Geometrical Transformations of Space-Time Diagrams

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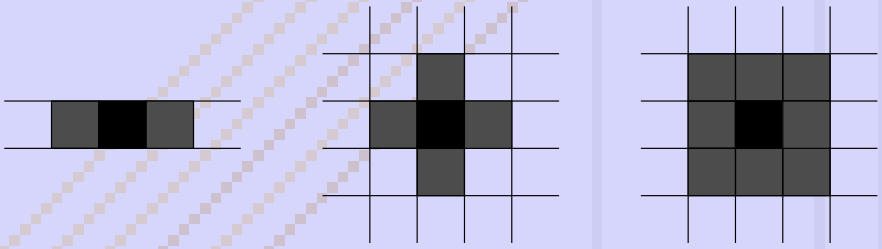
Cellular Automata (1)

Definition. A d-*CA* $\mathcal{A}$ is a 4-uple $\left(\mathbb{Z}^{\mathrm{d}}, S, \mathrm{N}, \delta\right)$ where:

- S is the finite state set of $\mathcal{A}$;
- $\mathrm{N} \subset \mathbb{Z}^{\mathrm{d}}$, finite, is the neighborhood of $\mathcal{A}$;



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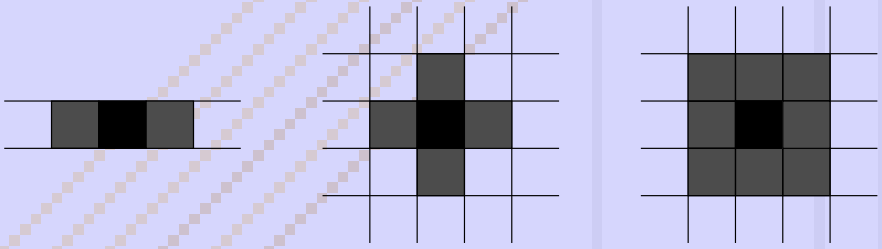

- $\delta : S^{|N|} \rightarrow S$ is the local rule of $\mathcal{A}$.

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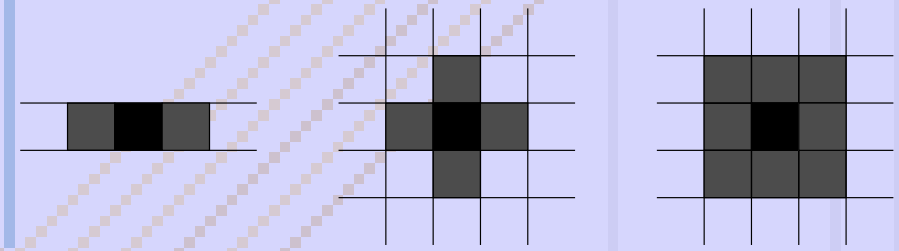

- $\delta : S^{|N|} \to S$ is the local rule of $\mathcal{A}$.

A *configuration* C is a mapping from $\mathbb{Z}^d$ to S.

The *global rule* applies δ uniformly according to N:

$$\forall p \in \mathbb{Z}^d, \quad G(C)_p = \delta\left(C_{p+N_1}, \ldots, C_{p+N_\nu}\right)$$

Cellular Automata (2)

- The *space-time diagram* Δ of $\mathcal{A}$ starting from C is the mapping from $\mathbb{N} \times \mathbb{Z}^d$ to S defined for any time t and any position p by $\Delta(t, p) = G^t(C)_p$ where $G^0(C) = C$.

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Theorem[Hedlund]. A map $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is the global rule of a cellular automaton if and only if G is continuous for the product topology of the trivial topology on S and G commutes with shifts (for all ν, we have $\sigma_\nu \circ G = G \circ \sigma_\nu$).

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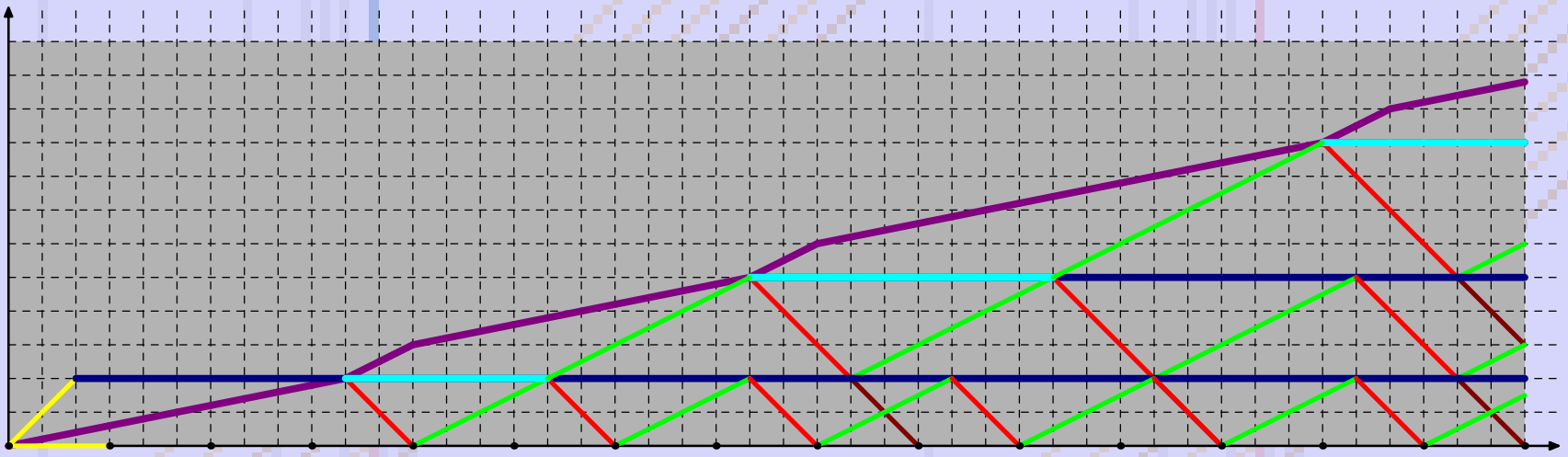
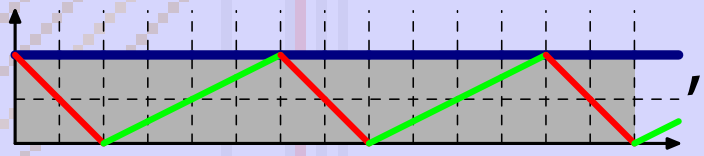
Consequences. If we compose cellular automata, or inverse bijective ones, we obtain cellular automata.

Example - Fisher (1)

P. C. Fisher, *Generation of Primes by a One-Dimensional Real-Time Iterative Array*, Journal of the ACM, 1965



Idea:



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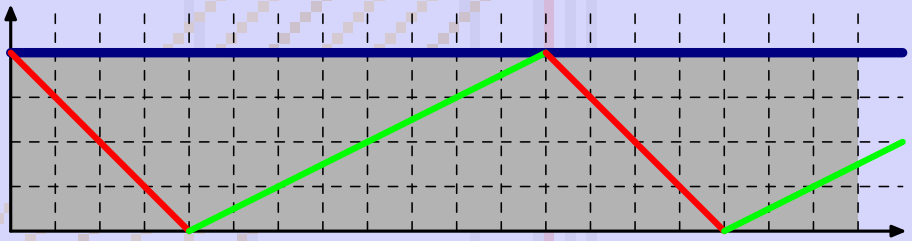
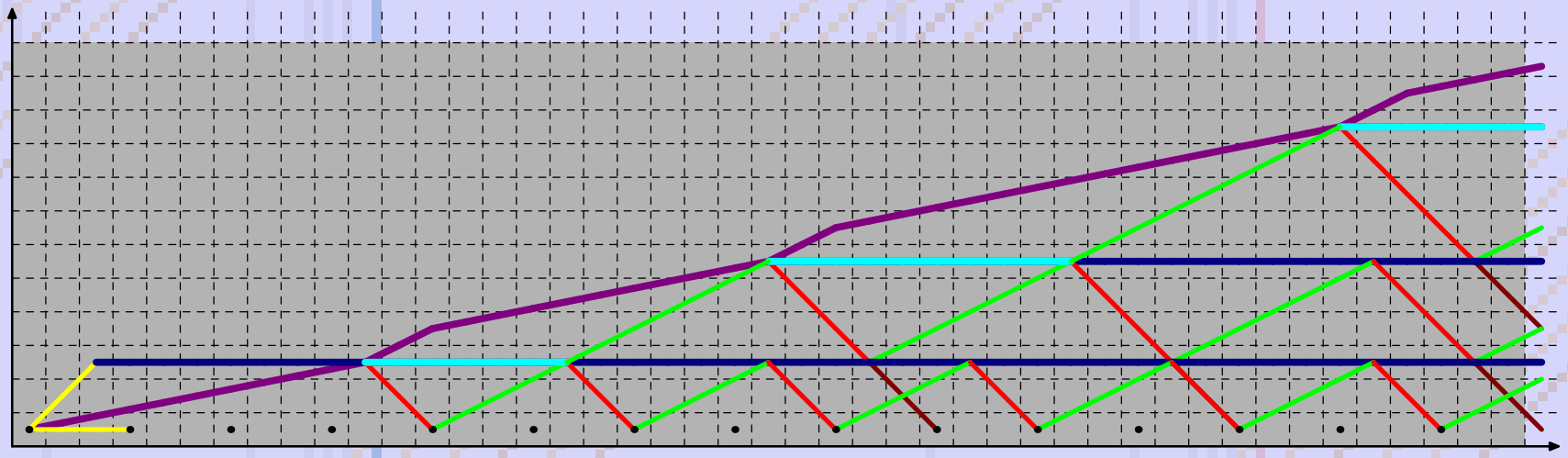
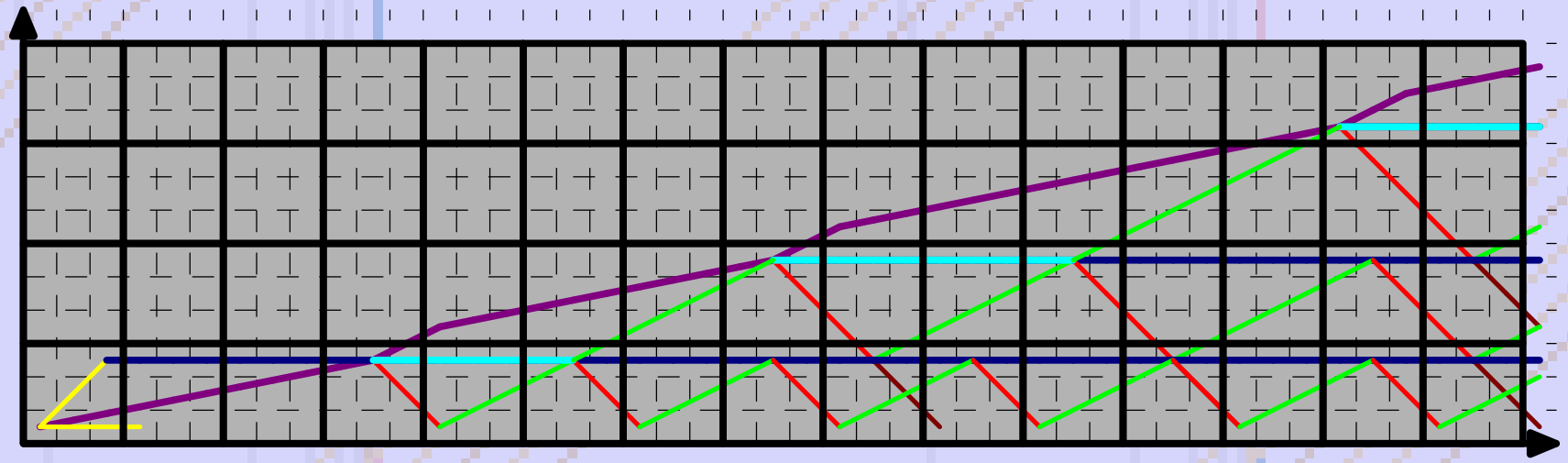


Example - Fisher (2)



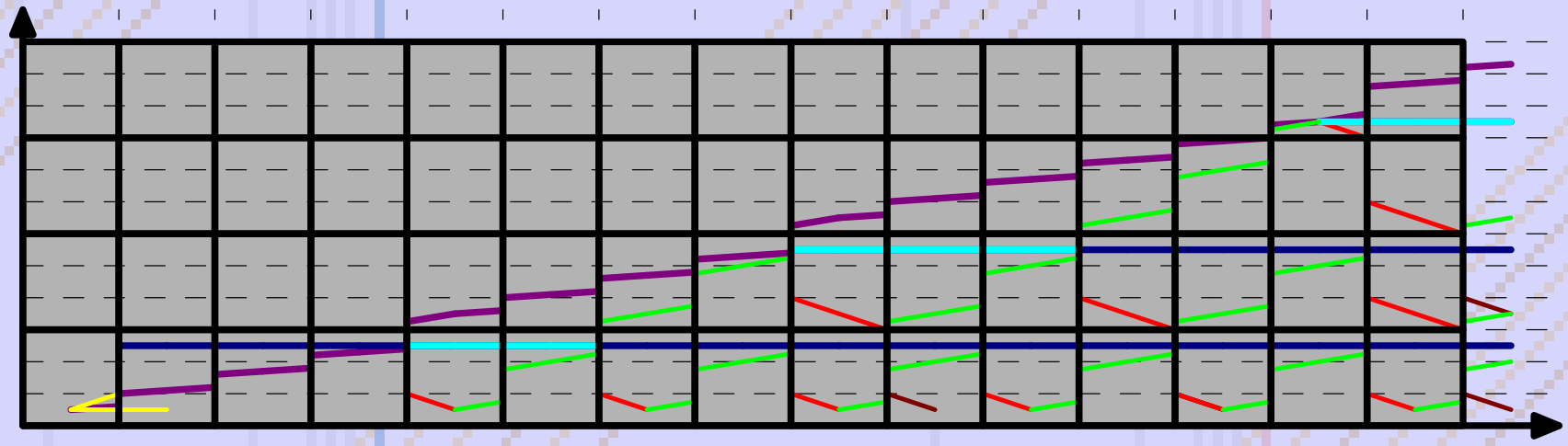
Pb: it recognizes $3p$ such that p is prime... not p!

Example - Fisher (3)



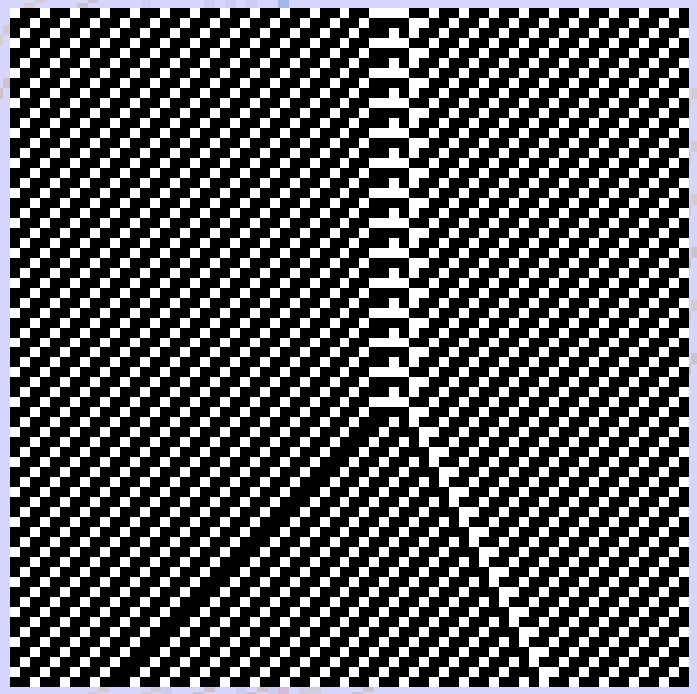
Fisher scales the CA by making 3×3 blocks of cells...

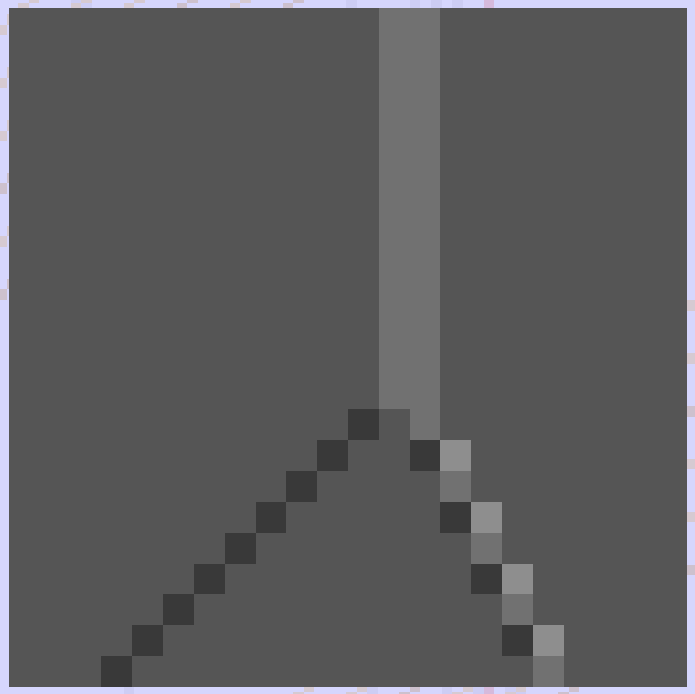
Example - Fisher (4)



...but we can also keep just the bottom cells.

Example - Particles (1)

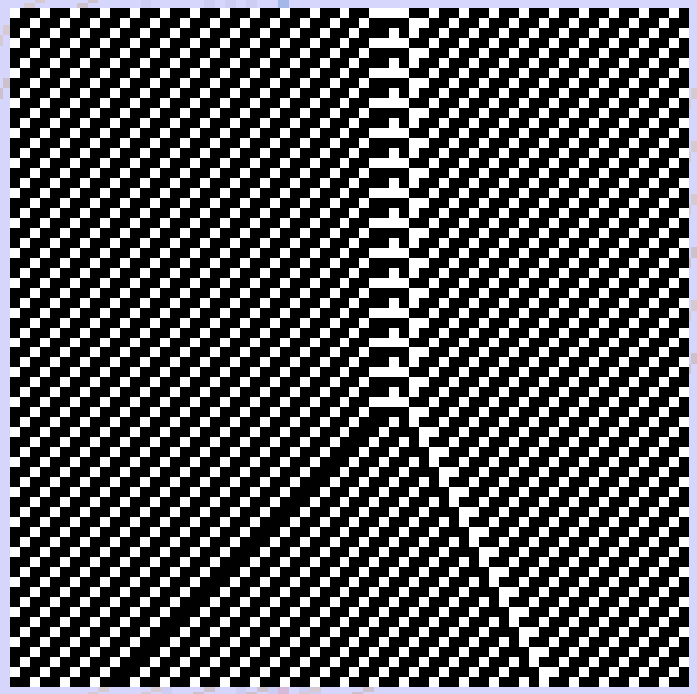


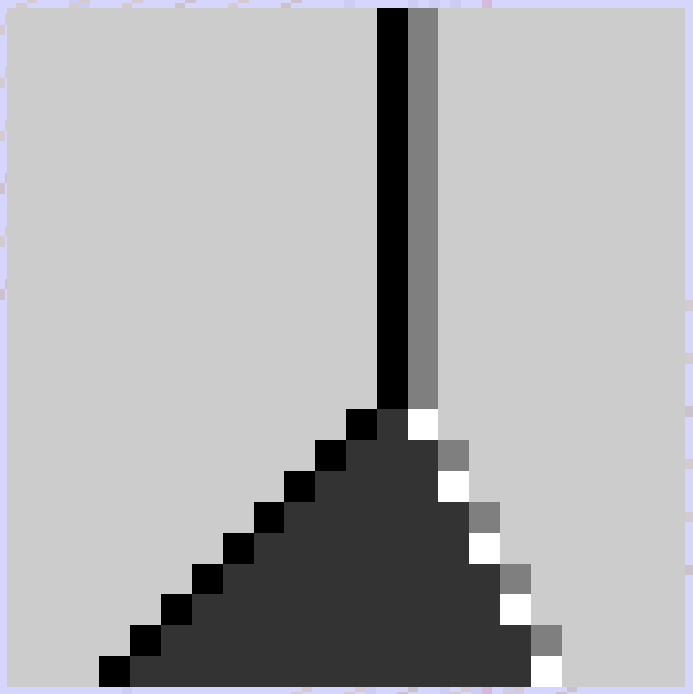


How to eliminate the periodic background pattern ?
You can zoom out and use shades of grey...

$$C'_p = 1/9 \sum_{\nu \in [\![0,2]\!]^2} C_{3p+\nu}$$

Example - Particles (2)



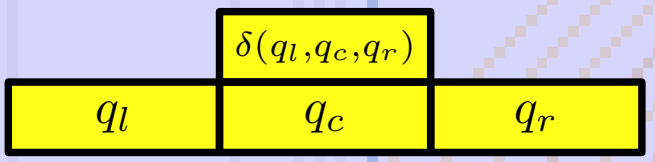


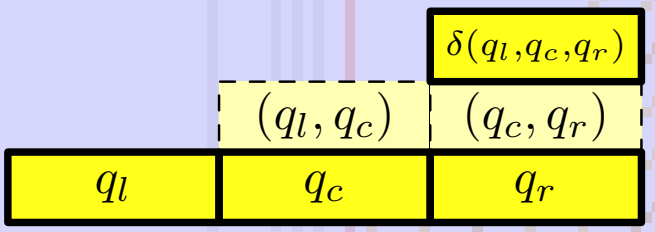
How to eliminate the periodic background pattern ?

...but also make blocks of bottom cells of the squares

$$C'_p = \left(C_{3p+(0,0)}, C_{3p+(1,0)}, C_{3p+(2,0)}\right)$$

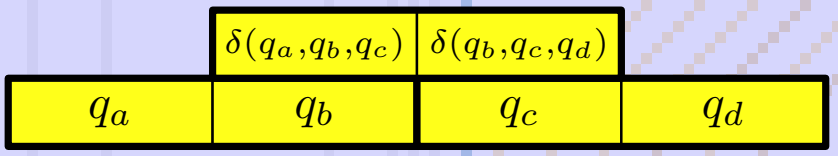
Example - from CA to OCA (1)

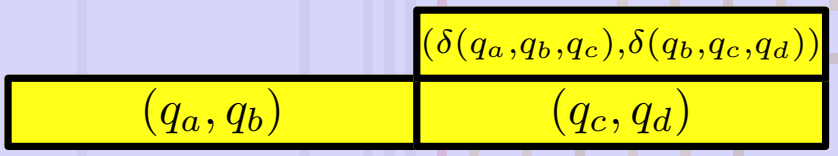




C. Choffrut and K. Culik II, *On real-time cellular automata and trellis automata*, Acta Informatica, 1984

Example - from CA to OCA (2)





O. H. Ibarra, *Sequential machine characterizations of trellis and cellular automata and applications*, SIAM Journal on Computing, 1985

Formalization (1)

- A *geometrical transformation* on space-time diagrams **transforms a cellular automaton into a new one** by **combining cells** of a space-time diagram of the first one to construct a space-time diagram of the second one.

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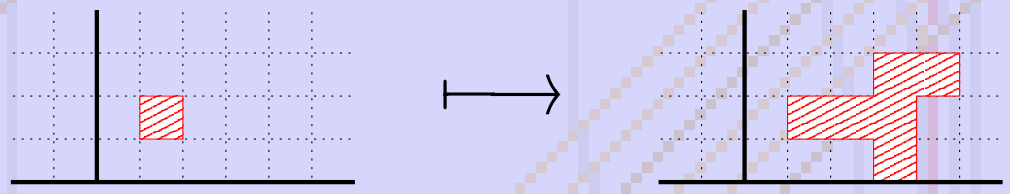
- Formally, it is a pair (k, Λ) where

$$\Lambda : \quad \mathbb{N} \times \mathbb{Z}^d \longrightarrow \left(\mathbb{N} \times \mathbb{Z}^d\right)^k$$



Formalization (2)

- To apply a transformation (k, Λ) to a space-time diagram Δ over S, we define $\overline{\Lambda}_S : S^{\mathbb{N}\times\mathbb{Z}^d} \to \left(S^k\right)^{\mathbb{N}\times\mathbb{Z}^d}$ by

$$\overline{\Lambda}_S(\Delta)(t,p) = (\Delta(\Lambda(t,p)_1), \ldots, \Delta(\Lambda(t,p)_k)) \quad .$$

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$$\overline{\boldsymbol{\Lambda}}_S(\Delta)(t,p) = (\Delta(\boldsymbol{\Lambda}(t,p)_1), \ldots, \Delta(\boldsymbol{\Lambda}(t,p)_k)) \quad .$$

- We define an operation rather similar to composition :

$$(k', \boldsymbol{\Lambda}') \circ (k, \boldsymbol{\Lambda}) = (kk', \boldsymbol{\Lambda}' \circ \boldsymbol{\Lambda})$$

where

$$\left(\boldsymbol{\Lambda}' \circ \boldsymbol{\Lambda}\right)(t,p) = \left(\boldsymbol{\Lambda}'\left(\boldsymbol{\Lambda}(t,p)_1\right)_1 \ldots, \boldsymbol{\Lambda}'\left(\boldsymbol{\Lambda}(t,p)_k\right)_{k'}\right)$$

Formalization (3)

- We also introduce $\widetilde{\Lambda}$ as

$$\begin{aligned} \widetilde{\Lambda}: \quad 2^{\mathbb{N}\times\mathbb{Z}^d} &\longrightarrow 2^{\mathbb{N}\times\mathbb{Z}^d} \\ X &\longmapsto \bigcup_{(t,p)\in X} \{\Lambda(t,p)_1,\ldots,\Lambda(t,p)_k\} \end{aligned}$$

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- A *good* geometrical transformation satisfies

1. $\forall \mathcal{A}, \exists \mathcal{B}, \quad \left\{\overline{\Lambda}_{S_{\mathcal{A}}}(\Delta)\right\}_{\Delta\in \mathrm{Diag}(\mathcal{A})} = \mathrm{Diag}(\mathcal{B}) \quad ;$

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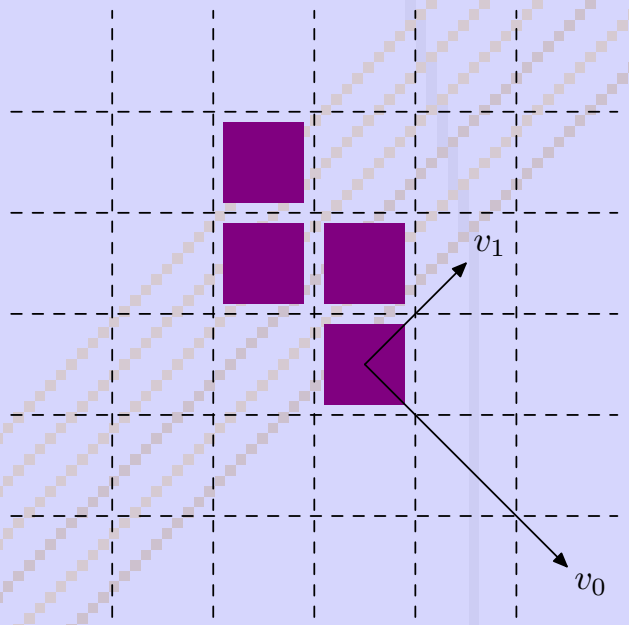
2. $\forall t \in \mathbb{N}, \widetilde{\Lambda}\left(\{t+1\}\times\mathbb{Z}^d\right) \not\subseteq \widetilde{\Lambda}\left(\{t\}\times\mathbb{Z}^d\right) \quad .$

Packing

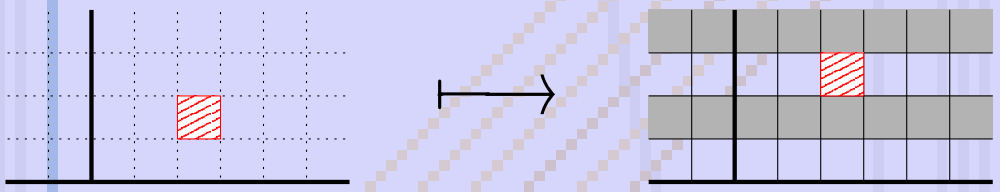


$$\mathbf{P}_{F,v}(t,p) = t \circledast (F \oplus (p \odot v))$$

Transformed CA global rule:

$$o_{F,v}^{-1} \circ G \circ o_{F,v}$$

Cutting

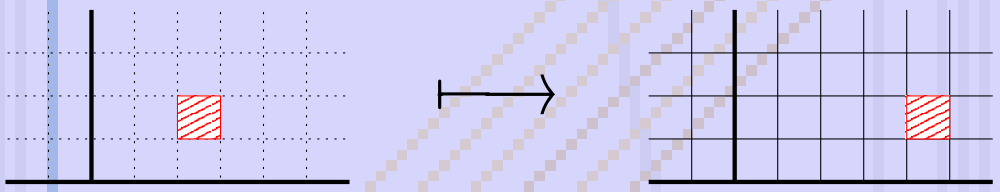


$$\mathbf{C}_T\,(t, p) = (tT, p)$$

Transformed CA global rule:

$$G^T$$

Shifting



$$\mathbf{S}_s\,(\mathrm{t},\mathrm{p}) = (\mathrm{t},\mathrm{p} \oplus \mathrm{ts})$$

Transformed CA global rule:

$$\sigma_s \circ \mathrm{G}$$

Composition

We define **PCS** transformations as

$$\mathbf{PCS}_{F,\nu,T,s} = \mathbf{P}_{F,\nu} \circ \mathbf{S}_s \circ \mathbf{C}_T$$

$$\mathbf{PCS}_{F,\nu,T,s}(t,p) = tT \circledast (F \oplus (p \odot \nu \oplus ts))$$

Transformed CA global rule:

$$o_{F,\nu}^{-1} \circ \sigma_s \circ G^T \circ o_{F,\nu}$$

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PCS transformations are closed under composition.

Characterization

Theorem. A geometrical transformation is a good geometrical transformation if and only if it can be expressed as a **PCS** transformation.