended to continuity as in [1]:

$$\begin{cases} dG &= \dfrac{dt}{2.G} - \dfrac{dt}{2.D} \ , \\[2ex] dD &= -\dfrac{dt}{2.G} + 2\dfrac{dt}{2.D} \ . \end{cases}$$

These equations can be solved with the hypothesis $D = \sqrt{2}\,G$ which comes from the observations of section 4. It leads to:

$$G.dG \;=\; \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)\ dt \ .$$

With this hypothesis, the possible solutions are:

$$\begin{aligned} G &= \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}}\,t + c} \ , \\ D &= \sqrt{2}\,G \;=\; \sqrt{\left(\sqrt{2}-1\right)\,t + 2c} \ . \end{aligned}$$

Where t is the time (or number of fallen grains or number of iterations) and c is a constant.

From Figure 5, the number of fallen grains n is also the total area, that is, of the two triangles and of the rectangle. We get the following approximations:

$$\begin{gathered} n \approx \frac{D^2}{2} + G.D + G^2 \approx (2+\sqrt{2})\,G^2 \ , \\ G \approx \sqrt{\frac{n}{2+\sqrt{2}}} \ , \\ D \approx \sqrt{2}\,G \approx \sqrt{\frac{n}{\sqrt{2}+1}} \ . \end{gathered} \tag{1}$$

It should be noted that both triangles of Figure 5 have almost the same area, G^2. This is coherent with the surface observations of section 4. The rectangle is equally parted by the diagonal, and even and odd grains are equally parted on both sides of the diagonal.

6 Conclusion

A more random distribution of odd and even grains might have been expected, on the contrary grains are sorted. This is important, because if even and odd grains, or tasks, are very different, in a one-dimensional processor array sequentially fed using a SPM load balancing technique, disparities arise. When taken modulo 3, 5, or more, there is no such segregated location as before, grains are more uniformly spread.

The way grains spread as a wave and are fixed in the silhouette is very interesting. It gives a physical meaning to the signals. When L goes right it spreads grains. When it goes left, it makes a one over two selection. Signal L is going right and left while the grains are always running to the right.

The signal R is acting similarly. When it goes right it is spreading the grains on a new layer, opening it. When it goes left it fixes them. When grains and signals are going in opposite directions, since they have speed one, signals only meet every other grain.

These signals, from a physical point of view, are very interesting because they correspond to waves on a pile of sand that can be seen when you dig at the bottom.

We have proved that the pile is expanding in the square root of the number of fallen grains (or iterations). This is absolutely normal when one considers that the grains (linear) are filling a surface (quadratic). The relative length of the two parts is $\sqrt{2}$.

To compare with the work in [1], on the one hand, they found a quadratic relaxation time for the CFG starting with the pile $\ldots 0\,0\,n\,0\,0 \ldots$ and final pile $\ldots 0\,0\,1\,1 \ldots 1\,1\,0 \ldots$. But when considered as stacks of grains, they correspond to $\ldots n\,n\,n\,0\,0 \ldots$ and $\ldots n\,n\,(n{-}1)\,(n{-}2) \ldots 2\,1\,0 \ldots$ respectively. This is a very different case because of the influence of the left border which is high and feeds grains to the right part. On the other hand, they also observed geometric patterns and signal propagations.

Acknowledgements

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