Packing, Cutting, Shifting, and Twisting

space-time diagrams
of Cellular Automata

Nicolas Ollinger – Automata 2001 – 27 september 2001

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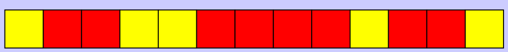
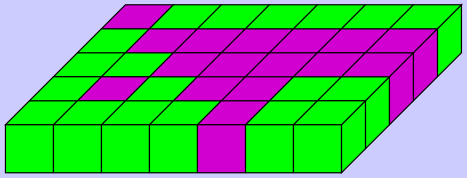
 ,  , ...

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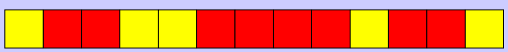
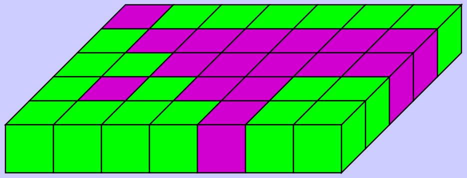
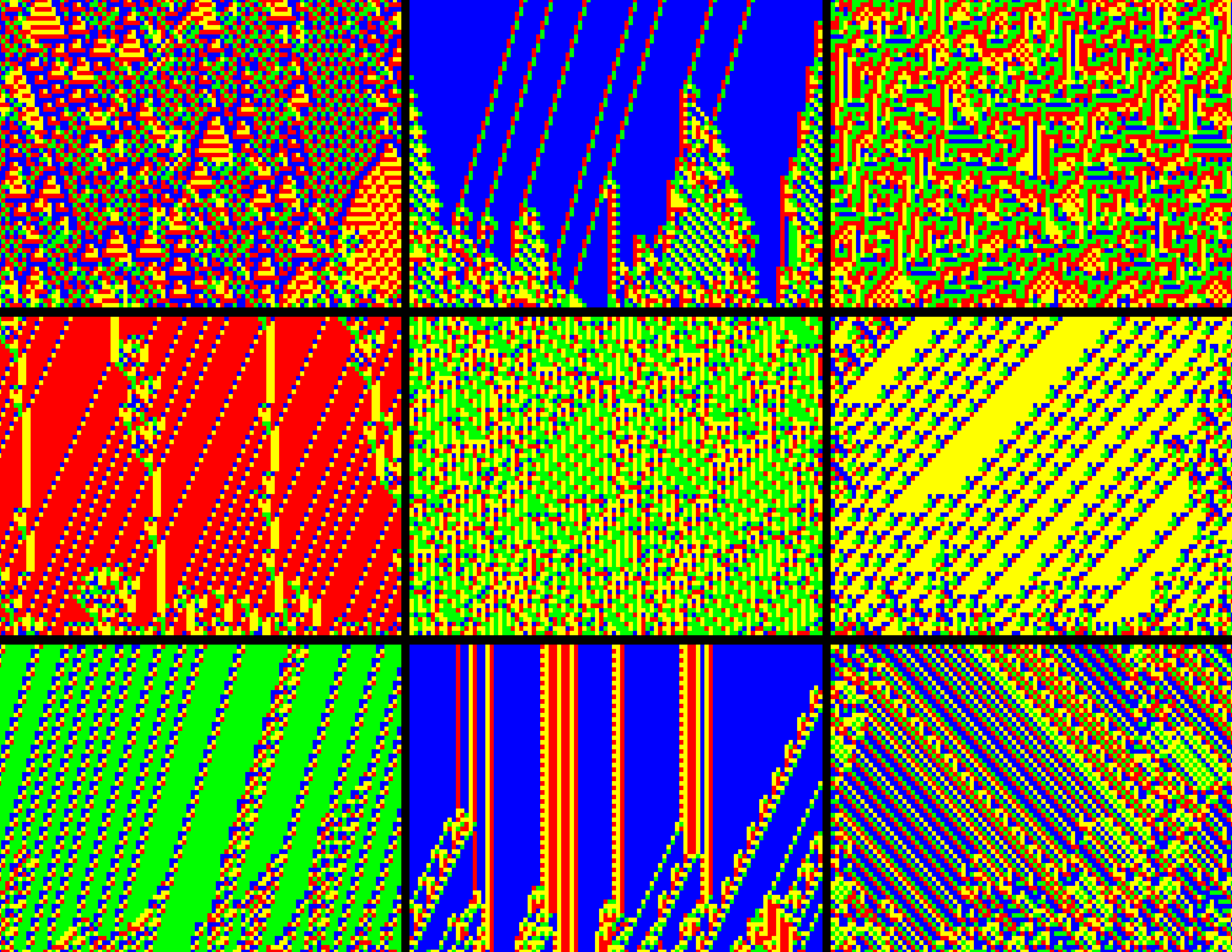
 ,  , $\ldots$

- The *global rule* applies δ uniformly:

$$G_{\mathcal{A}}(C)_i = \delta\left(C_{i+v_1}, \ldots, C_{i+v_\nu}\right)$$

where $N = \{v_1, \ldots, v_\nu\}$.



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Theorem[Hedlund 69]. *A map $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$ is the global rule of a d-CA if and only if it is continue and commute with the shifts.*

A (naive) way to use CA

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- Choose a set of initial configurations.
- Construct their space-time diagrams.
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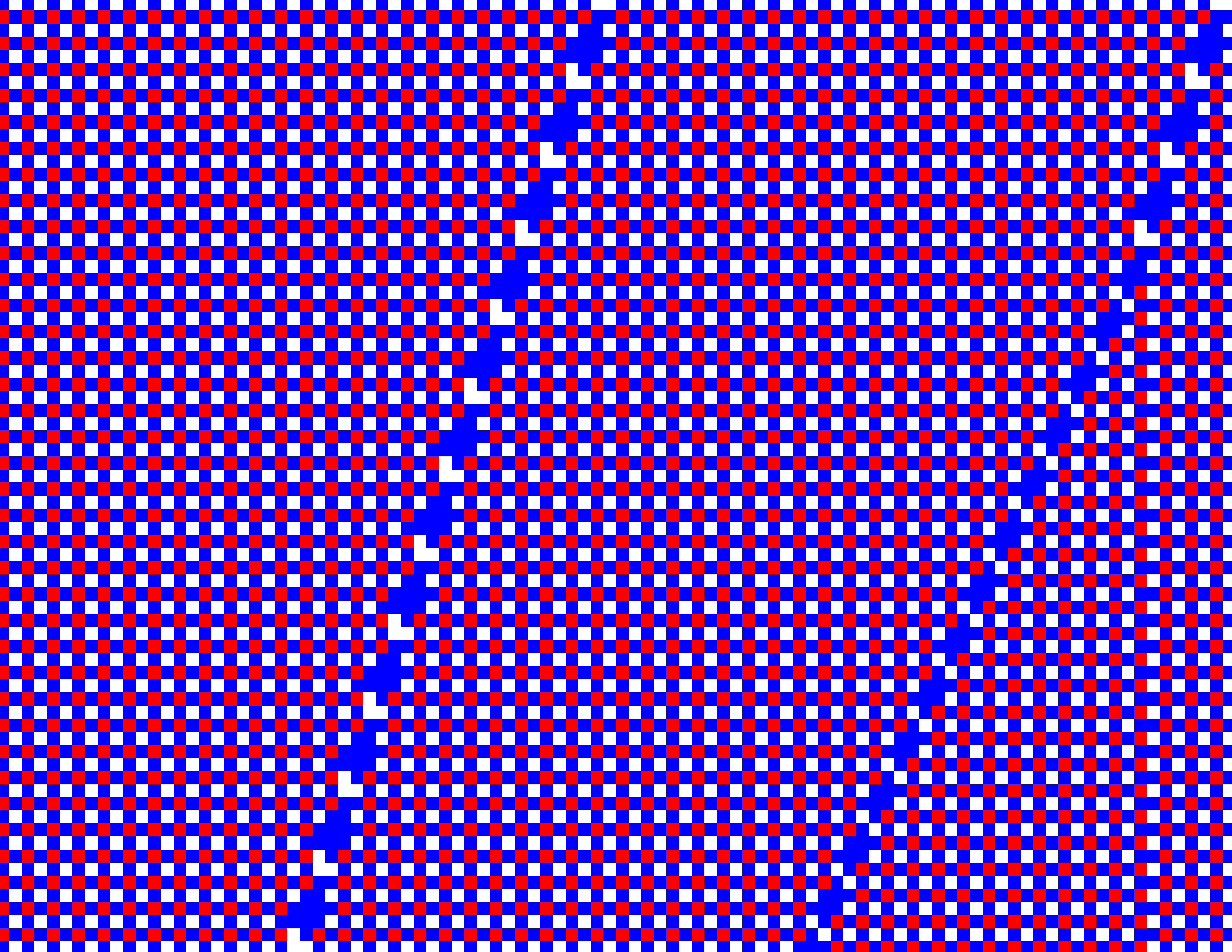
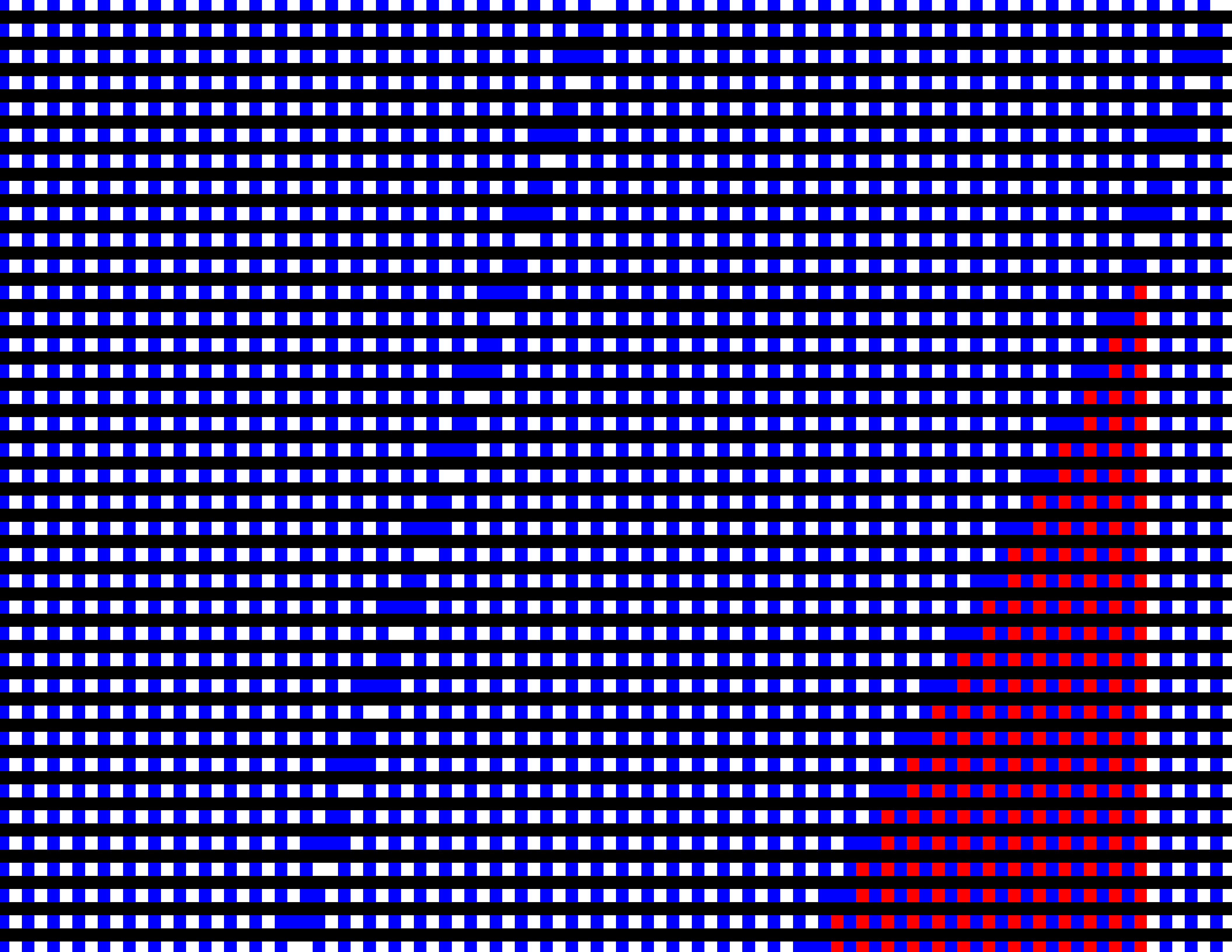
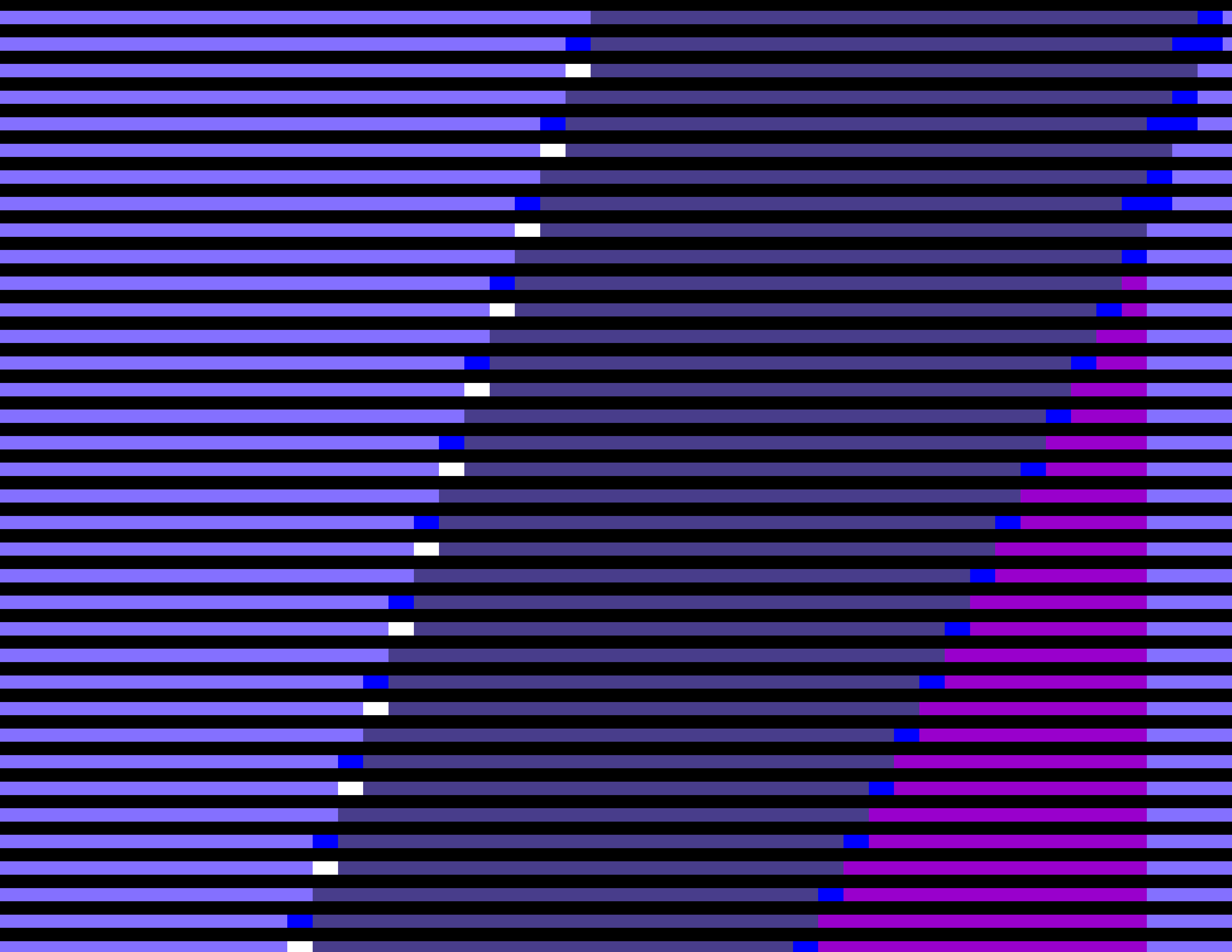
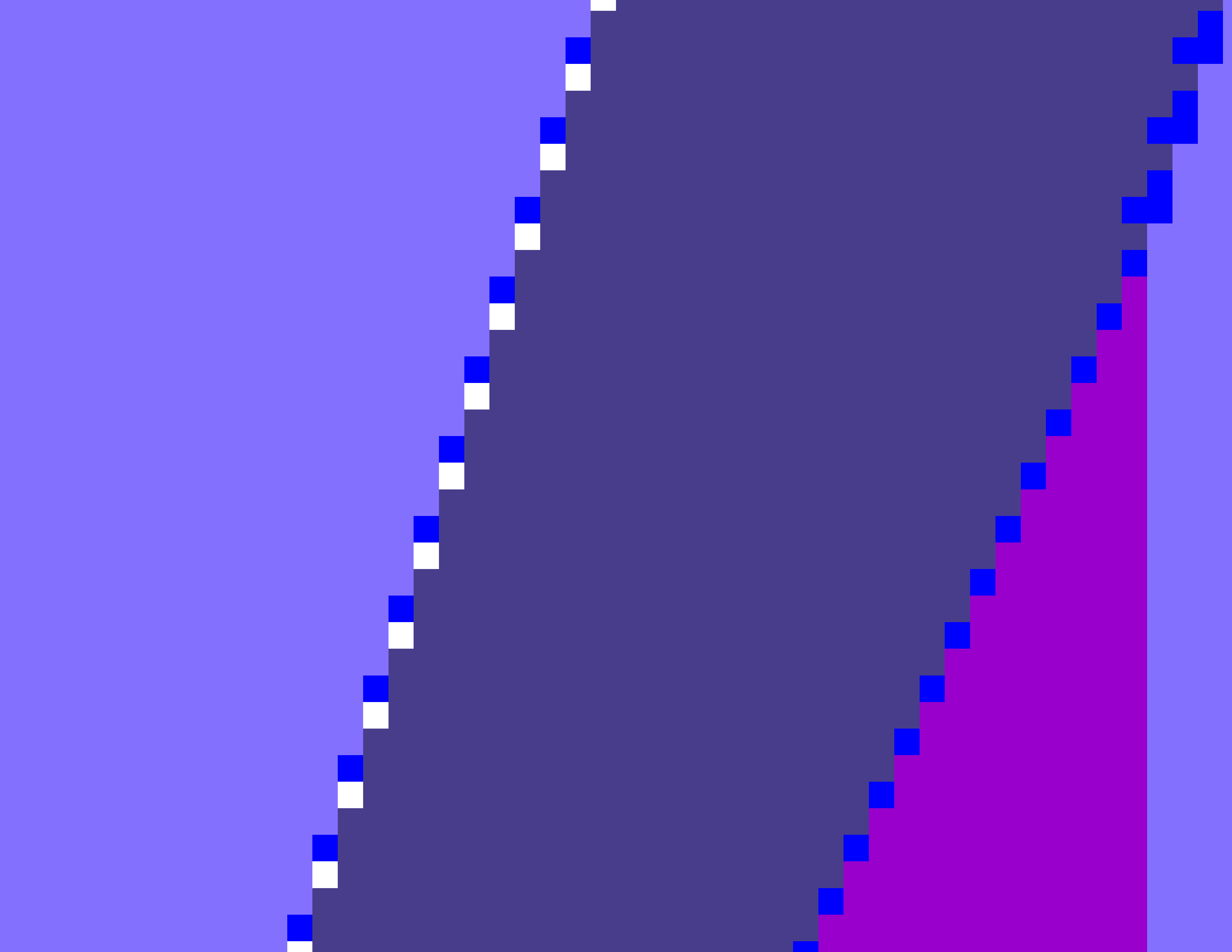
Some properties depend on
the granularity of the model!



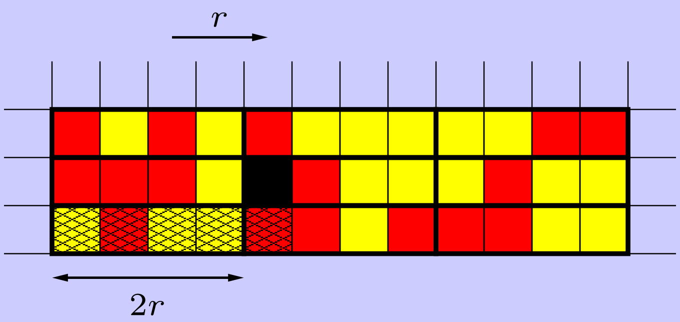






Classical Simulations

- Any CA can be simulated by an OCA *(Cole 69, Ibarra 85)*



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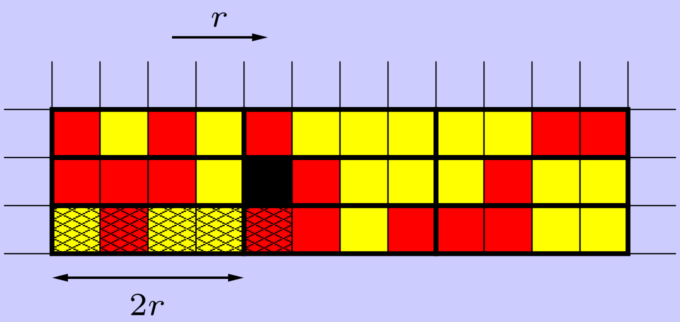
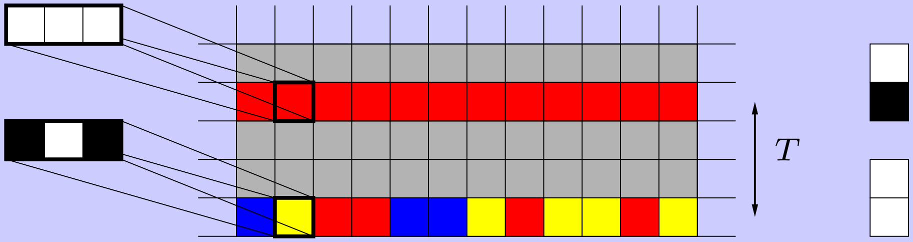
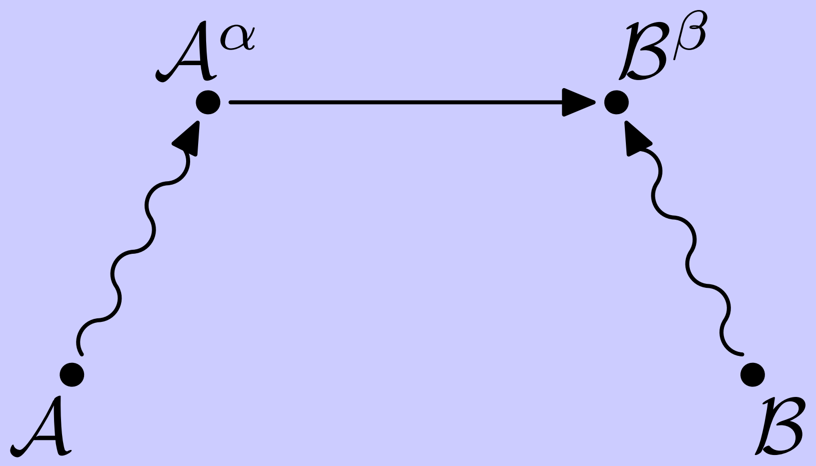


- Any Nil CA can be simulated by the trivial one



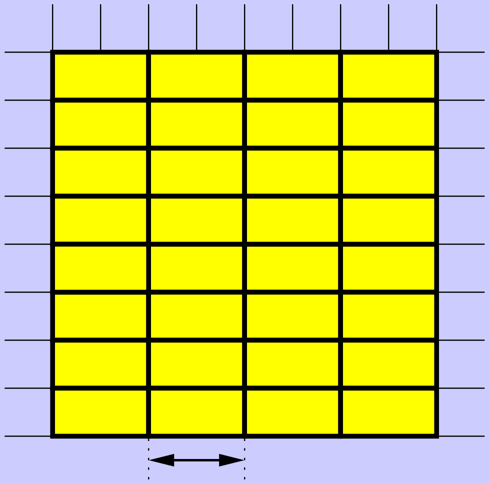
Simulation and Transformations

Idea. *A CA $\mathcal{A}$ simulates another CA $\mathcal{B}$ if, up to geometrical transformations, any space-time diagram from $\mathcal{B}$ is a space-time diagram from $\mathcal{A}$.*



Good transformations

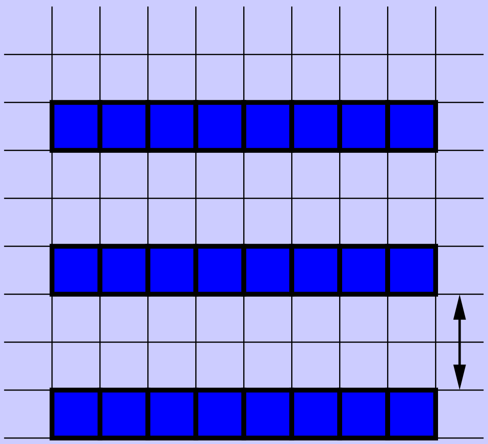
packing



$o^m \circ G_{\mathcal{A}} \circ o^{-m}$

spatial organization

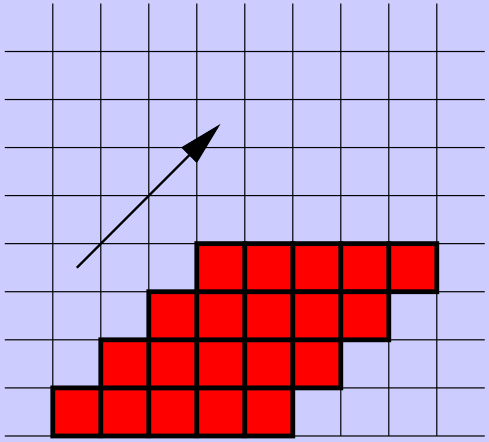
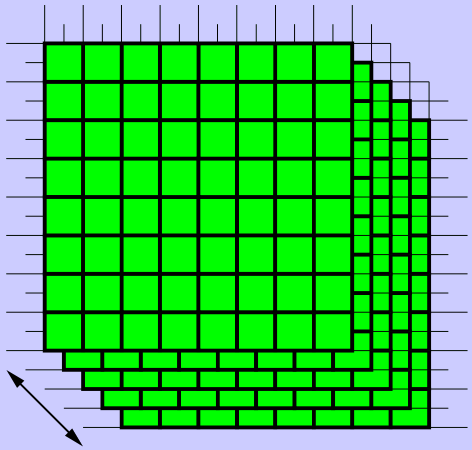
cutting



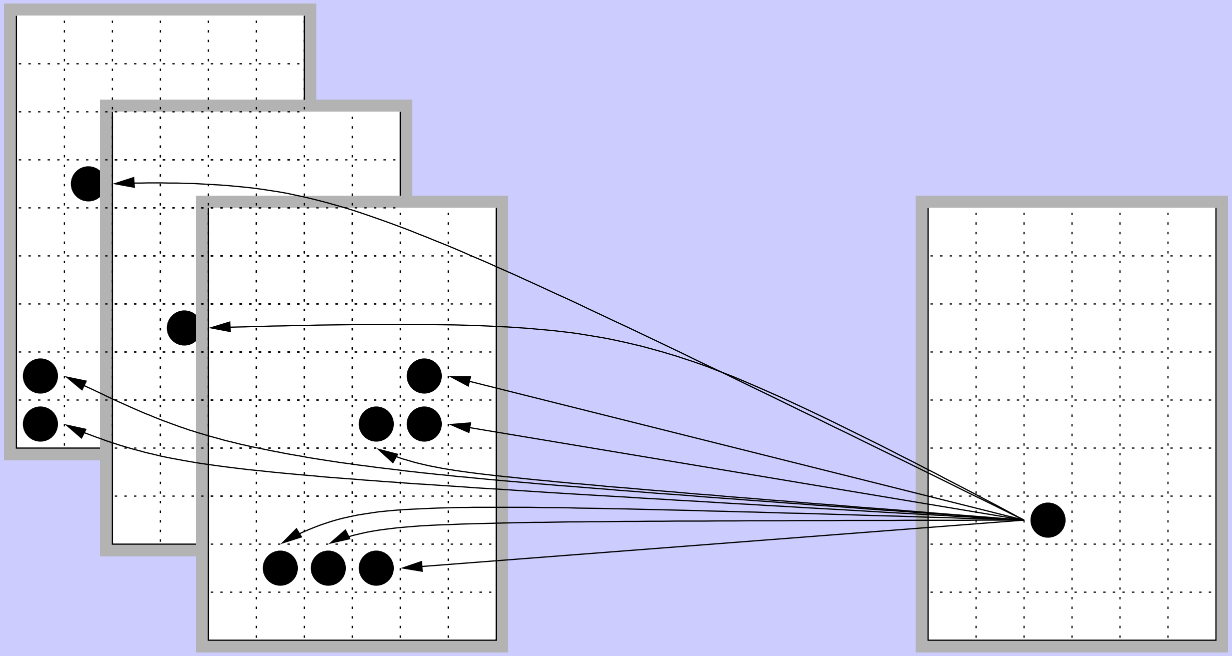
$G_{\mathcal{A}}^n$

temporal organization

Good transformations (2)

shifting	twisting
	
$G_{\mathcal{A}} \circ \sigma_k$	$\prod_i G_{\mathcal{A}_i}$
information mixing	*independent*

Generalizing Transformations

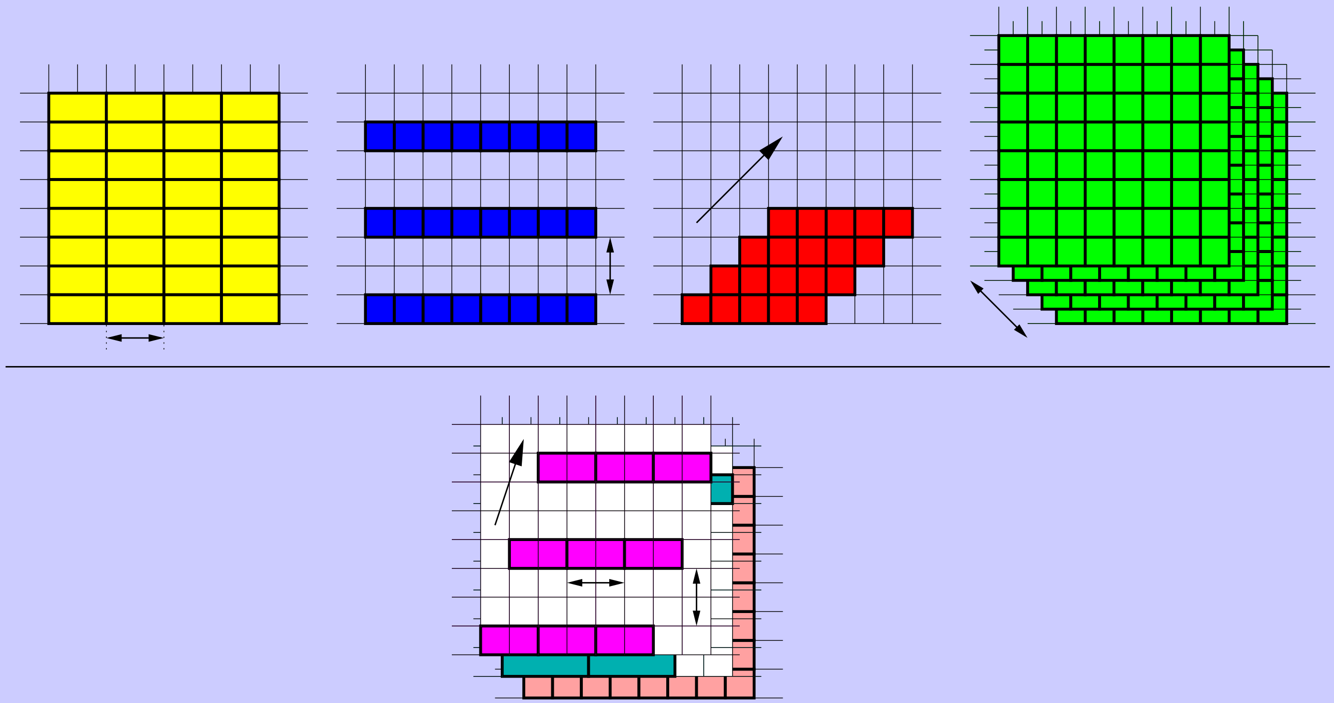


$$\varphi : \mathbb{N} \times \mathbb{Z} \longrightarrow 2^{\{1,\ldots,k\}\times\mathbb{N}\times\mathbb{Z}}$$

- The new CA must be *completely* defined.

Geometrical Characterization

Theorem. *There exist no transformation but compositions of the 4 good previous ones.*



$\bar{P}CST$ transformations

Definition. *The* $\prod \langle m_i, n_i, k_i \rangle$ regular $\bar{P}CST$ *transformation of a CA* $\mathcal{A}$ *is the CA* $\mathcal{A}^{\prod \langle m_i, n_i, k_i \rangle}$ *where*

$$\mathcal{A}^{\prod \langle m_i, n_i, k_i \rangle} = \prod_i o^{m_i} \circ \mathcal{A}^{m_i n_i} \circ o^{-m_i} \circ \sigma^{k_i}.$$

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Theorem. *The* P̄CST *relation of simulation is a quasi-order with a maximal equivalence class.*

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✥ Definition of CA rules from elementary rules by an algebraic closure.

The semi-lattice

Theorem. $\mathrm{P\bar{C}ST}$ *induces a sup semi-lattice with the natural operation* $\mathcal{A} \times \mathcal{B}$ *as a sup operation.*

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❖ Ideals capture interesting notions:

- reversibility (principal),
- ultimately periodic,
- simple signal,
- naive non-chaoticity.