Intrinsic Simulation between Cellular Automata

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SEMINARIO MATEMÁTICAS DISCRETAS

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(revised and commented version)

Cellular Automata

Definition. A d-*CA* $\mathcal{A}$ is a 4-uple $\left(\mathbb{Z}^d, S, N, \delta\right)$ with:

- S the finite set of states of $\mathcal{A}$,
- $N \subseteq \mathbb{Z}^d$ finite, the neighborhood of $\mathcal{A}$,
- $\delta : S^{|N|} \to S$ the local rule of $\mathcal{A}$.

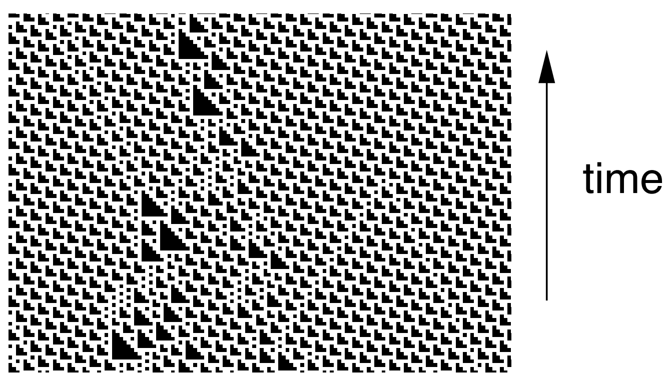
► A *configuration* C is a mapping from $\mathbb{Z}^d$ to S.

Cellular Automata (2)

- The *global rule* applies δ uniformly according to N:

$$G_{\mathcal{A}}(C)_p = \delta\left(C_{p+N_1}, \ldots, C_{p+N_n}\right).$$

- A *space-time diagram* is a graphical representation of an orbit.



Topological Characterization

▶ We endow S with the trivial topology.

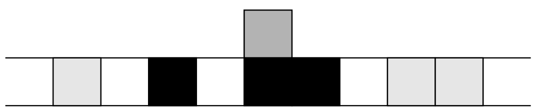
▶ We endow $S^{\mathbb{Z}^d}$ with the induced product topology.

▶ The *shift* $\sigma_p : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is defined as $\sigma_p(C)_i = C_{i+p}$.

Theorem[Hedlund 69]**.** A map $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is the global rule of a d-CA if and only if it is continue and commute with the shifts.

Simulation

- Simulation = analysis of the computational power.
- *Extrinsic simulation:* the CA simulates another device.



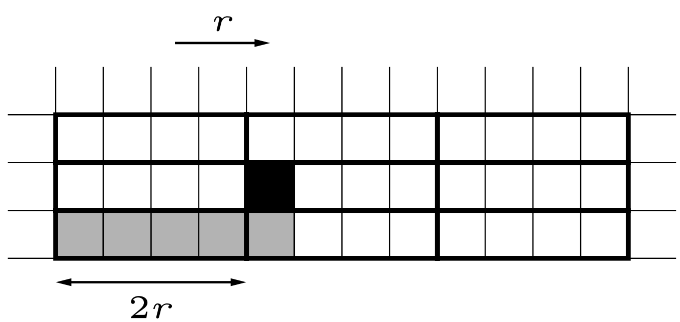
Turing machine

1–CA

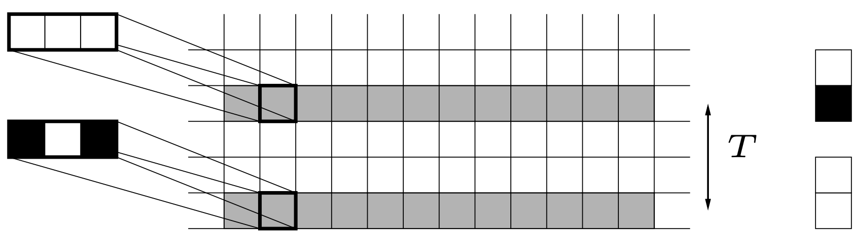
- *Intrinsic simulation:* the CA simulates another CA.

Classical Intrinsic Simulations

- Any CA can be simulated by an OCA *(Cole 69, Ibarra 85)*

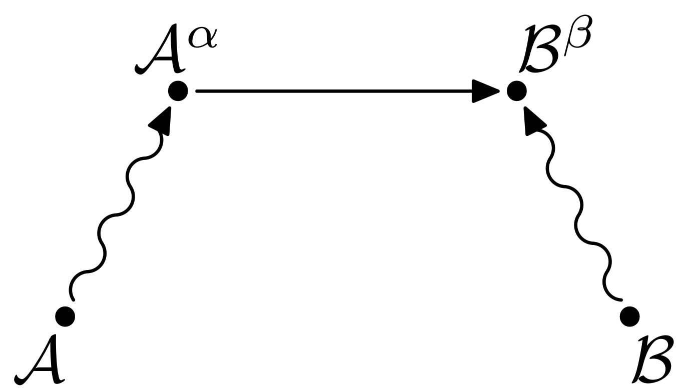


- Any nilpotent CA can be simulated by the trivial nilpotent CA



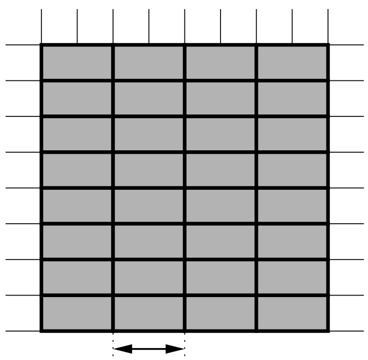
Simulation by Geometrical Transformation

Idea. A CA $\mathcal{A}$ simulates another CA $\mathcal{B}$ if, up to geometrical transformations, any space-time diagram from $\mathcal{B}$ is a space-time diagram from $\mathcal{A}$.



5 good transformations

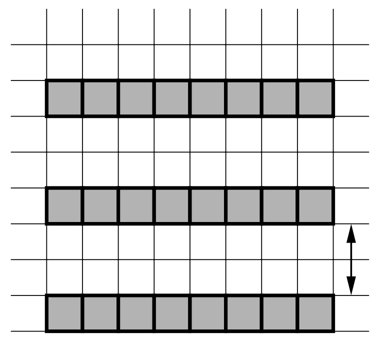
packing



$$o^{\mathcal{T}} \circ \mathcal{A} \circ o^{-\mathcal{T}}$$

spatial organization

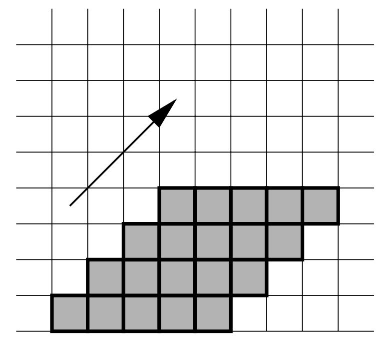
cutting



$$\mathcal{A}^n$$

temporal organization

shifting

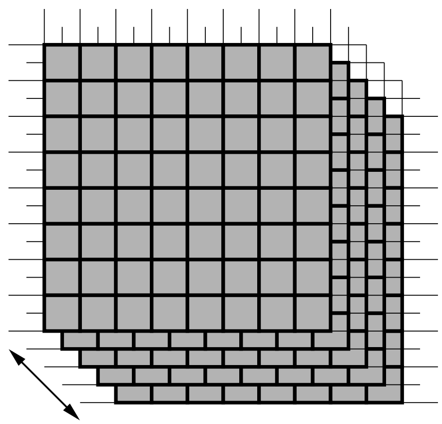


$$\mathcal{A} \circ \sigma_k$$

information mixing

5 good transformations (2)

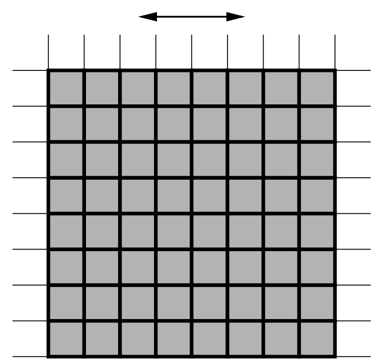
twisting



$\prod_i \mathcal{A}_i$

independent layers

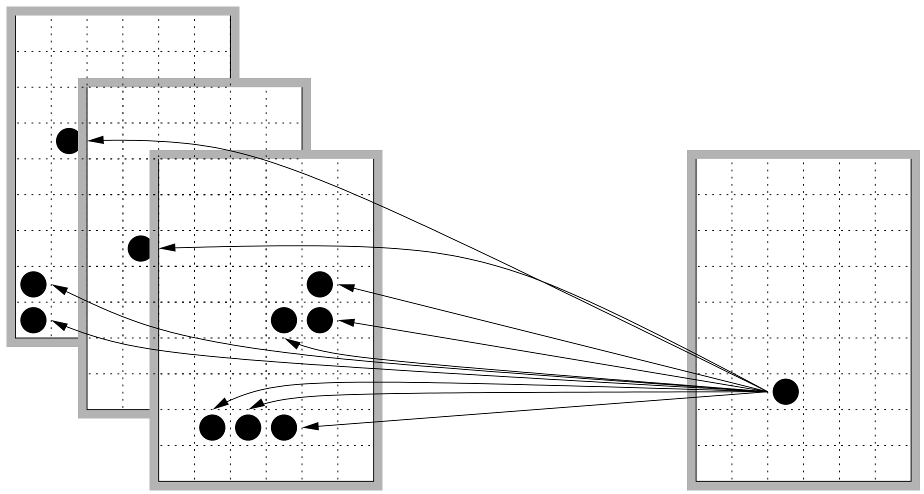
mirroring



$\leftrightarrow \circ \mathcal{A} \circ \leftrightarrow$

symmetry

Generalizing Geometrical Transformations



$$\varphi : \mathbb{N} \times \mathbb{Z} \rightarrow 2^{\{1,\ldots,k\}\times\mathbb{N}\times\mathbb{Z}}$$

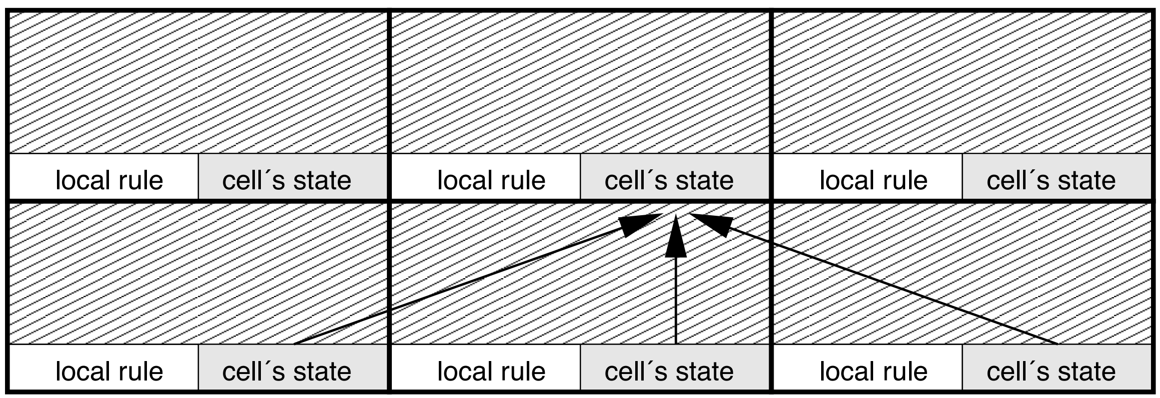
► The new CA must be *completely* defined for any initial CA.

Theorem. There exist no geometrical transformation but compositions of the 5 good previous ones.

Universality

Definition. A *universal* CA is a CA which can simulate any CA.

► There are universal CA for the $\mathsf{P}^{-}\mathsf{C}$ simulation.



► Therefore, there are universal CA for the PCSTM simulation.

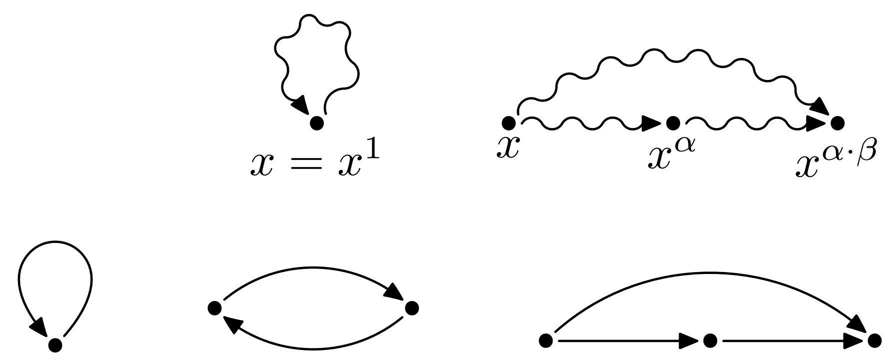
Theorem. The universal CA for PCSTM and $\mathsf{P}^{-}\mathsf{CS}$ coincide.

Abstract Bulking

Definition. An *abstract bulking* is a two-sorted first-order structure

$$\mathfrak{A} = \big(\mathsf{Obj}, \mathsf{Pow};\quad f : \mathsf{Obj} \times \mathsf{Pow} \to \mathsf{Obj},\quad R \subseteq \mathsf{Obj} \times \mathsf{Obj},\quad \cdot : \mathsf{Pow} \times \mathsf{Pow} \to \mathsf{Pow}\big)$$

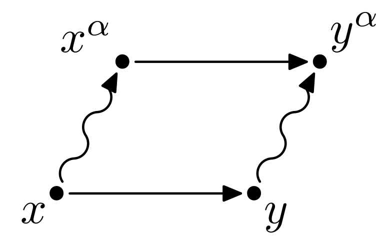
- $(\mathsf{Pow}, \cdot)$ is a monoid,
- f is compatible with $\cdot$,
- R is a partial order.



More Axioms

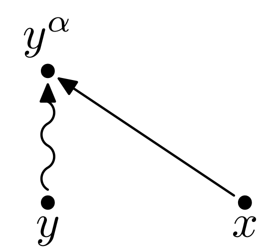
- R is compatible with f.

$\forall x, y, \alpha, R(x, y) \Rightarrow R(f(x, \alpha), f(y, \alpha))$



- f preserves richness.

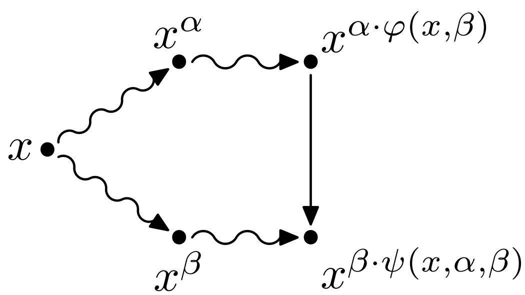
$\forall x, \alpha, \exists y, R(x, f(y, \alpha))$



The last axiom

- f keeps objects nearby.

$$\exists \varphi, \psi, \forall x, \alpha, \beta, R(f(x, \alpha \cdot \varphi(x,\beta)), f(x, \beta \cdot \psi(x,\alpha,\beta))$$



The way to conduct proofs

- Let $\mathbf{\Phi}$ denotes the set of axioms.

- An object y simulates an object x if $\phi(x, y)$ is satisfied.

$$\phi(x, y) = \exists\alpha\exists\beta, R(f(x, \alpha), f(y, \beta))$$

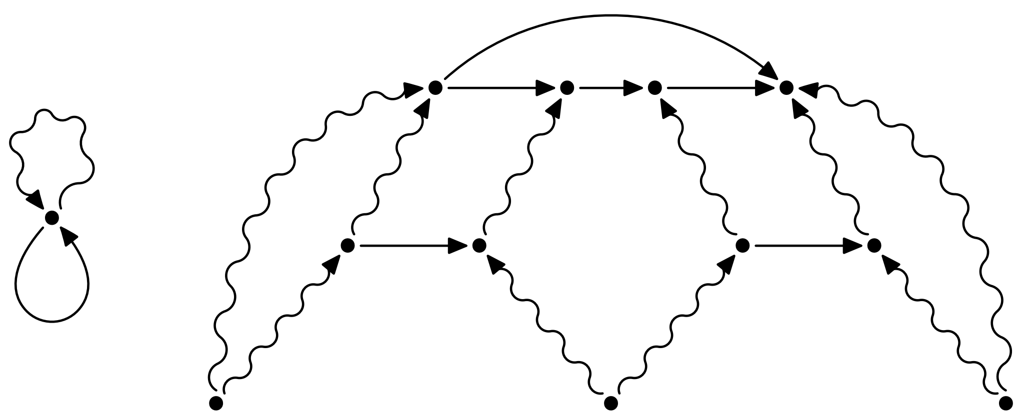
Definition. A *bulking property* is a property on ϕ expressed by a formula φ such that $\mathbf{\Phi} \models \varphi$.

The quasi-order

Theorem. "ϕ is a quasi-order" is a bulking property.

$$\varphi = \forall x, \phi(x,x) \wedge \forall x,y,z, (\phi(x,y) \wedge \phi(y,z)) \Rightarrow \phi(x,z)$$

Proof.



Universality

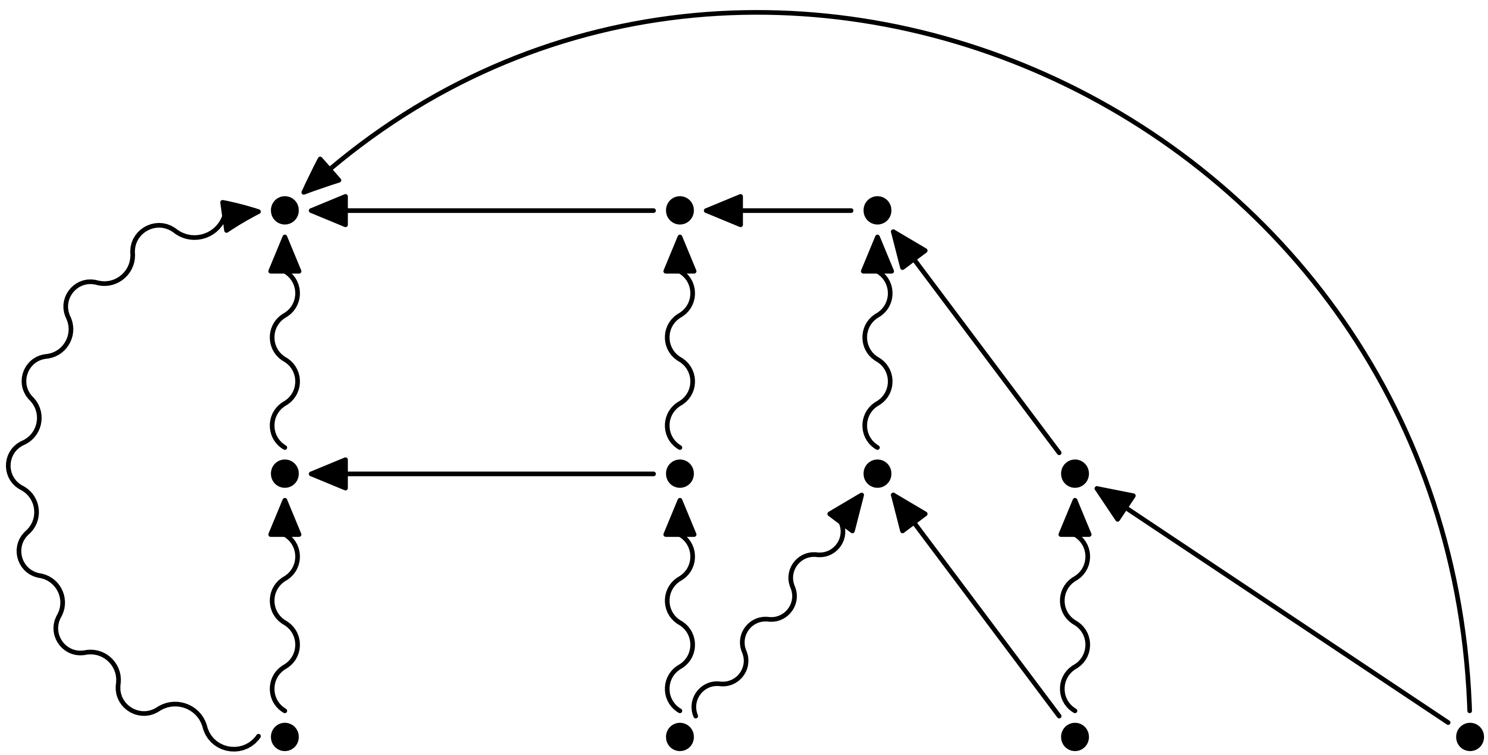
Definition. An object x is *strongly universal* if it can simulate any other object directly (*ie* without transformation).

$$\psi(x) = \forall y, \exists \alpha, R(y, f(x, \alpha))$$

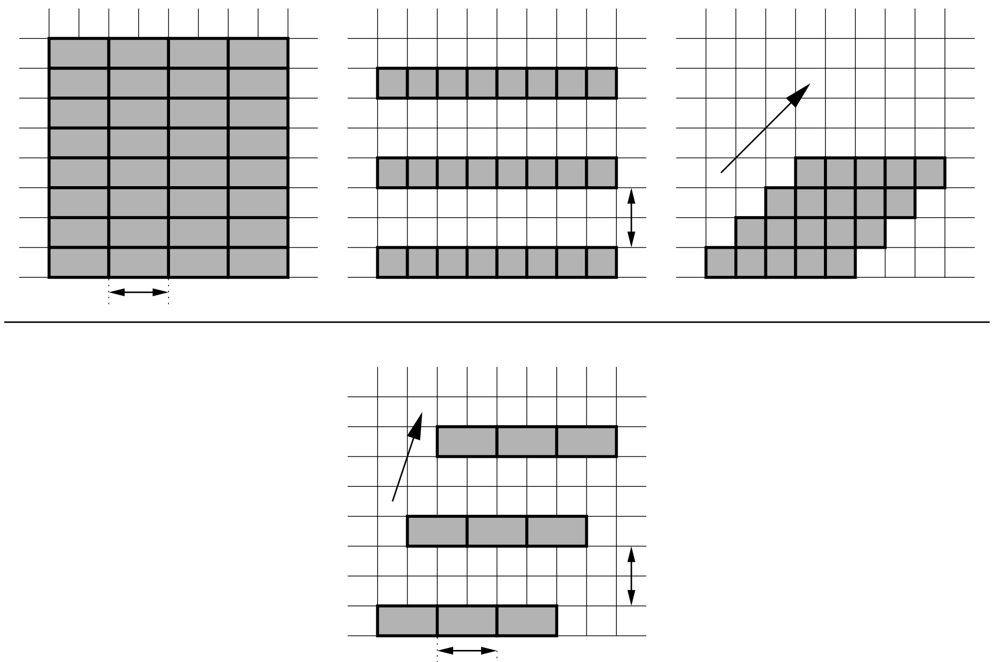
Theorem. “If there exists a strongly universal object, then any universal object is strongly universal” is a bulking property.

$$\varphi = (\exists x, \psi(x)) \Rightarrow \forall x, ((\forall y, \phi(y, x)) \Rightarrow \psi(x))$$

Proof



P^{-}CS



$$o^{m} \circ \mathcal{A}^{n} \circ \sigma^{k} \circ o^{-m}$$

$\mathsf{P}^{-}\mathsf{CS}$ and Bulking

Definition. The $\langle m, n, k\rangle$ *regular* $\mathsf{P}^{-}\mathsf{CS}$ transformation of a CA $\mathcal{A}$ is the CA $\mathcal{A}^{\langle m,n,k\rangle}$ where

$$\mathcal{A}^{\langle m,n,k\rangle} = o^{m} \circ \mathcal{A}^{mn} \circ o^{-m} \circ \sigma^{k}.$$

Lemma. $\mathcal{A}$ simulates $\mathcal{B}$ if and only if $\mathcal{A}$ simulates $\mathcal{B}$ regularly.

Theorem. Regular $\mathsf{P}^{-}\mathsf{CS}$ induces a Bulking.

Proof

- Obj is the set of CA and $\mathsf{Pow} = \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{Z}^*$.
- $f\left(\mathcal{A}, \langle m,n,k\rangle\right) = \mathcal{A}^{\langle m,n,k\rangle}$
- $R\left(\mathcal{A},\mathcal{B}\right)$ if there is an injection φ from $S_\mathcal{A}$ into $S_\mathcal{B}$ such that

$$\bar{\varphi} \circ \mathcal{A} = \mathcal{B} \circ \bar{\varphi}$$

- $\langle m,n,k\rangle \cdot \langle m',n',k'\rangle = \langle mm',nn',k'+n'k\rangle$
- Then we need to check the axioms...

Intrinsic Universality

- There exists a strongly universal CA on $\mathsf{P}^{-}\mathsf{C}$.

- Universal CA coincide with universal CA on $\mathsf{P}^{-}\mathsf{CS}$.

Robust Definition. An intrinsically universal CA is a strongly universal CA on $\mathsf{P}^{-}\mathsf{CS}$.

Theorem[Rapaport 98]**.** There is no real-time universal CA.

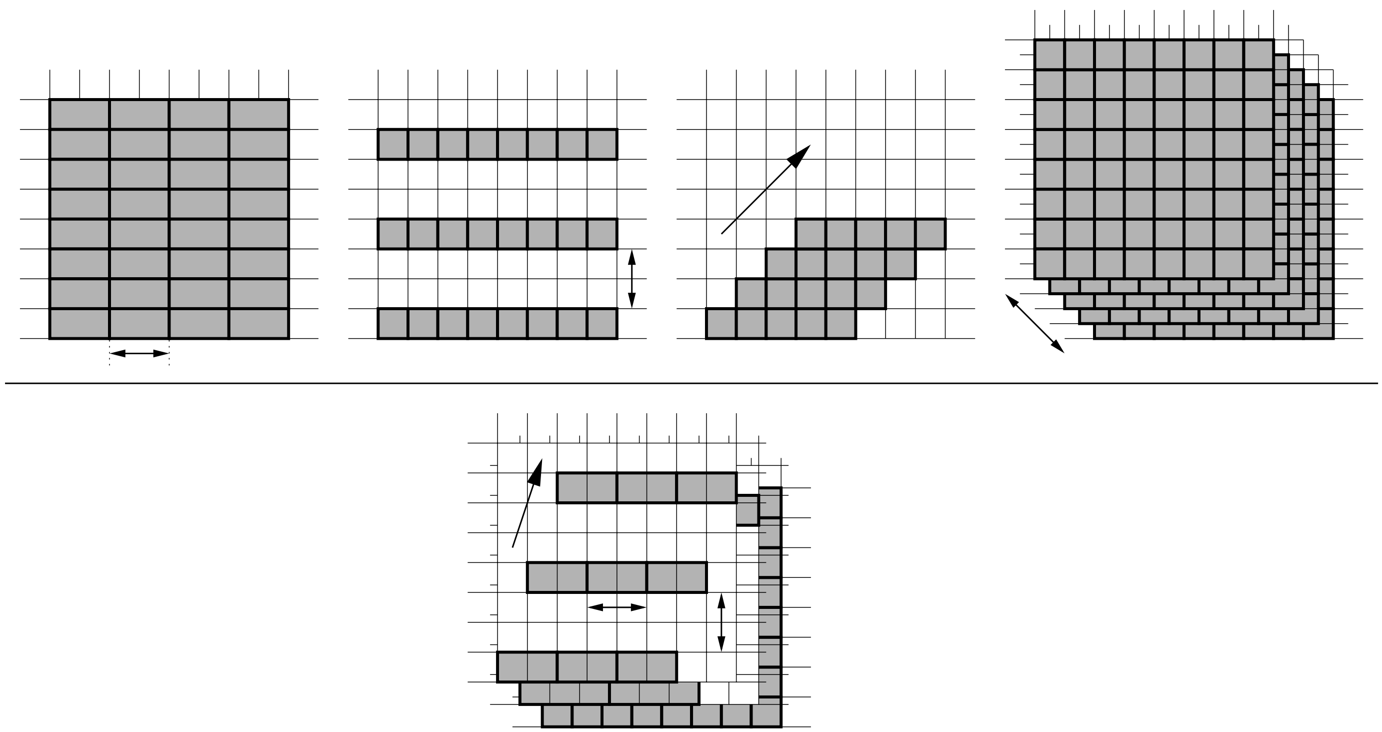
Some results on $\mathsf{P}^{-}\mathsf{CS}$

- Undecidability anywhere (thanks to Nilpotent CA),
- Infinitely many equivalence classes,
- Classical properties are compatible with $\mathsf{P}^{-}\mathsf{CS}$:

trivial CA, two-state CA, first-neighbors CA,
Number-conserving CA, Totalistic CA, Reversible CA.

- **But** the preorder induces no semi-lattice.

P⁻CST



$$\prod_{i=1}^{j} o^{m_i} \circ \mathcal{A}^{n_i} \circ \sigma^{k_i} \circ o^{-m_i}$$

$\mathsf{P}^{-}\mathsf{CST}$ and Bulking

Definition. The $\prod \langle m_i, n_i, k_i \rangle$ *regular* $\mathsf{P}^{-}\mathsf{CST}$ transformation of a CA $\mathcal{A}$ is the CA $\mathcal{A}^{\prod \langle m_i, n_i, k_i \rangle}$ where

$$\mathcal{A}^{\prod \langle m_i, n_i, k_i \rangle} = \prod_i o^{m_i} \circ \mathcal{A}^{m_i n_i} \circ o^{-m_i} \circ \sigma^{k_i}.$$

Lemma. $\mathcal{A}$ simulates $\mathcal{B}$ if and only if $\mathcal{A}$ simulates $\mathcal{B}$ regularly.

Theorem. Regular $\mathsf{P}^{-}\mathsf{CST}$ induces a Bulking.

Proof

- Obj is the set of CA and Pow $= \cup_j \left(\mathbb{N}^* \times \mathbb{N}^* \times \mathbb{Z}^*\right)^j$.
- $f\left(\mathcal{A}, \prod \langle m_i, n_i, k_i\rangle\right) = \mathcal{A}^{\prod \langle m_i, n_i, k_i\rangle}$
- $R\left(\mathcal{A}, \mathcal{B}\right)$ if there is an injection φ from $S_{\mathcal{A}}$ into $S_{\mathcal{B}}$ such that

$$\bar{\varphi} \circ \mathcal{A} = \mathcal{B} \circ \bar{\varphi}$$

- $\prod \langle m_i, n_i, k_i\rangle \cdot \prod \left\langle m'_j, n'_j, k'_j\right\rangle =$
$\prod \left\langle m_i m'_j, n_i n'_j, k'_j + n'_j k_i\right\rangle$
- Then we need to check the axioms...

The semi-lattice

Theorem. $\mathsf{P}^{-}\mathsf{CST}$ induces a sup semi-lattice with the natural operation $\mathcal{A} \times \mathcal{B}$ as a sup operation.

► Ideals play an interesting role:

- Reversible CA build a principal ideal,
- Non-chaoticity build an ideal.

Going further...

- Continue studying $\mathsf{P}^{-}\mathsf{CST}$.
- Continue the exploration of classical properties.
- Is the ideal of non-universal CAs principal ?