Automates cellulaires : structures

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Cellular Automata: Structures

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19 december 2002 / Ph.D. defence

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Approach

- **A main concern of “Complex Systems”:**

a relatively simple microscopic rule

completely defined local rule (given)

may produce

a very complex macroscopic behavior

far more complex global rule (induced)

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- **A main concern of "Complex Systems":**

a relatively simple microscopic rule

completely defined local rule (given)

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far more complex global rule (induced)

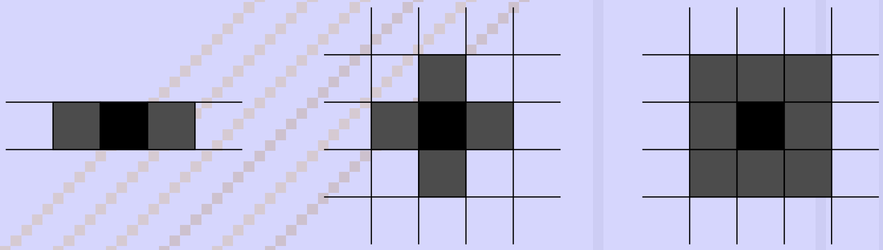
- Cellular Automata provide a simple – not simplistic – and uniform model for studying this problem.

Table of Content

Cellular Automata

Definition. A d-*CA* $\mathcal{A}$ is a 4-uple $(\mathbb{Z}^{\mathrm{d}}, S, \mathrm{N}, \delta)$ where:

- S is the finite state set of $\mathcal{A}$;
- $\mathrm{N} \subset \mathbb{Z}^{\mathrm{d}}$, finite, is the neighborhood of $\mathcal{A}$;



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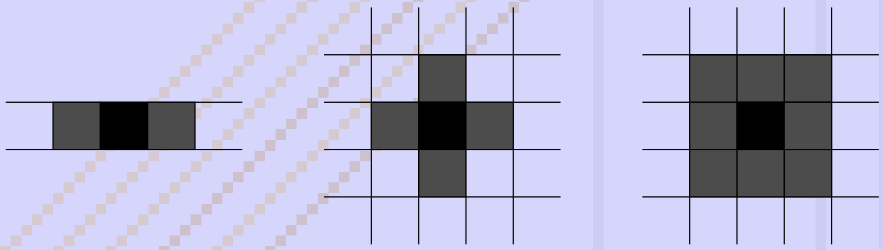

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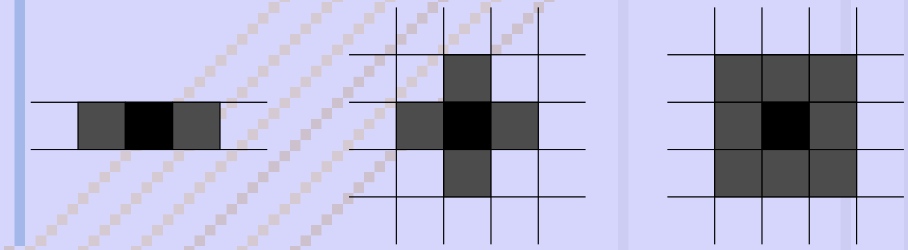


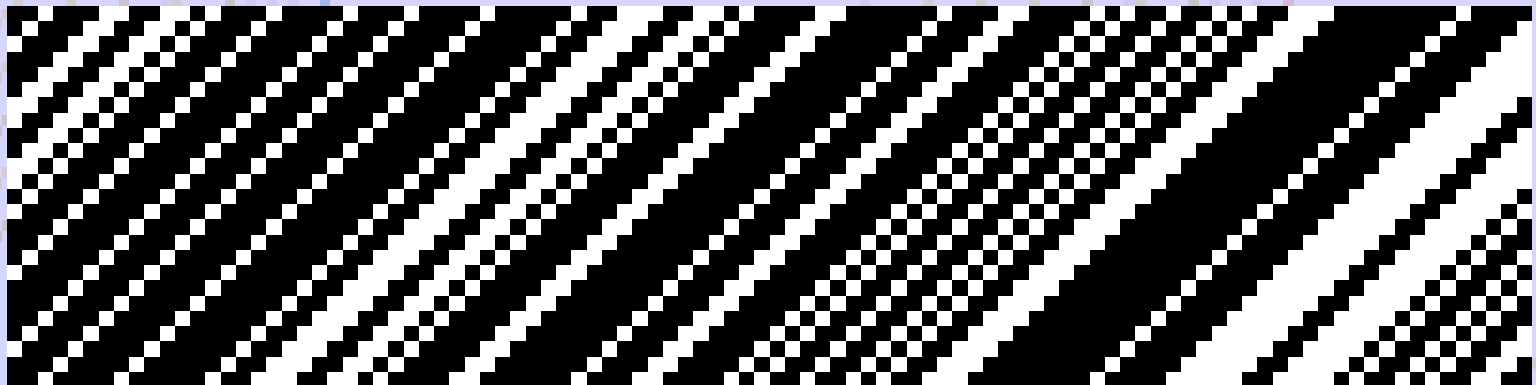
- $\delta : S^{|N|} \to S$ is the local rule of $\mathcal{A}$.

A *configuration* C is a mapping from $\mathbb{Z}^d$ to S.

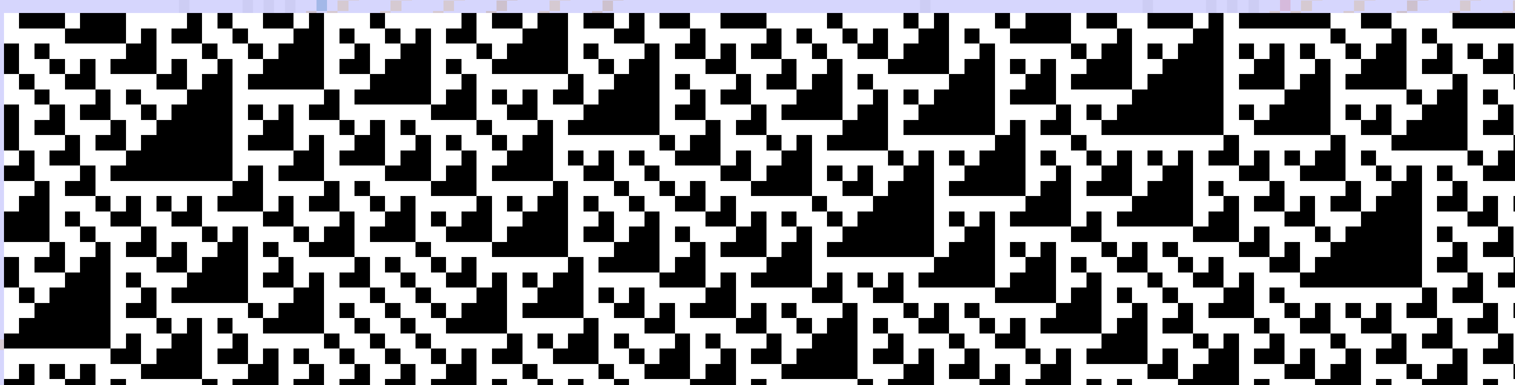
The *global rule* applies δ uniformly according to N:

$$\forall p \in \mathbb{Z}^d, \quad G(C)_p = \delta\left(C_{p+N_1}, \ldots, C_{p+N_\nu}\right)$$

Examples (1)



$$\sigma = (\mathbb{Z}, \{\blacksquare, \square\}, \{-1\}, q \mapsto q)$$

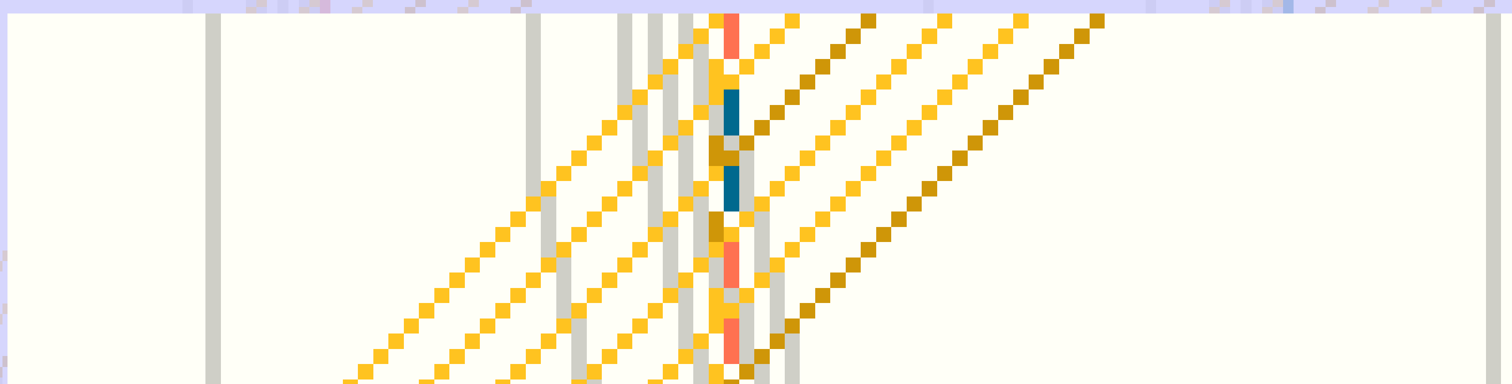


$$\Sigma_2 = (\mathbb{Z}, \{\blacksquare, \square\}, [\![-1, 0]\!], (q, q') \mapsto q \oplus q'),$$

where $(\{\blacksquare, \square\}, \oplus)$ is isomorphic to $(\mathbb{Z}_2, +)$

Examples (2)

$(\mathbb{Z}, \{\blacksquare, \square\}, [\![-1, 1]\!], \text{maj})$,
where maj is majority between 3



$(\mathbb{Z}, \{\square, \square, \square, \square, \square, \square\}, [\![-1, 1]\!], \delta_6)$

Topological Charact.

- Endow S with the trivial topology.
- Endow $S^{\mathbb{Z}^d}$ with the induced product topology.
- The *shift* $\sigma_v : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is defined as

$$\sigma_v(C)_{p+v} = C_p \, .$$

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Theorem[Hedlund 69]**.** A map $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$ is the global rule of a d-CA if and only if it is continue and commutes with shifts.

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Theorem[Hedlund 69]**.** A map $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is the global rule of a d-CA if and only if it is continue and commutes with shifts.

Consequences. We can freely compose CA and invert bijective CA to obtain new CA.

Subautomaton

- A CA $\mathcal{A}$ is *isomorphic* to a CA $\mathcal{B}$ ($\mathcal{A} \cong \mathcal{B}$) if there exists a **bijective** map $\varphi : S_{\mathcal{A}} \to S_{\mathcal{B}}$ such that

$$\overline{\varphi} \circ G_{\mathcal{A}} = G_{\mathcal{B}} \circ \overline{\varphi}$$

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Definition. $\mathcal{A} \subseteq \mathcal{B}$ if there exists an **injective** map $\varphi : S_{\mathcal{A}} \to S_{\mathcal{B}}$ such that this diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \overline{\varphi}(C) \\ {\scriptstyle G_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle G_{\mathcal{B}}} \\ G_{\mathcal{A}}(C) & \xrightarrow[\varphi]{} & \overline{\varphi}(G_{\mathcal{A}}(C)) \end{array}$$

Closure (1)

- An *autarkic* CA $\overline{\psi}$ is a CA with neighborhood $\{0\}$ and local rule $\psi : S \to S$. (notice that $\overline{\psi}$ is ultimately periodic)

- An *elementary shift* is a shift σ_v such that $\|v\|_1 = 1$.

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- The *composition* $\mathcal{A} \circ \mathcal{B}$ of two CA $\mathcal{A}$ and $\mathcal{B}$ satisfies
$$G_{\mathcal{A}\circ\mathcal{B}} = G_{\mathcal{A}} \circ G_{\mathcal{B}} .$$

- The *Cartesian product* $\mathcal{A} \times \mathcal{B}$ of two CA satisfies
$$G_{\mathcal{A}\times\mathcal{B}} = G_{\mathcal{A}} \times G_{\mathcal{B}} .$$

Closure (2)

- A new characterization of CA

Theorem. The set of CA is the closure of the set of autarkic CA and elementary shifts by the operations of composition, Cartesian product and subautomaton.

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Theorem. The set of **reversible** (bijective) CA is the closure of the set of **bijective** autarkic CA and elementary shifts by the operations of composition, Cartesian product and subautomaton.

Table of Content

Experimental Work

Wolfram (1984) First classification.

“ [. . .] In class 1, the behavior is very simple, and almost all initial conditions lead to exactly the same uniform final state.

In class 2, there are many different possible final states, but all of them consist just of a certain set of simple structures that either remain the same forever or repeat every few steps.

In class 3, the behavior is more complicated, and seems in many respects random, although triangles and other small-scale structures are essentially always at some level seen.

And finally [. . .] class 4 involves a mixture of order and randomness: localized structures are produced which on their own are fairly simple, but these structures move around and interact with each other in very complicated ways. [. . .] ”

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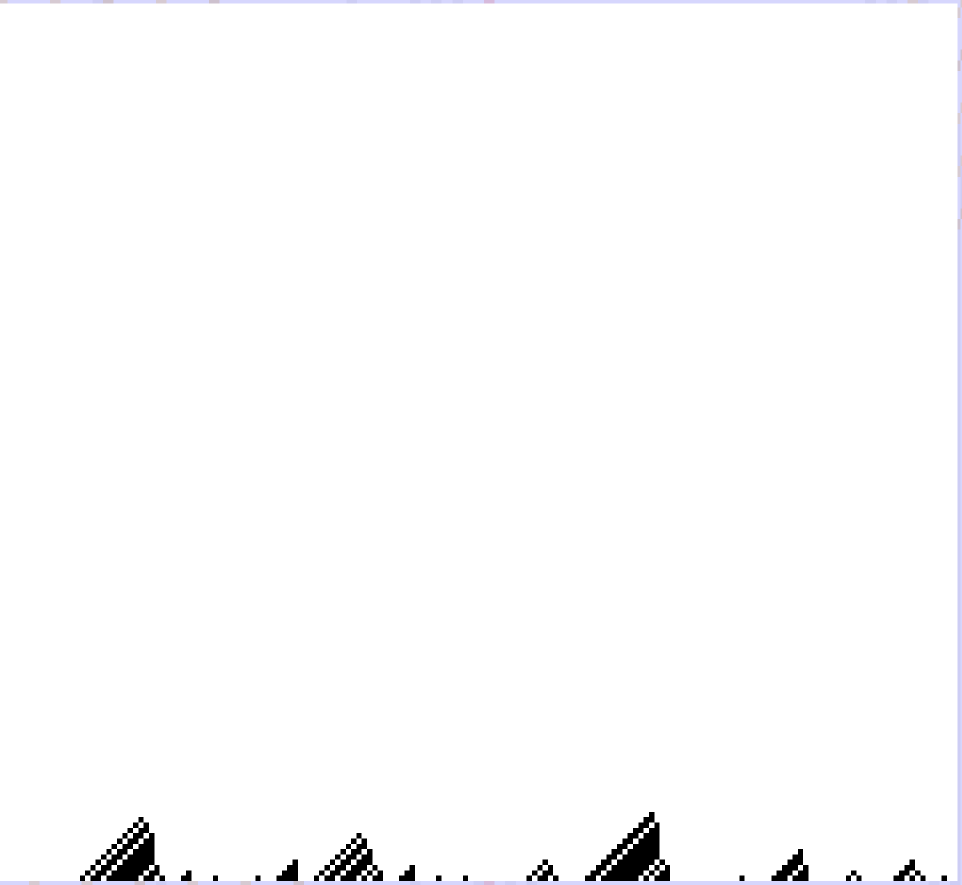

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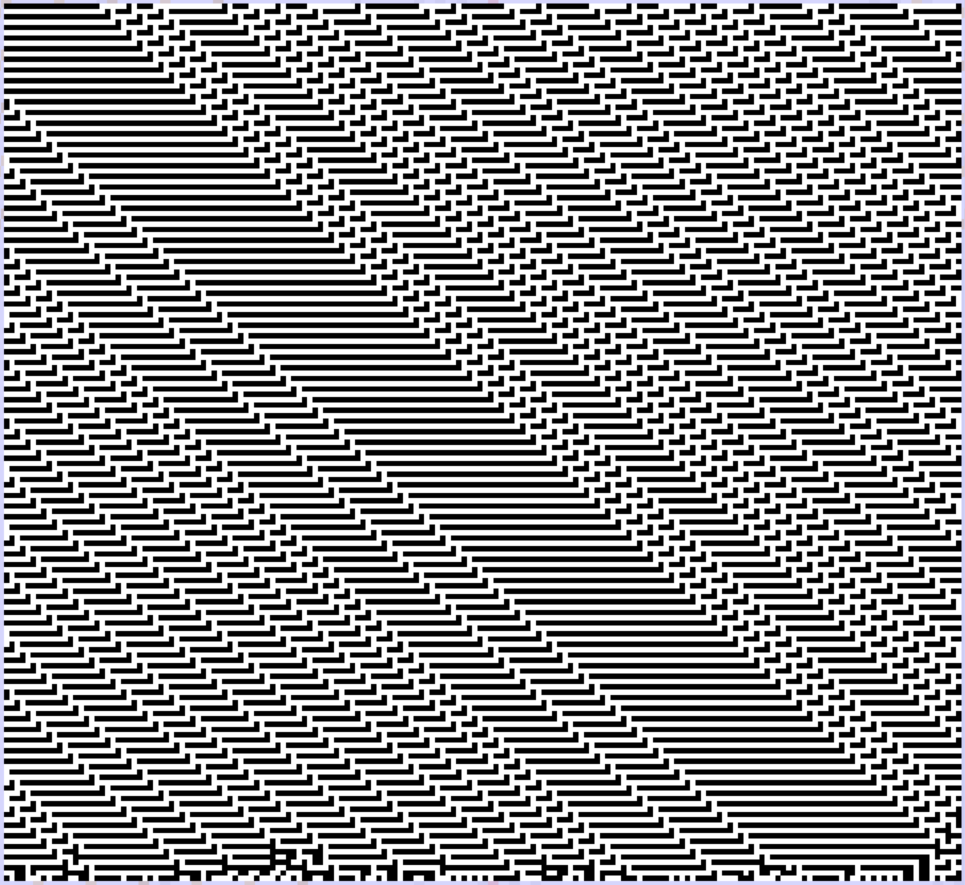

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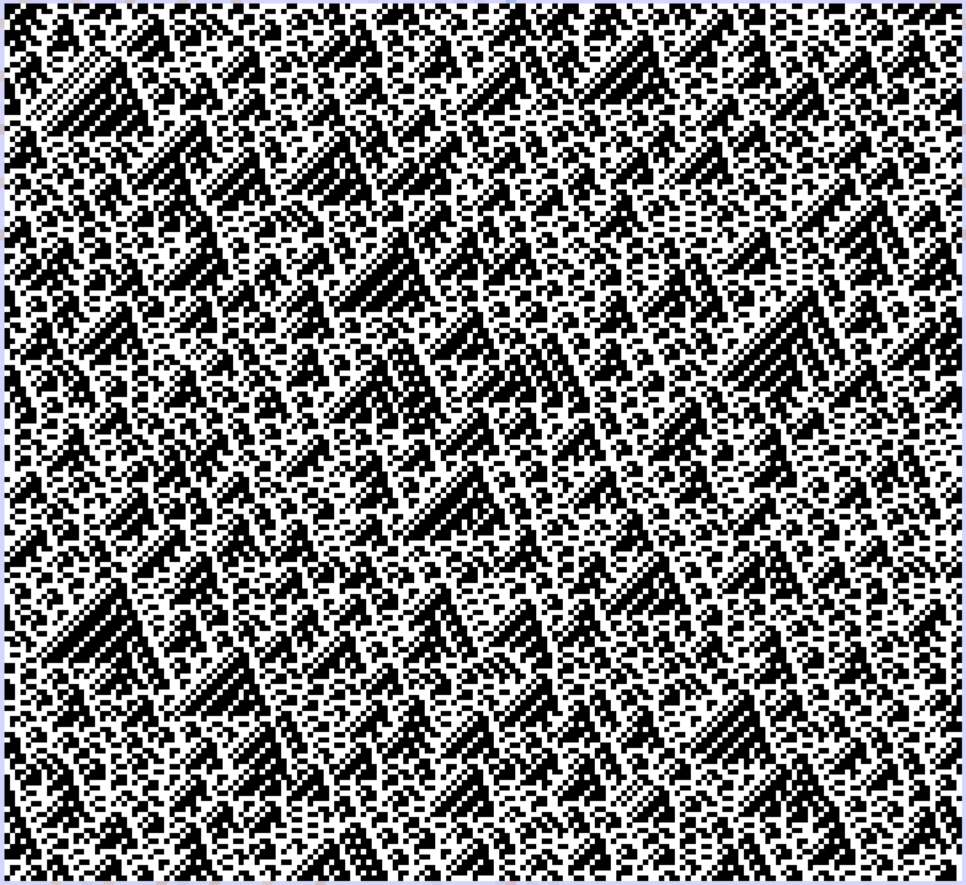

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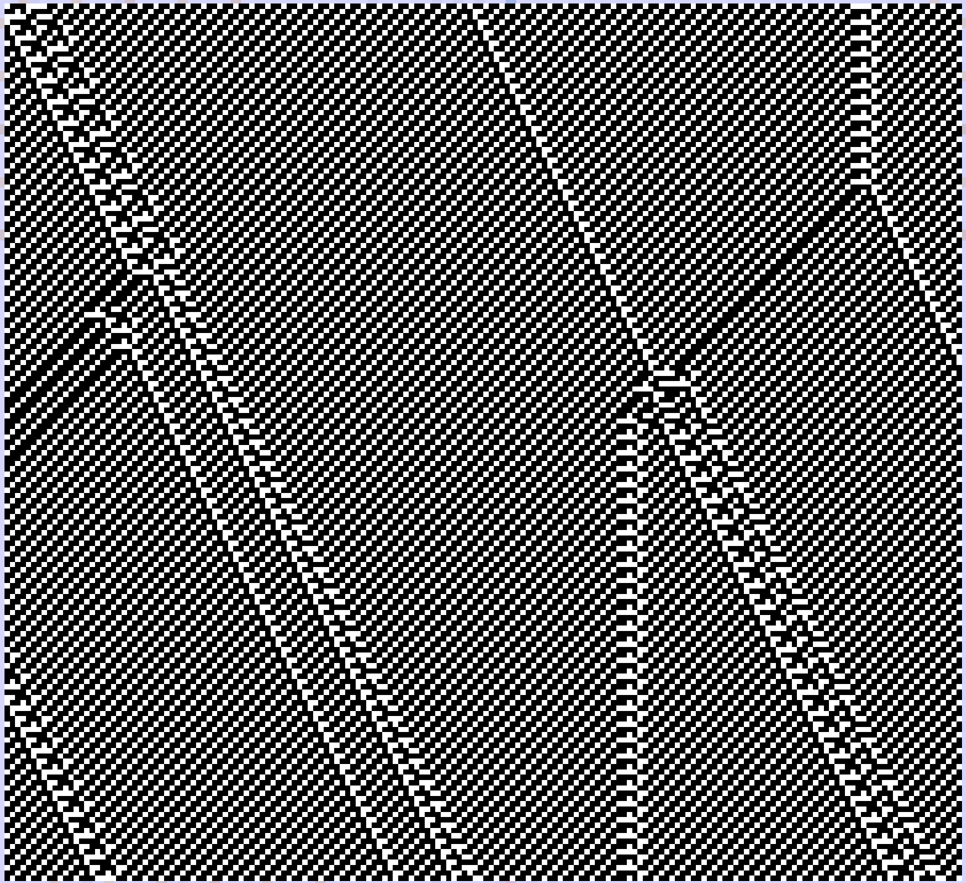

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- **Grouping** relies on an algebraic approach

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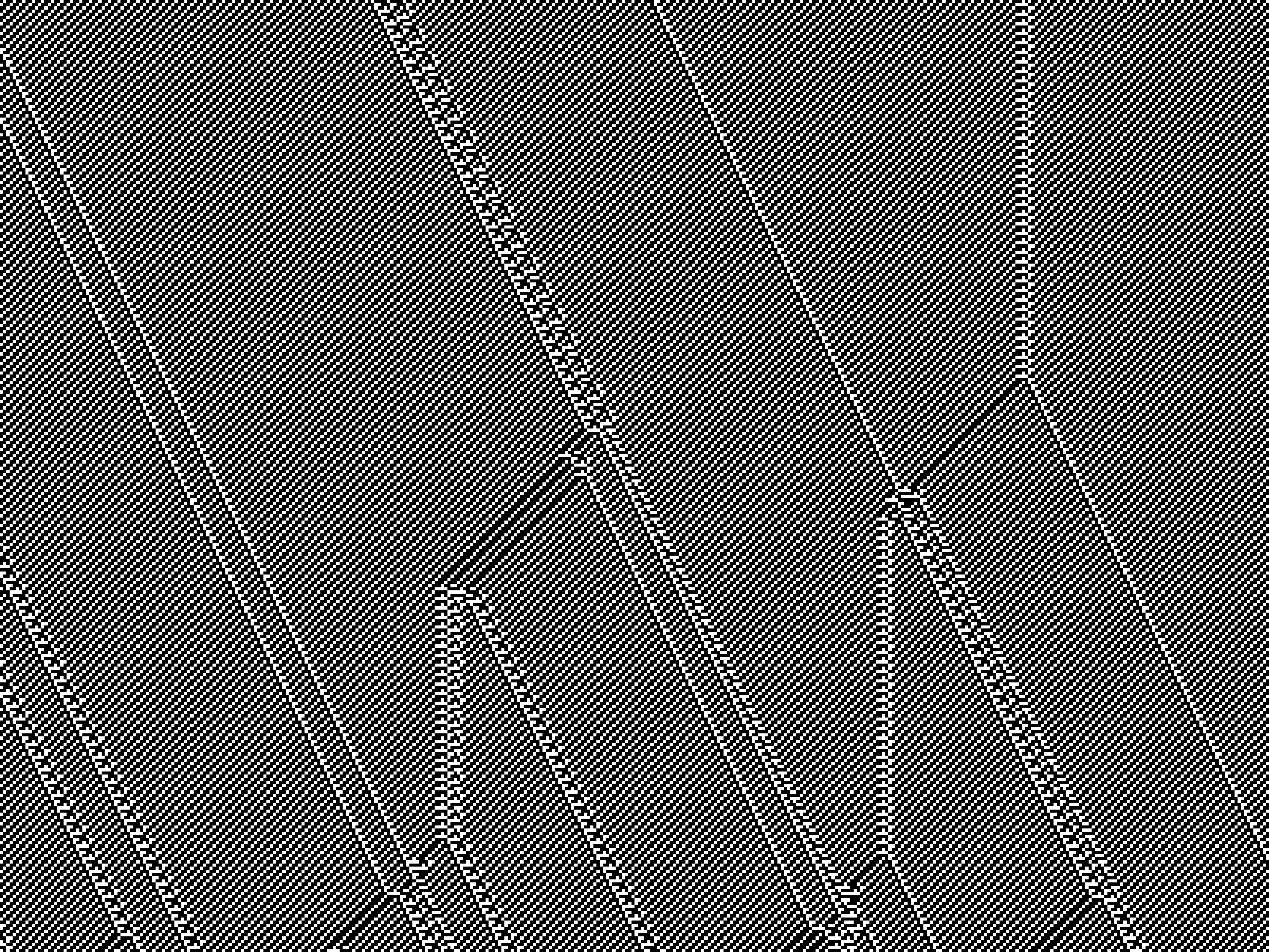
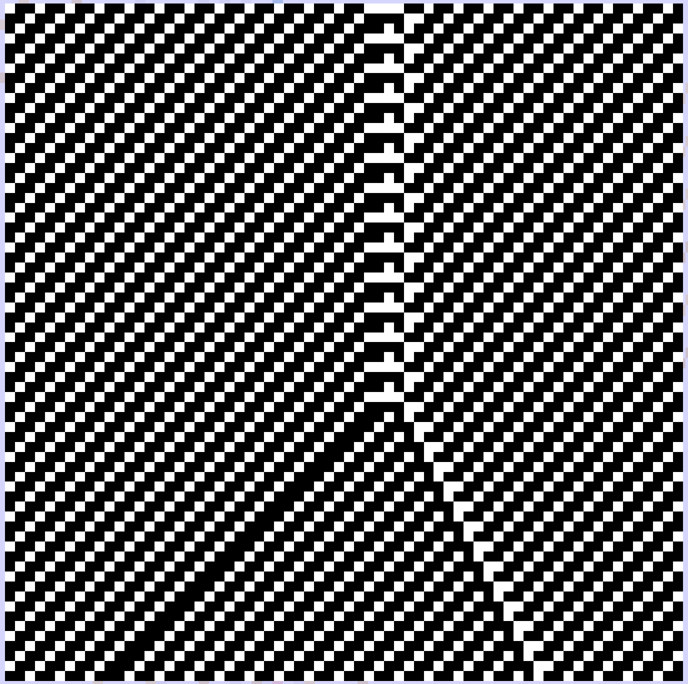
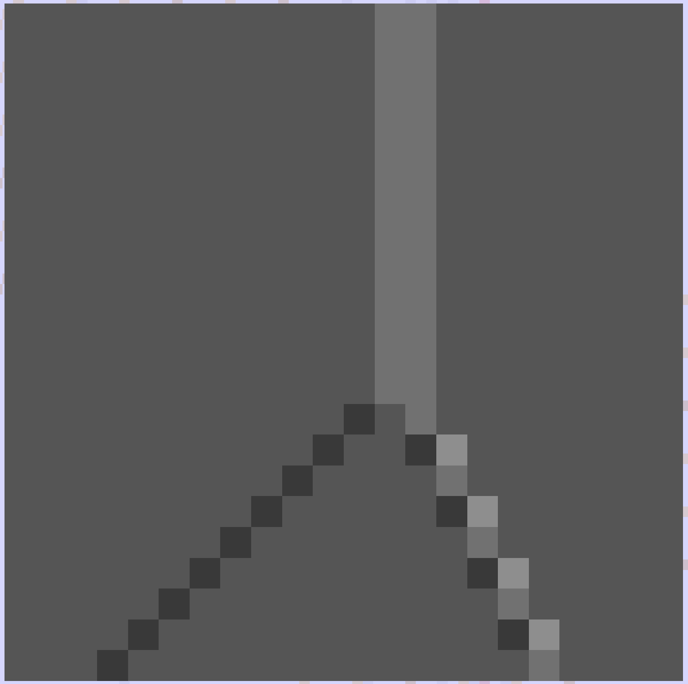
- **Grouping** relies on an algebraic approach

Idea. Define a quasi-order on CA using the subautomaton relation, up to some geometrical transformation of these CA.



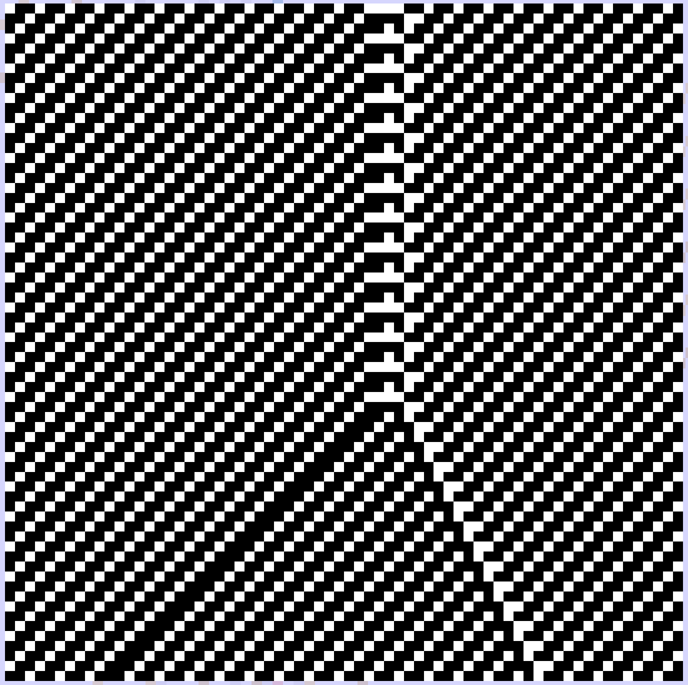
Example - Particles (1)

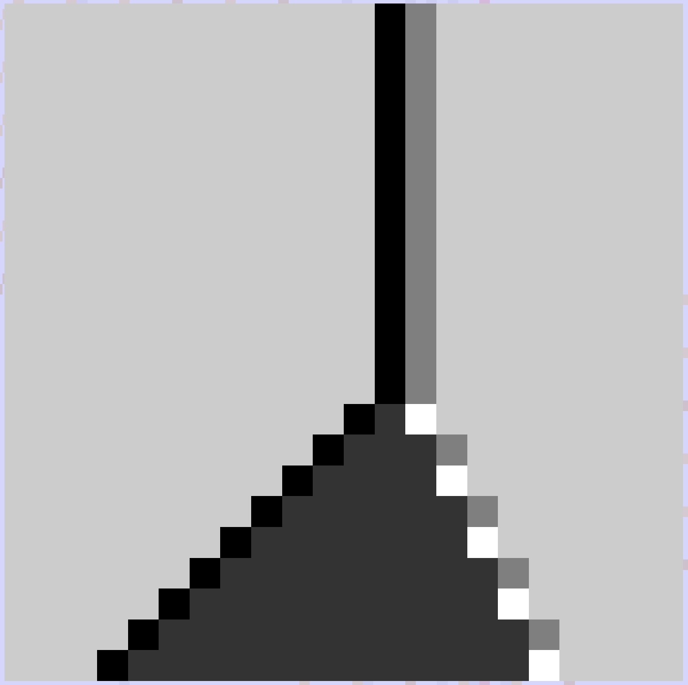
 

How to eliminate the periodic background pattern ?
You can zoom out and use shades of grey...

$$C'_p = 1/9 \sum_{v \in [\![0,2]\!]^2} C_{3p+v}$$

Example - Particles (2)





How to eliminate the periodic background pattern ?
...but also make blocks of bottom cells of the squares

$$C'_p = \left(C_{3p+(0,0)}, C_{3p+(1,0)}, C_{3p+(2,0)}\right)$$

Grouping

We consider 1D CA with neighborhood $[\![-1, 1]\!]$.

- Define the kth power $\mathcal{A}^k$ of a CA $\mathcal{A}$.

Definition. A CA $\mathcal{B}$ simulates a CA $\mathcal{A}$, $\mathcal{A} \leqslant_{\square} \mathcal{B}$, if there exists m and n such that $\mathcal{A}^m \subseteq \mathcal{B}^n$.

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Theorem. The relation $\leqslant_{\square}$ is a quasi-order.

It admits a global minimum, some equivalence classes at the bottom of the order correspond to simple known CA families. It admits no global maximum.

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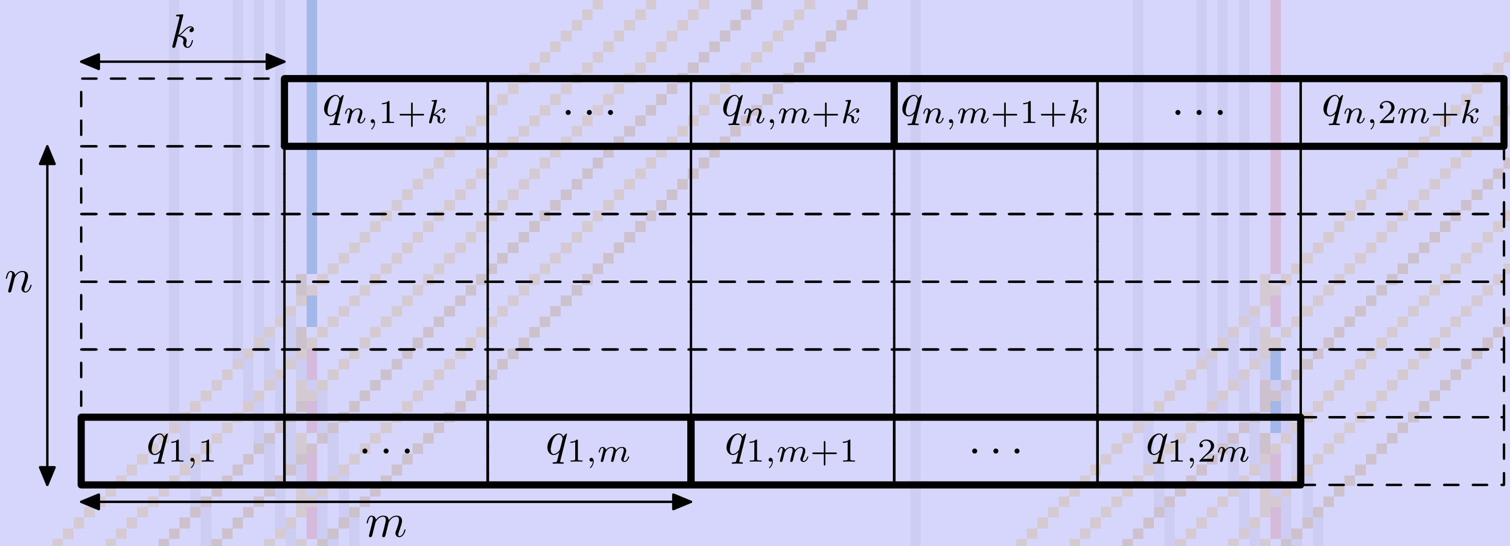
Extension

Claim. The grouping operation doesn't take into account some classical geometrical transformations of the literature, natural in the context of:

- Transformation from CA to OCA,
- Nilpotency,
- Intrinsic Universality.

Classical transform.

- Classical transformations are usually of the type:



$$G_{\mathcal{A}}^{\langle m,n,k\rangle} = o_m^{-1} \circ \sigma_k \circ G_{\mathcal{A}}^{n} \circ o_m$$

Formalization (1)

- A *geometrical transformation* on space-time diagrams **transforms a cellular automaton into a new one** by **combining cells** of a space-time diagram of the first one to construct a space-time diagram of the second one.

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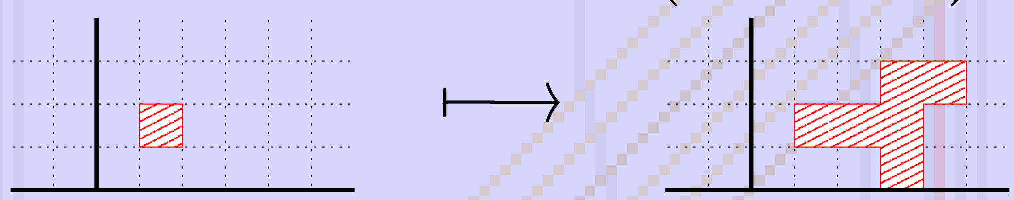
- Formally, it is a pair $(k, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\Lambda} : \quad \mathbb{N} \times \mathbb{Z}^d \longrightarrow \left(\mathbb{N} \times \mathbb{Z}^d\right)^k$$



Formalization (2)

- To apply a transformation $(k, \boldsymbol{\Lambda})$ to a space-time diagram Δ over S, we define $\overline{\boldsymbol{\Lambda}}_S : S^{\mathbb{N}\times\mathbb{Z}^d} \to \left(S^k\right)^{\mathbb{N}\times\mathbb{Z}^d}$ by

$$\overline{\boldsymbol{\Lambda}}_S(\Delta)(t,p) = (\Delta(\boldsymbol{\Lambda}(t,p)_1), \ldots, \Delta(\boldsymbol{\Lambda}(t,p)_k)) \quad .$$

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$$\overline{\Lambda}_S(\Delta)(t,p) = (\Delta(\Lambda(t,p)_1), \ldots, \Delta(\Lambda(t,p)_k)) \quad .$$

- We define an operation rather similar to composition:

$$(k', \Lambda') \circ (k, \Lambda) = (kk', \Lambda' \circ \Lambda)$$

where

$$(\Lambda' \circ \Lambda)(t,p) = \left(\Lambda\left(\Lambda'(t,p)_1\right)_1 \ldots, \Lambda\left(\Lambda'(t,p)_{k'}\right)_k\right)$$

Formalization (3)

- We also introduce $\widetilde{\Lambda}$ as

$$\begin{array}{rcl} \widetilde{\Lambda}: \quad 2^{\mathbb{N}\times\mathbb{Z}^d} & \longrightarrow & 2^{\mathbb{N}\times\mathbb{Z}^d} \\ X & \longmapsto & \bigcup_{(t,p)\in X} \{\Lambda(t,p)_1, \ldots, \Lambda(t,p)_k\} \end{array}$$

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- A *good* geometrical transformation satisfies

1. $\forall \mathcal{A}, \exists \mathcal{B}, \quad \{\overline{\Lambda}_{S_\mathcal{A}}(\Delta)\}_{\Delta\in \mathrm{Diag}(\mathcal{A})} = \mathrm{Diag}(\mathcal{B}) \quad ;$

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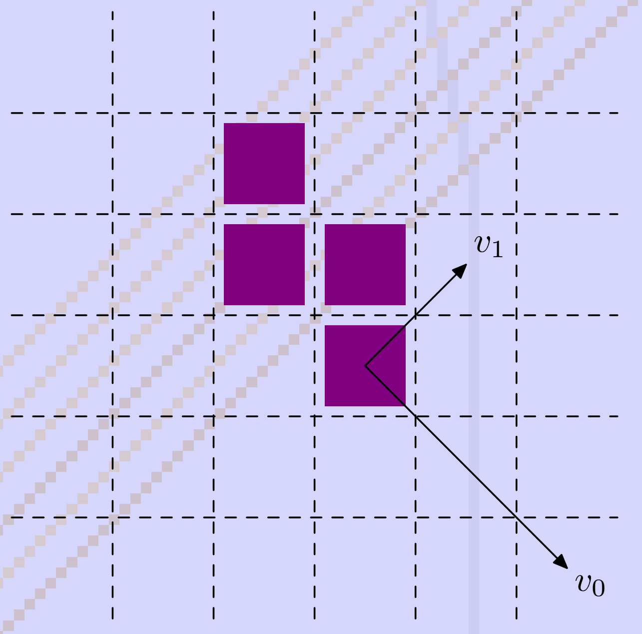
- A *good* geometrical transformation satisfies

1. $$\forall \mathcal{A}, \exists \mathcal{B}, \quad \{\overline{\Lambda}_{S_{\mathcal{A}}}(\Delta)\}_{\Delta\in \mathrm{Diag}(\mathcal{A})} = \mathrm{Diag}(\mathcal{B}) \quad ;$$

2. $$\forall t \in \mathbb{N}, \widetilde{\Lambda}\left(\{t+1\}\times\mathbb{Z}^d\right) \not\subseteq \widetilde{\Lambda}\left(\{t\}\times\mathbb{Z}^d\right) \quad .$$

Packing

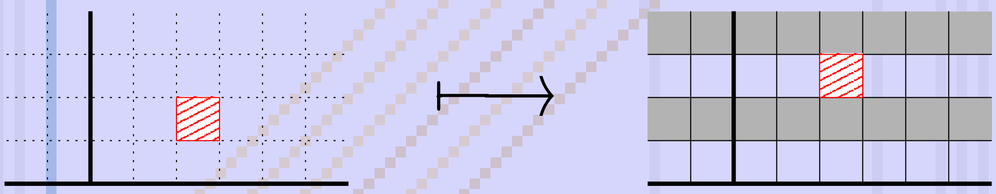


$$\mathbf{P}_{F,\nu}(t,p) = t \circledast (F \oplus (p \odot \nu))$$

Transformed CA global rule:

$$o_{F,\nu}^{-1} \circ G \circ o_{F,\nu}$$

Cutting

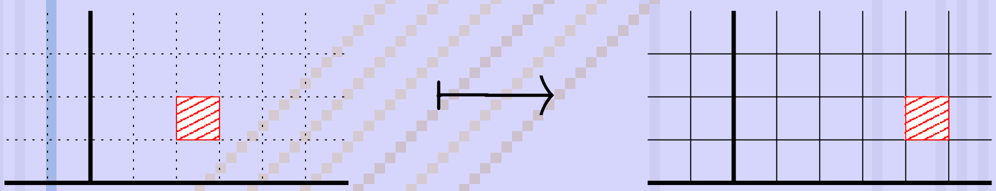


$$\mathbf{C}_T\,(t,p) = (tT, p)$$

Transformed CA global rule:

$$G^T$$

Shifting



$$\mathbf{S}_s\,(t,p) = (t, p \oplus ts)$$

Transformed CA global rule:

$$\sigma_s \circ G$$

Composition

We define **PCS** transformations as

$$\mathbf{PCS}_{F,v,T,s} = \mathbf{P}_{F,v} \circ \mathbf{S}_s \circ \mathbf{C}_T$$

$$\mathbf{PCS}_{F,v,T,s}(t,p) = tT \circledast (F \oplus (p \odot v \oplus ts))$$

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Transformed CA global rule:

$$o_{F,\nu}^{-1} \circ \sigma_s \circ G^T \circ o_{F,\nu}$$

PCS transformations are closed under composition.

Characterization

Theorem. A geometrical transformation is a good geometrical transformation if and only if it can be expressed as a **PCS** transformation.

The proof highly relies on the uniformity of cellular automata and the construction of counter-examples.

Table of Content

Abstract Bulking

- We don't want to reproof that we have a quasi-order for each kind of grouping we introduce.
- Some properties are generic and do not rely on painful computation at the level of geometrical transformations but come from more abstract properties.
- We introduce a logical theory to uniformize the work with grouping.

Definition

Definition. An *abstract bulking* $\mathfrak{A}$ is a logical theory on the signature

$$\begin{aligned}\big(\mathsf{Obj}, \mathsf{Trans};\ &\mathbf{apply} : \mathsf{Obj} \times \mathsf{Trans} \to \mathsf{Obj},\\ &\mathbf{divide} \subseteq \mathsf{Obj} \times \mathsf{Obj},\\ &\mathbf{combine} : \mathsf{Trans} \times \mathsf{Trans} \to \mathsf{Trans}\big).\end{aligned}$$

Notation. An object y simulates an object x if they satisfy the formula

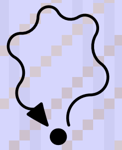
$$x \preccurlyeq y \quad \equiv \quad \exists\alpha\exists\beta\,\left(x^{\alpha} \mid y^{\beta}\right)$$

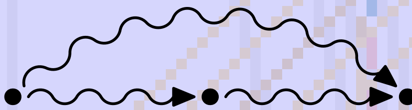
Axioms (1)

Combination. $(\mathsf{Trans}, \cdot)$ is a monoid.

$$\begin{aligned} \mathfrak{A} \vdash \quad & \exists 1 \forall \alpha \, (\alpha \cdot 1 = \alpha \wedge 1 \cdot \alpha = \alpha) \\ \wedge \quad & \forall \alpha \forall \beta \forall \gamma \, ((\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)) \end{aligned}$$

Compatibility. $(\mathsf{Trans}, \cdot)$ acts on Obj through **apply**.

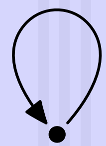


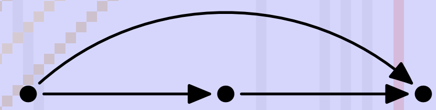


$$\mathfrak{A} \vdash \quad \forall x \, (x^1 = x) \quad \wedge \quad \forall x \forall \alpha \forall \beta \left((x^\alpha)^\beta = x^{\alpha \cdot \beta} \right)$$

Axioms (2)

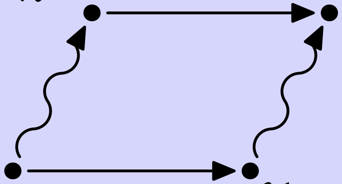
Divisibility. **divide** is a quasi-order on Obj.





$$\mathfrak{A} \vdash \quad \forall x\,(x \mid x) \quad \wedge \quad \forall x \forall y \forall z\,((x \mid y \wedge y \mid z) \to x \mid z)$$

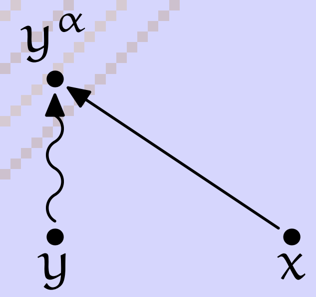
Transitivity. **apply** is compatible with **divide**.



$$\mathfrak{A} \vdash \quad \forall x \forall y \forall \alpha\,(x \mid y \to x^\alpha \mid y^\alpha)$$

Axioms (3)

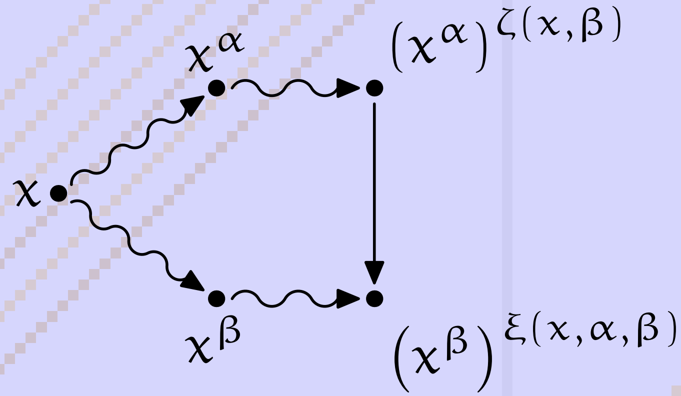
Surjectivity. **apply** preserve the richness of objects.



$$\mathfrak{A} \vdash \quad \forall\alpha\forall x\exists y\,(x \mid y^\alpha)$$

Axioms (4)

Proximity. **apply** keeps objects nearby. There exists two functions ζ and ξ such that



$$\mathfrak{A} \vdash \quad \forall x \forall \alpha \forall \beta \left((x^\alpha)^{\zeta(x,\beta)} \mid (x^\beta)^{\xi(x,\alpha,\beta)} \right)$$

Properties

Theorem. "$\preccurlyeq$ is a quasi-order" is a bulking property.

$$\mathfrak{A} \vdash \quad \forall x\,(x \preccurlyeq x) \quad \wedge \quad \forall x \forall y \forall z\,((x \preccurlyeq y \wedge y \preccurlyeq z) \rightarrow x \preccurlyeq z)$$

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- $\mathtt{u}$ is *universal* if $\forall \mathtt{x}\,(\mathtt{x} \preccurlyeq \mathtt{u})$.
- $\mathtt{u}$ is *strongly universal* if $\forall \mathtt{x} \exists \alpha\,(\mathtt{x} \mid \mathtt{u}^{\alpha})$.

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- u is *universal* if $\forall x\,(x \preccurlyeq u)$.
- u is *strongly universal* if $\forall x \exists \alpha\,(x \mid u^{\alpha})$.

Theorem. "If there exists a strongly universal objet then each universal object is strongly universal" is a bulking property.

Table of Content

First try

Idea. Use abstract bulking theory with:

Obj	the set of d-CA,
Trans	the set of **PCS** transformations,
apply	the transformation operator,
divide	the subautomaton relation,
combine	the composition of transformations.

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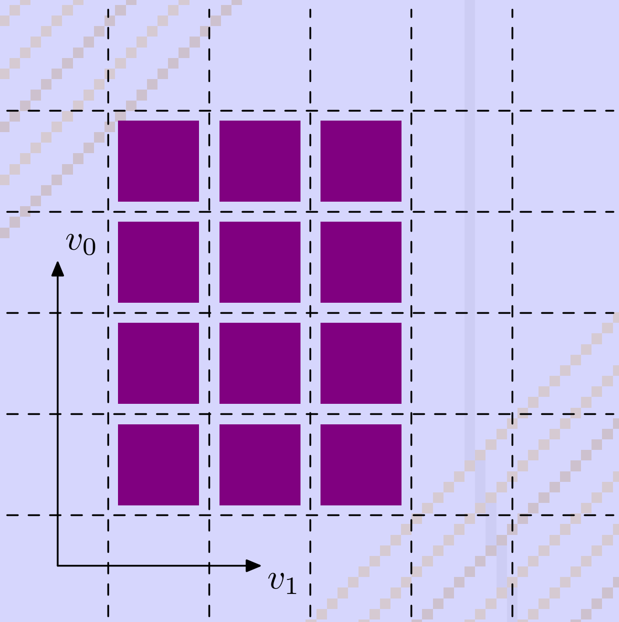
Argh! The **Proximity** axiom is **not** satisfied.

Regular Packing

$\tilde{\mathbf{P}}$: restriction on **P** transformations.

$$\tilde{\mathbf{P}}_{(m_1,\ldots,m_d),\tau} = \mathbf{P}_{\prod_{i=1}^{d}[\![0,m_i-1]\!],\left(\sigma_{\tau(1)},\ldots,\sigma_{\tau(d)}\right)\otimes m}$$



Second try

Idea. Use abstract bulking theory with:

Obj	the set of $\mathfrak{d}$-CA,
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(**P̃CS'** and **P̃CS** define the same relation of simulation)

It works: All the axioms are satisfied.

Summary

Applying a $\tilde{\textbf{P}}$**CS** transformation $\langle m_\tau, n, k\rangle$ to a CA $\mathcal{A}$:

$$G_{\mathcal{A}}^{\langle m_\tau, n, k\rangle} = o_{m_\tau}^{-1} \circ \sigma_k \circ G_{\mathcal{A}}^{n} \circ o_{m_\tau}$$

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$$\mathcal{A}^{\langle m_\tau, n, k\rangle} \subseteq \mathcal{B}^{\langle m'_\tau, n', k'\rangle}$$

Theorem. The relation $\leqslant$ is induced by an abstract bulking model.

Basic properties

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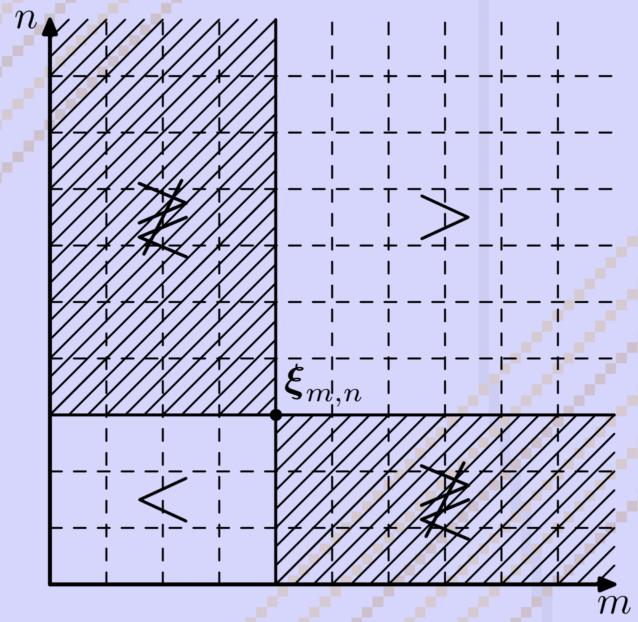
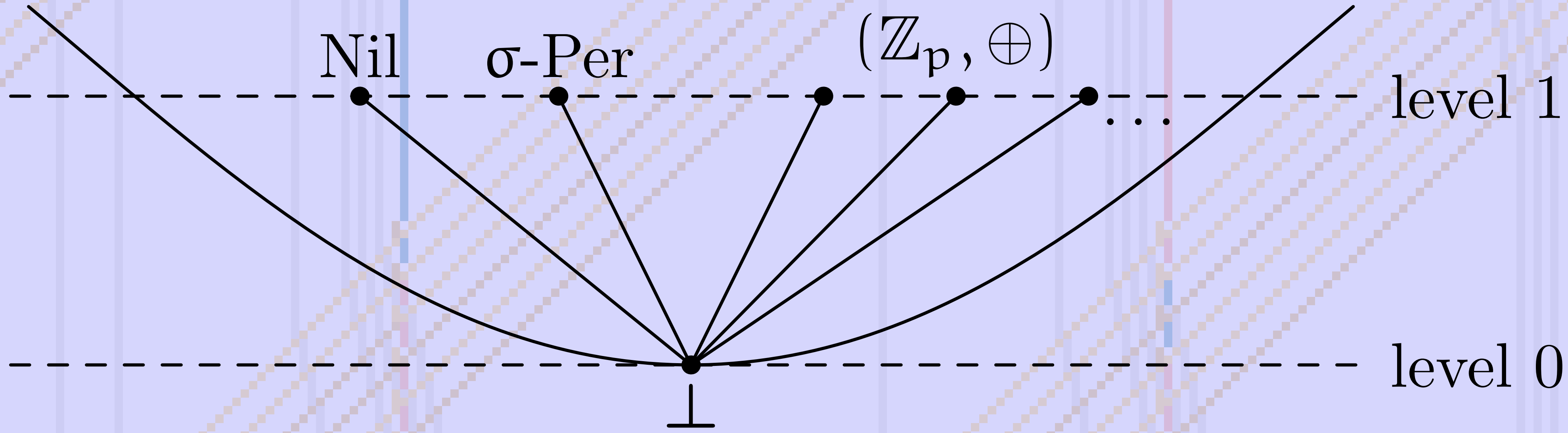
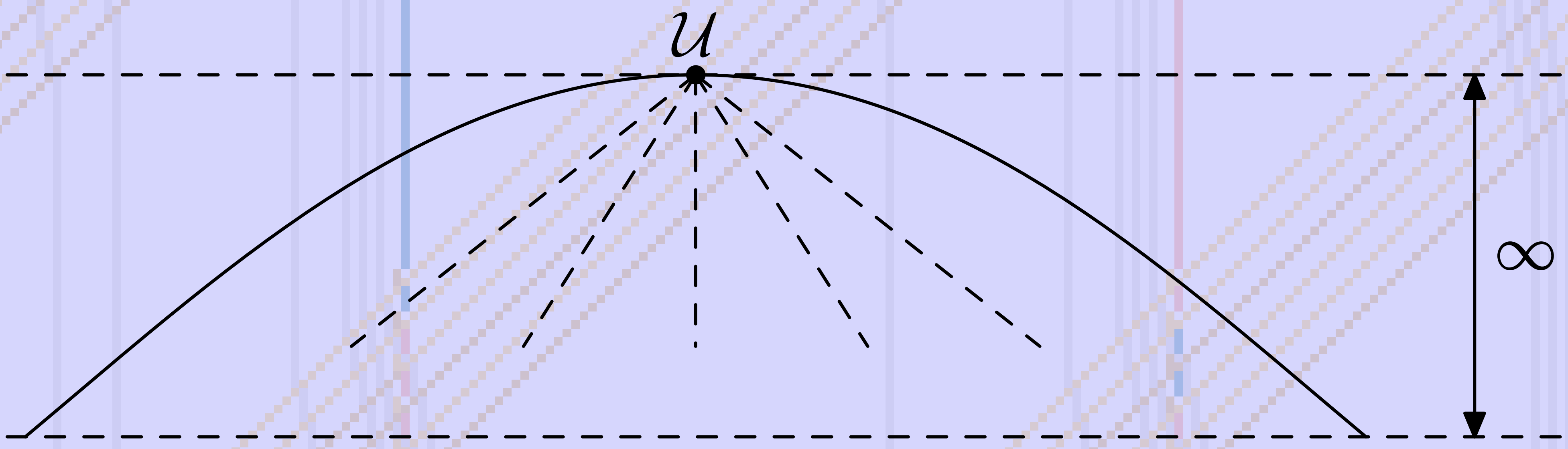
- In dimension 1, the relation $\leqslant_{\square}$ refines $\leqslant$.
- $\times$ corresponds to a local maximum.
- We still have infinite chains.



Bottom of the order



Top of the order



There is no quasi-universal CA.

Universality

Theorem. Given a CA, deciding whether it is intrinsically universal is undecidable.

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Universality

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Theorem. There exists no real-time intrinsically universal CA ($\forall \mathcal{A}, \exists n, \mathcal{A} \subseteq \mathcal{U}^{\langle n,n,0 \rangle}$).

- We can construct very small intrinsically universal CA (ex. 1D, von Neumann neighborhood, 6 states)

Structure

- The structure of products of shifts, $\prod_{i=1}^{k} \sigma_{v_i}$, and CA they simulate can be completely described.

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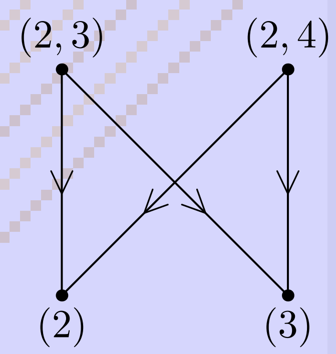

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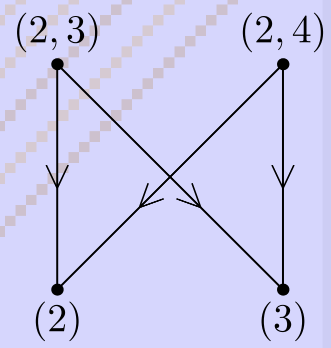


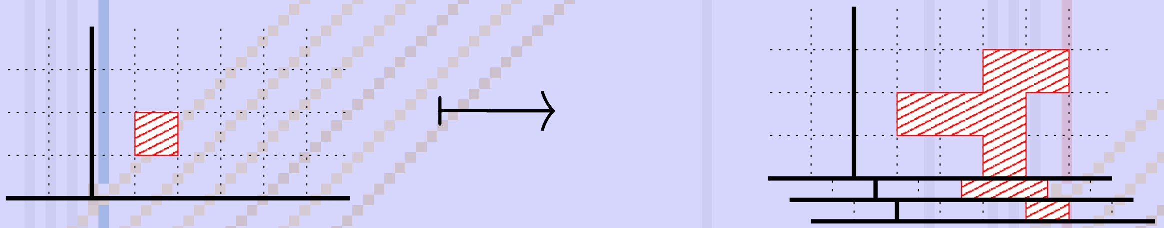
The relation $\leqslant$ induces no semi-lattice structure.

Idea. Modify bulking so that $\times$ defines a supremum.

A new bulking (1)

- New transformations: (k, l, Λ) where

$$\Lambda : \quad \mathbb{N} \times \mathbb{Z}^d \longrightarrow \left([\![1, l]\!] \times \mathbb{N} \times \mathbb{Z}^d \right)^k$$



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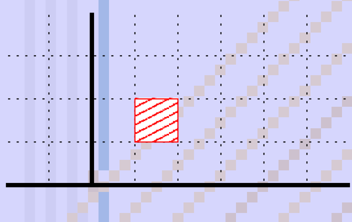
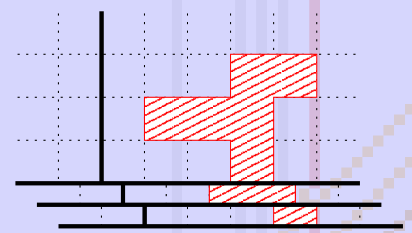


$\longmapsto$



- **PCST**$_{(F_i, \nu_i, T_i, s_i)_{i \in [\![1,l]\!]}}$ transforms $\mathcal{A}$ into

$$\left(o_{F_1,\nu_1}^{-1} \circ \sigma_{s_1} \circ G_{\mathcal{A}}^{T_1} \circ o_{F_1,\nu_1}\right) \times \cdots \times \left(o_{F_l,\nu_l}^{-1} \circ \sigma_{s_l} \circ G_{\mathcal{A}}^{T_l} \circ o_{F_l,\nu_l}\right)$$

A new bulking (2)

Idea. Use abstract bulking theory with:

Obj	the set of $\mathbb{d}$-CA,
Trans	the set of **$\tilde{\text{P}}$CST'** transformations,
apply	the transformation operator,
divide	the subautomaton relation,
combine	the composition of transformations.

A new bulking (2)

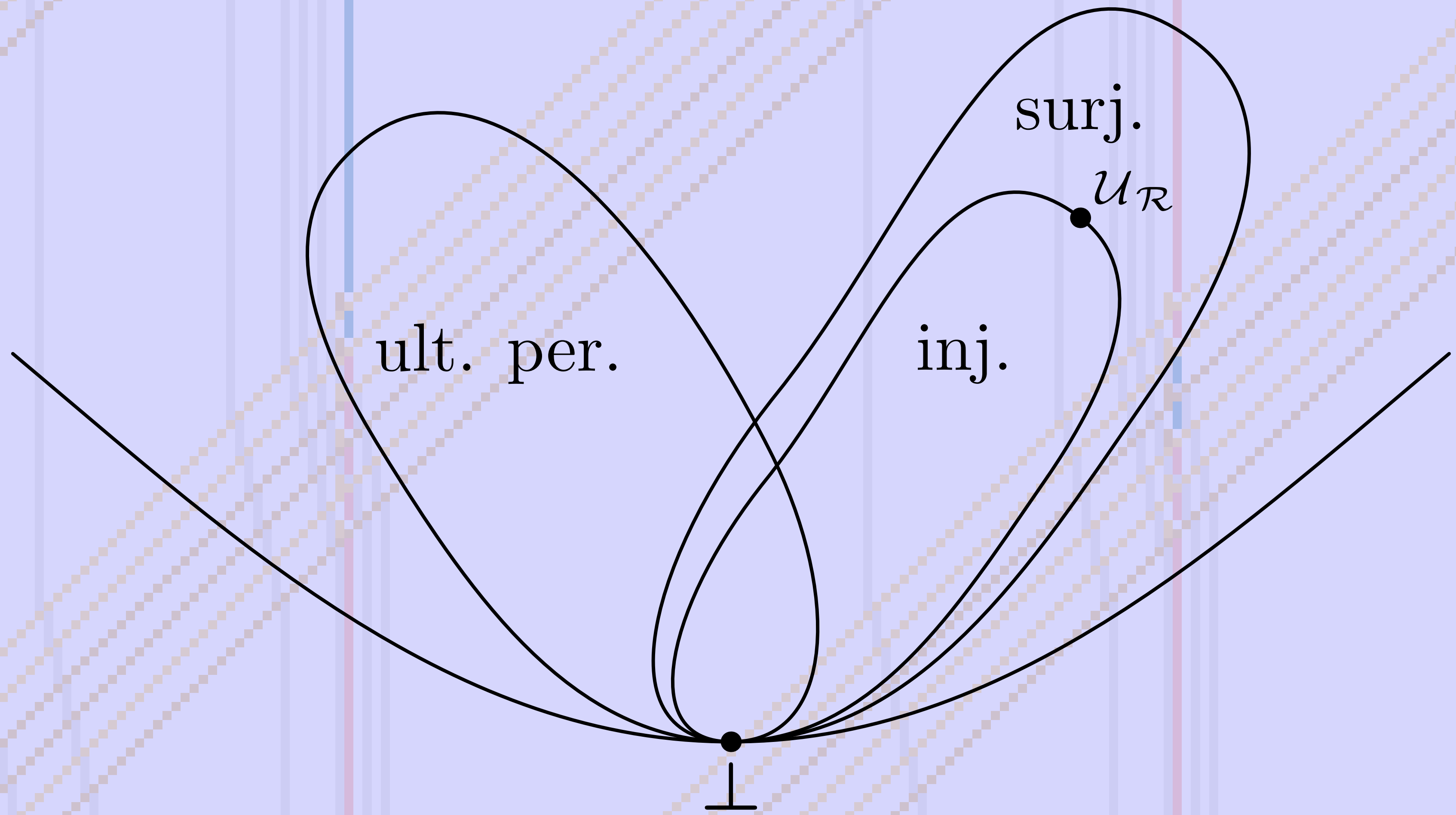
Idea. Use abstract bulking theory with:

Obj	the set of d-CA,
Trans	the set of $\tilde{\mathbf{P}}$**CST'** transformations,
apply	the transformation operator,
divide	the subautomaton relation,
combine	the composition of transformations.

- $\tilde{\mathbf{P}}$**CST** transformations are defined like $\tilde{\mathbf{P}}$**CS** ones.
- All the axioms are satisfied.
- The relation of simulation induces a sup-semi-lattice with $\times$ as a supremum operator.

Ideals

- An ideal is a set of equivalence classes stable by $\times$ and lower element by $\leqslant$.



Perspectives

- CA at the bottom and the top of the order seem to correspond to CA which are easy to describe. What about CA in “the middle” ?

- Links between stuctural properties of bulking and decidability questions have been presented. What about topological properties ?

- Study abstract bulking in the case of a different kind of dynamical system, refine the choice of axioms, generals properties.