The Intrinsic Universality Problem of 1D CA

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STACS 2003 / Berlin

Cellular Automata

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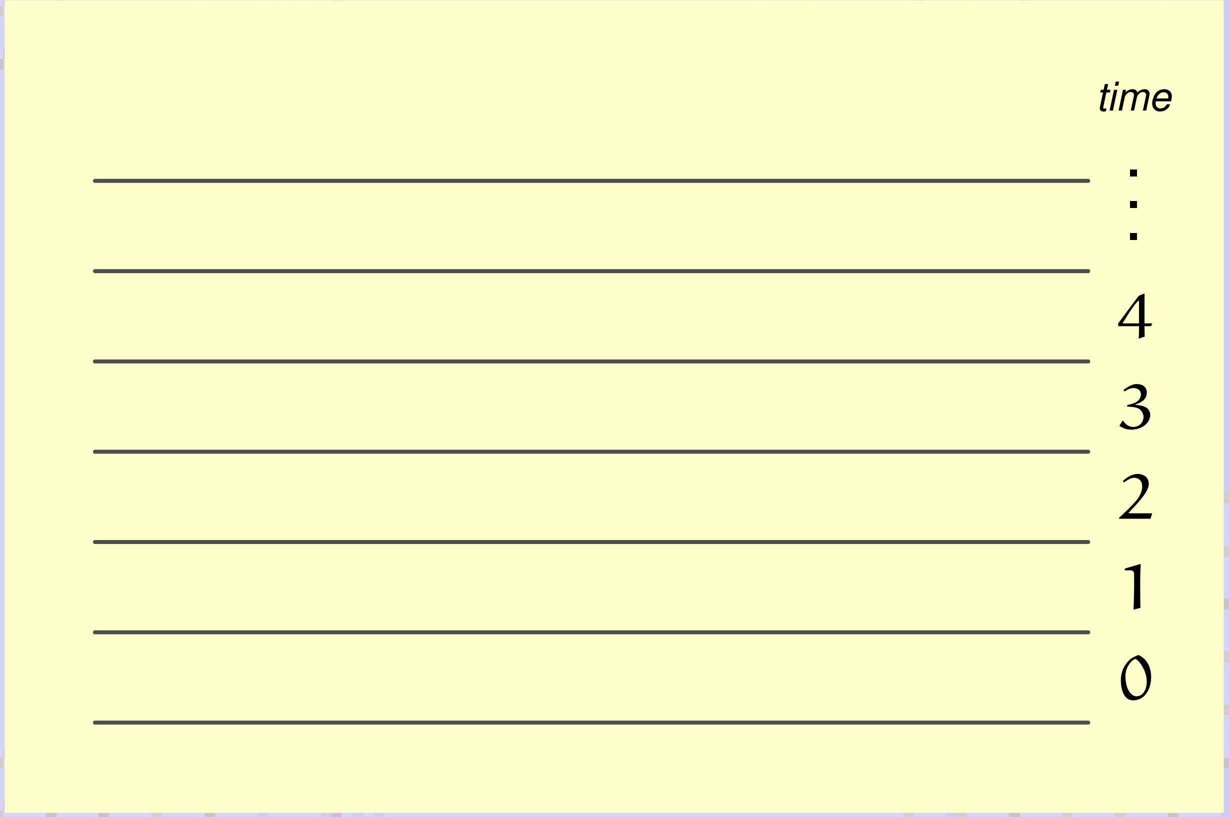

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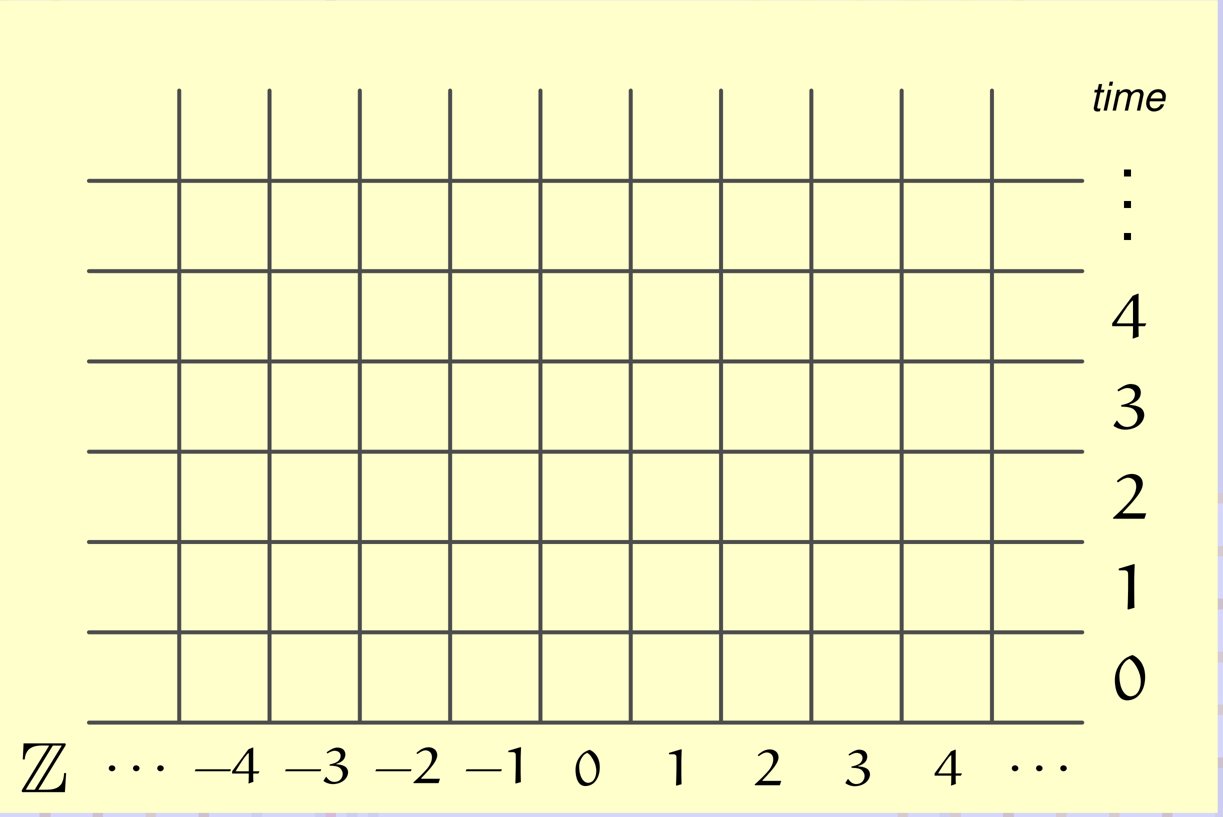

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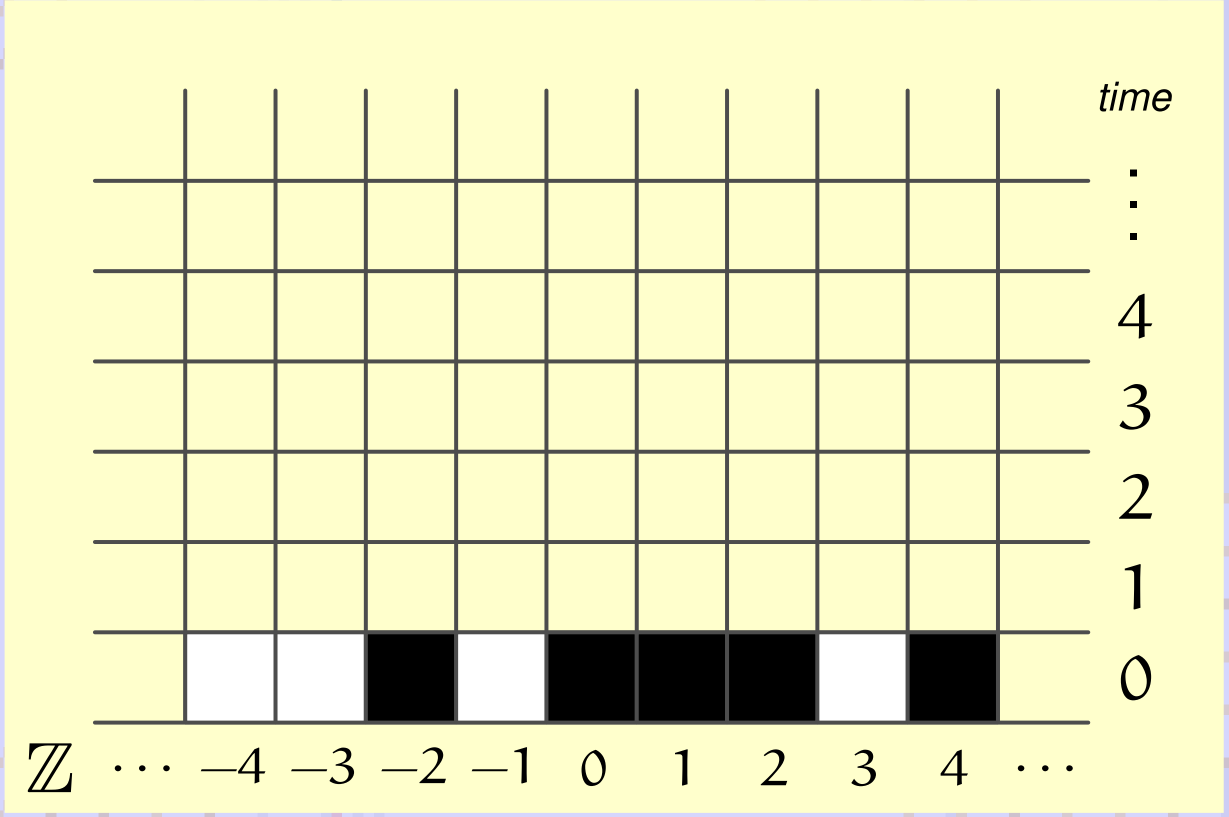

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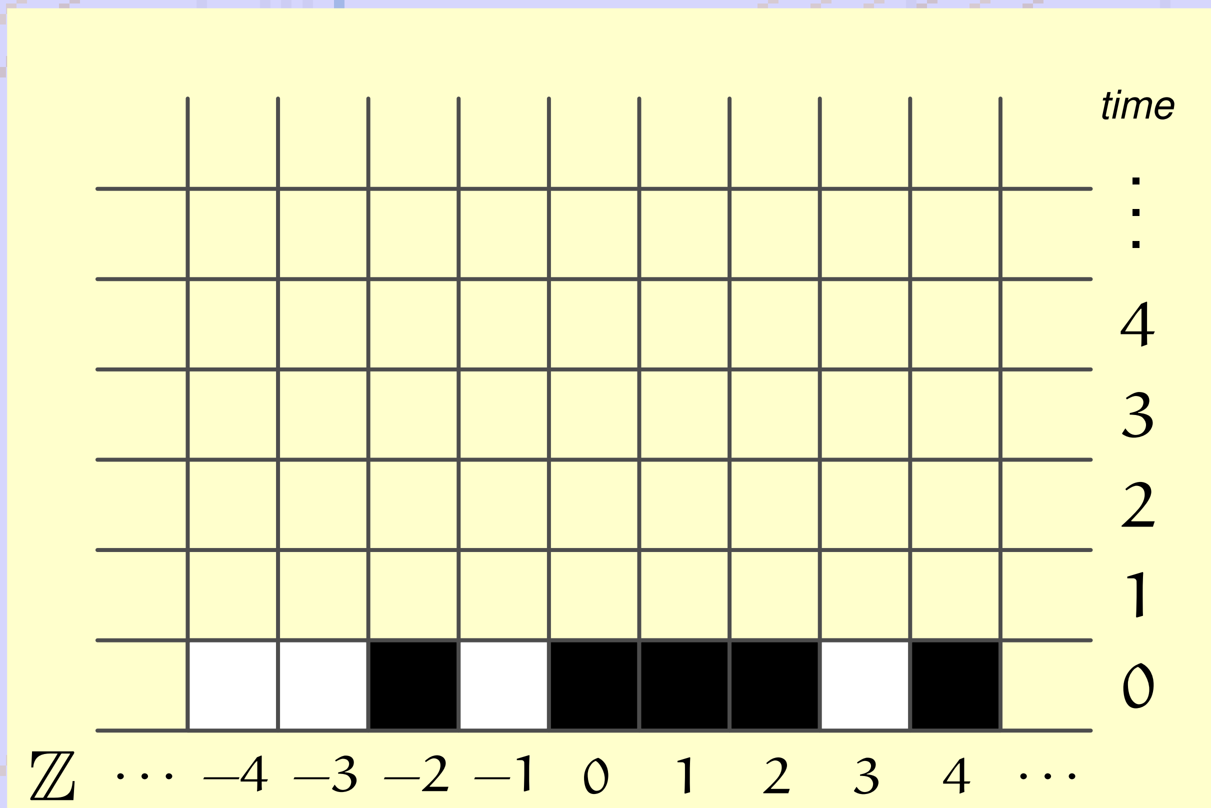

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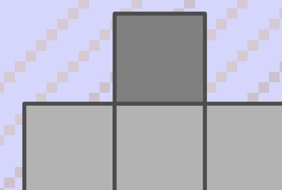

$\mathcal{N} \subseteq_{\text{finite}} \mathbb{Z}$

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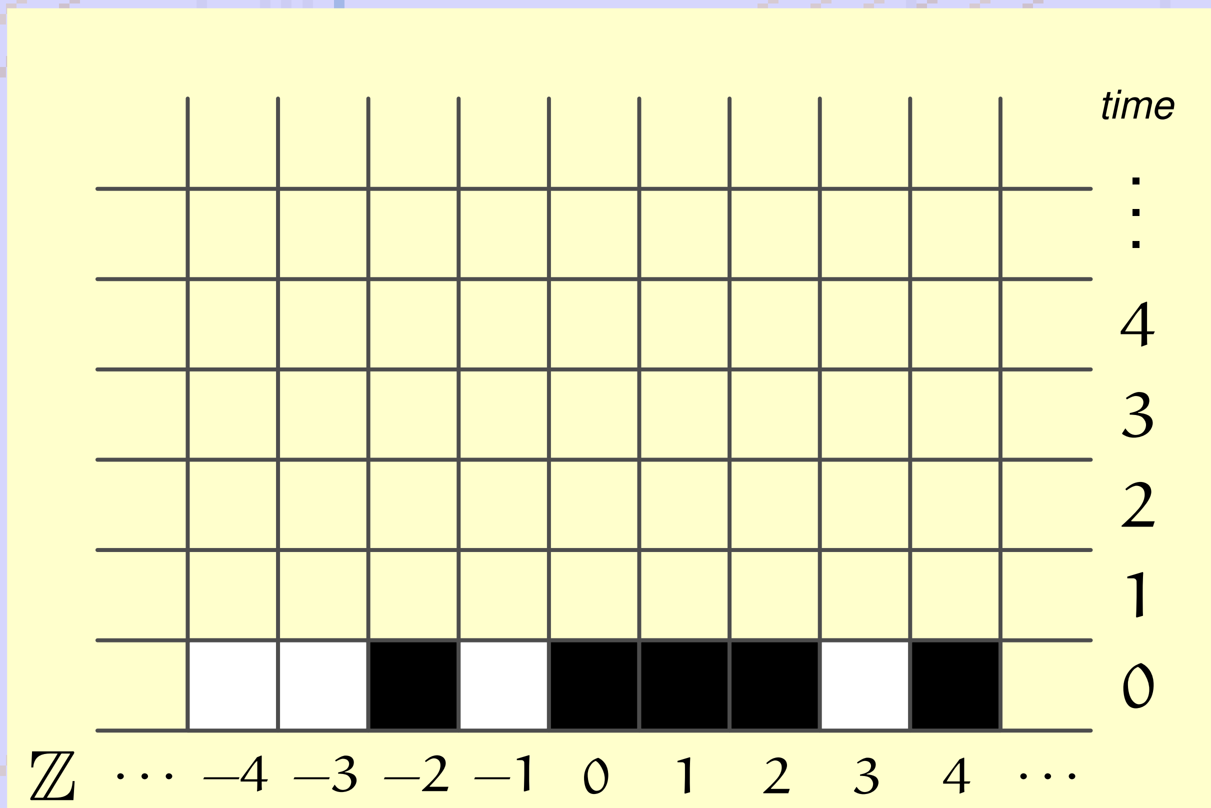

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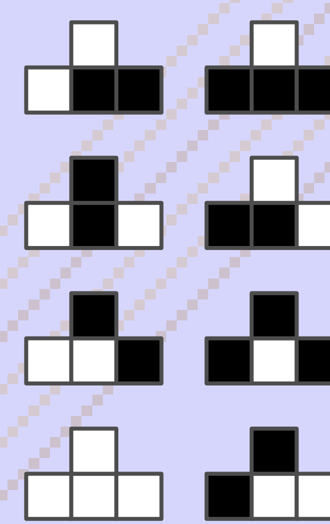

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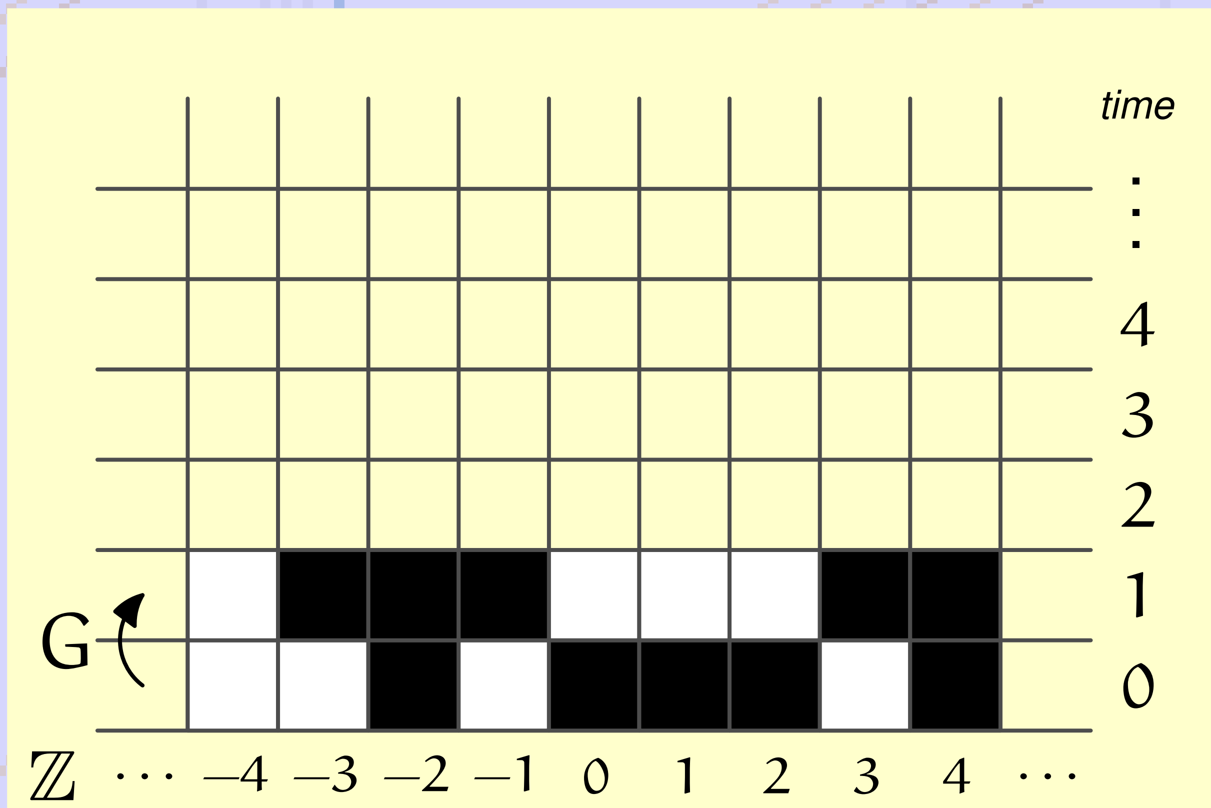

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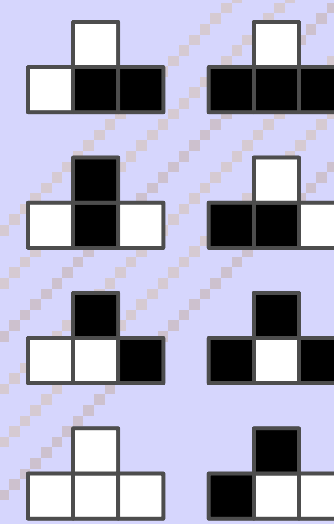

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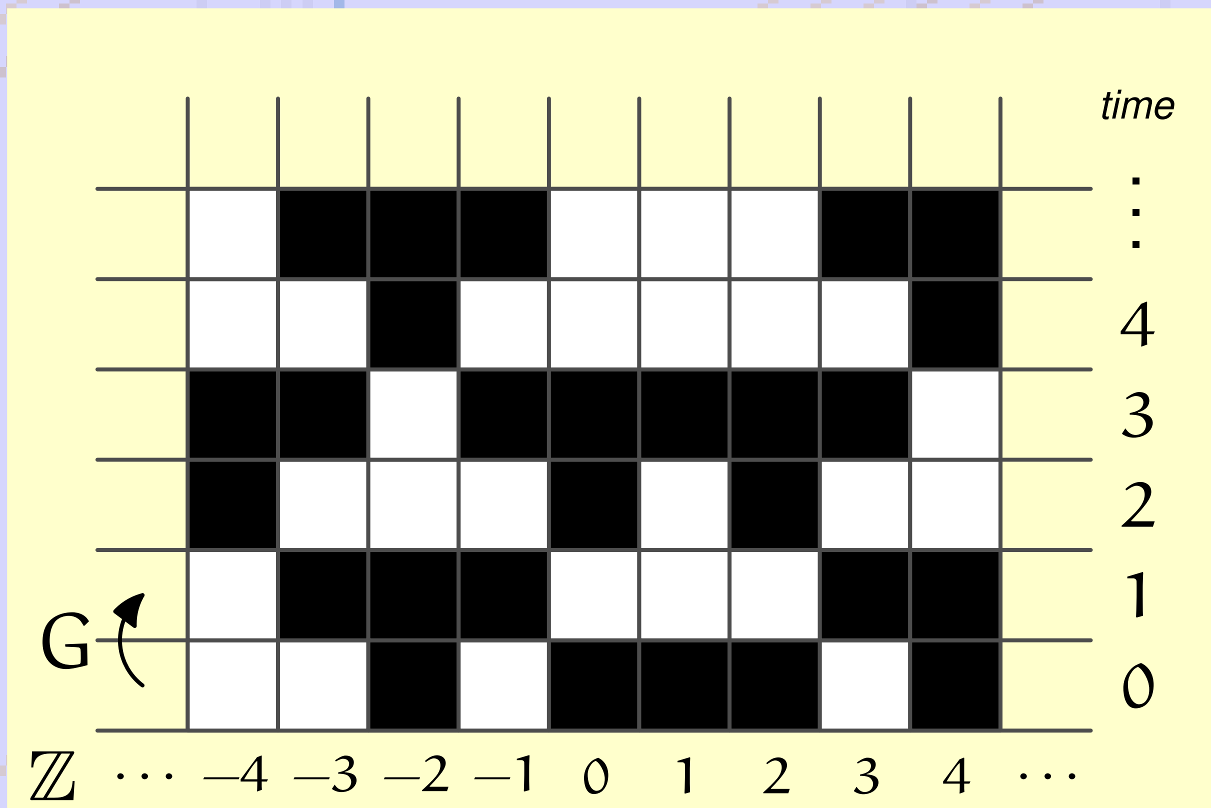

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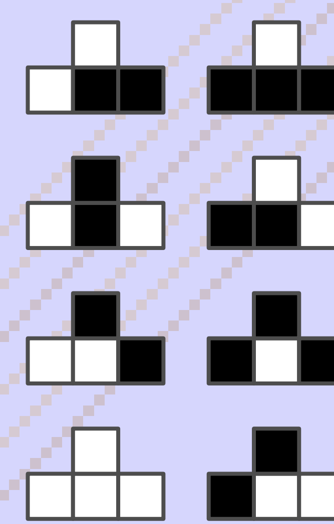

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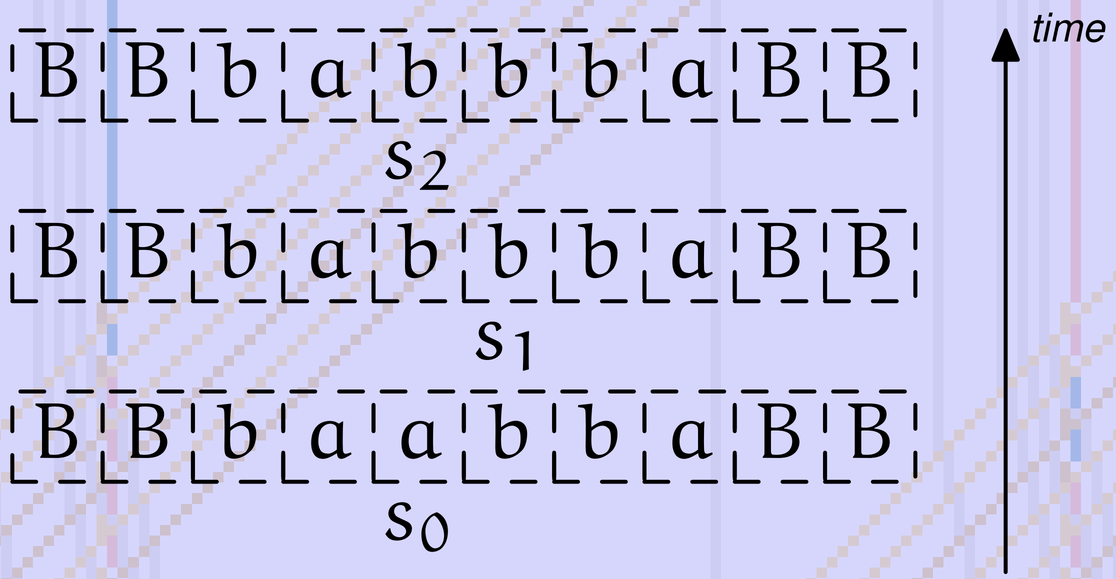
Computation Universality

Idea. Some CA can **compute** every recursive function, for example by **simulating** a universal TM.



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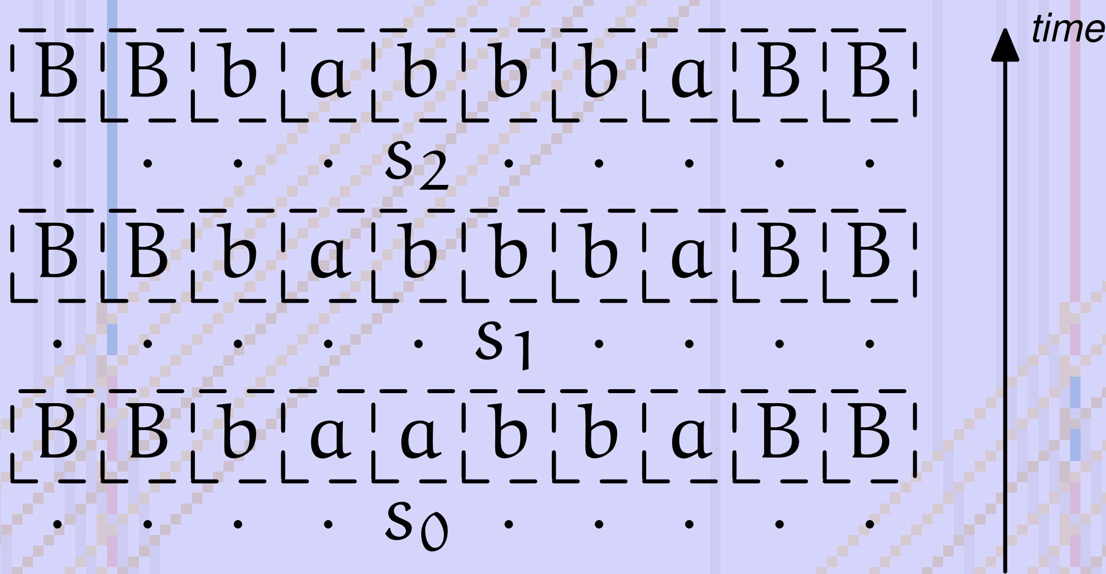

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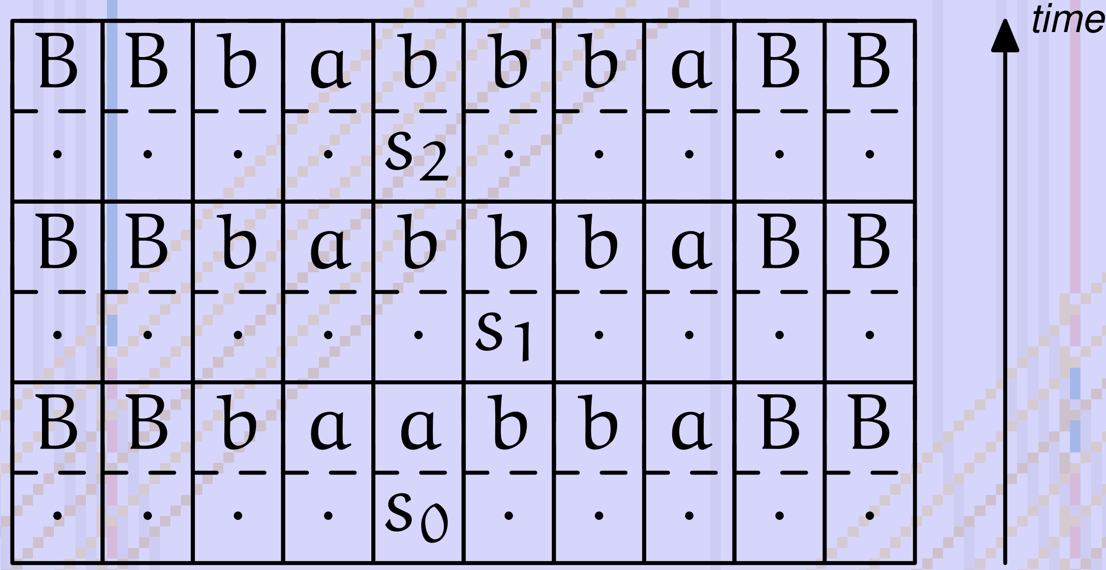

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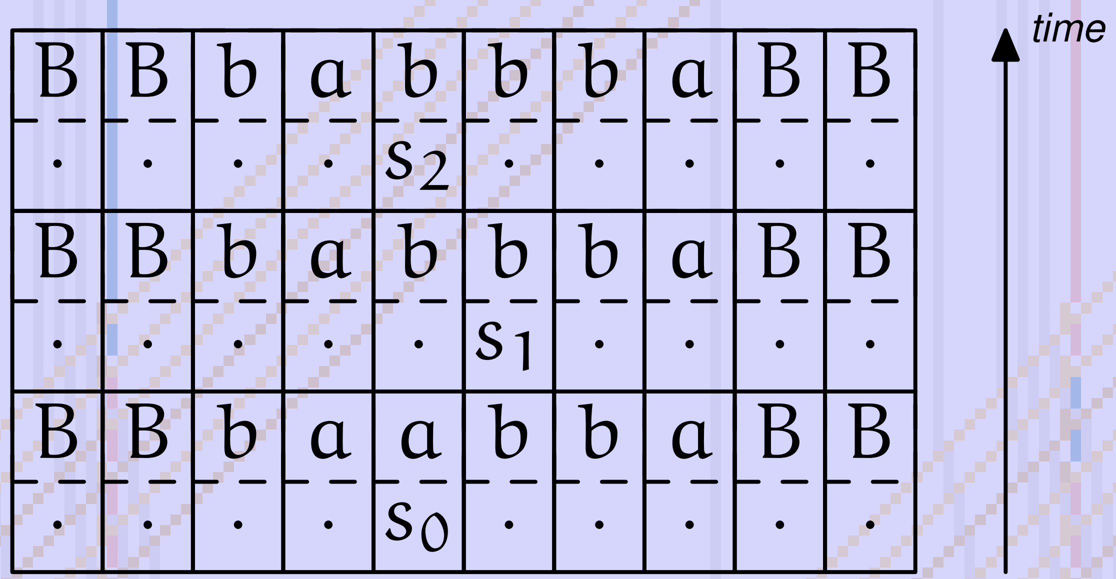
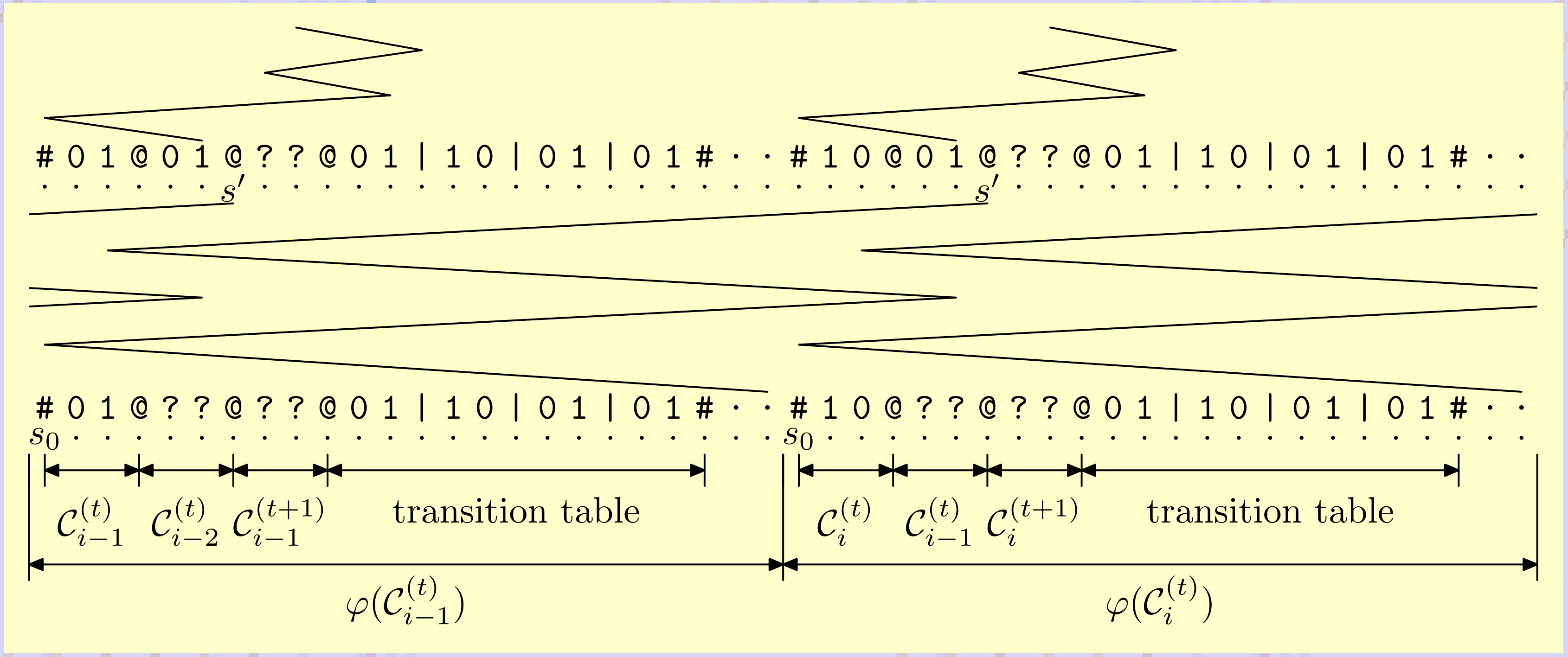


Problem. This is difficult to define formally and it involves extrinsic notions of simulation. . .

Intrinsic Universality

Idea. Some CA can **simulate** every possible CA.



Deciding Universality

- As computation universality for TM is certainly a property of the computed function, undecidability follows from Rice's theorem.

> **Theorem**[Rice] No non-trivial property on the computed function of TM is recursive.

- What about intrinsic universality for CA?

Undecidability and CA

One time step properties

(2D+) Injectivity of the global rule [Kari 94];

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  - *(1D)* Nilpotency [Kari 92];
  - *(1D)* Kind of “Rice’s theorem” for limit sets [Kari 94];
  - *(1D)* Nilpotency on periodic [Mazoyer and Rapaport 99];

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CA-1D-NIL-PER	
Input	A CA $\mathcal{A}$ and a state s of $\mathcal{A}$
Question	Is $\mathcal{A}$ s-nilpotent on periodic ?

Inducing an Order on CA (1)

Idea. A CA $\mathcal{A}$ is **less complex** than a CA $\mathcal{B}$ if, up to some renaming of states and some rescaling, every space-time diagram of $\mathcal{A}$ is a space-time diagram of $\mathcal{B}$.

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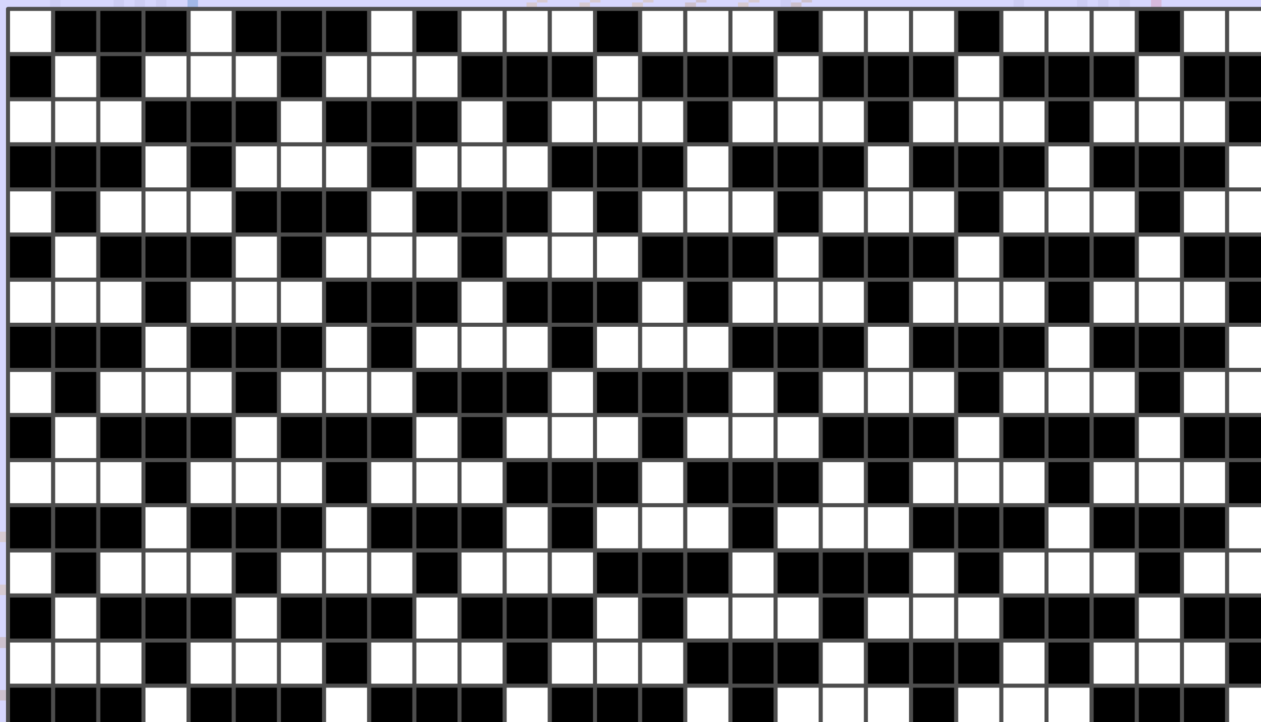
Definition. $\mathcal{A} \subseteq \mathcal{B}$ if there exists an injective mapping φ from $S_{\mathcal{A}}$ into $S_{\mathcal{B}}$ such that this diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\ \varphi\ } & \overline{\varphi}(C) \\ {\scriptstyle G_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle G_{\mathcal{B}}} \\ G_{\mathcal{A}}(C) & \xrightarrow[\ \varphi\ ]{} & \overline{\varphi}(G_{\mathcal{A}}(C)) \end{array}$$

Inducing an Order on CA (2)

Idea. Rescaling corresponds to an information preserving zoom out operation...



A sample $\langle 4, 4, 1 \rangle$ rescaling

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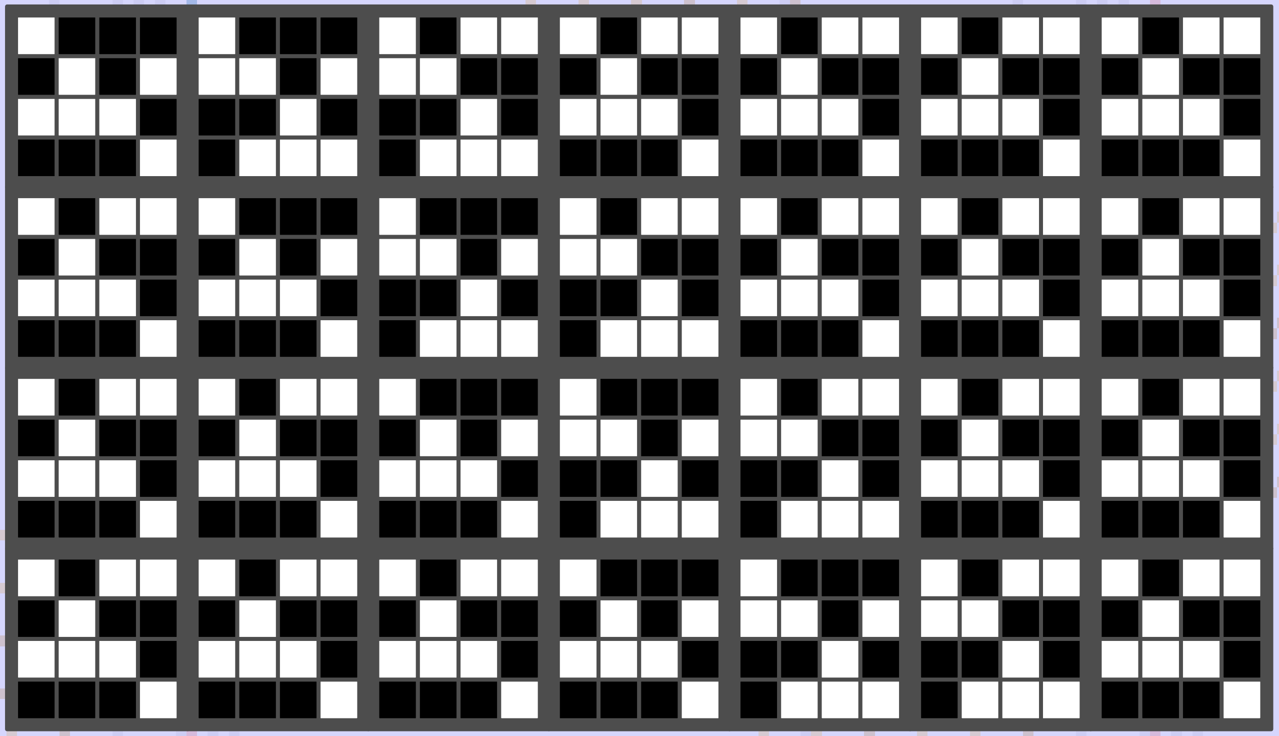

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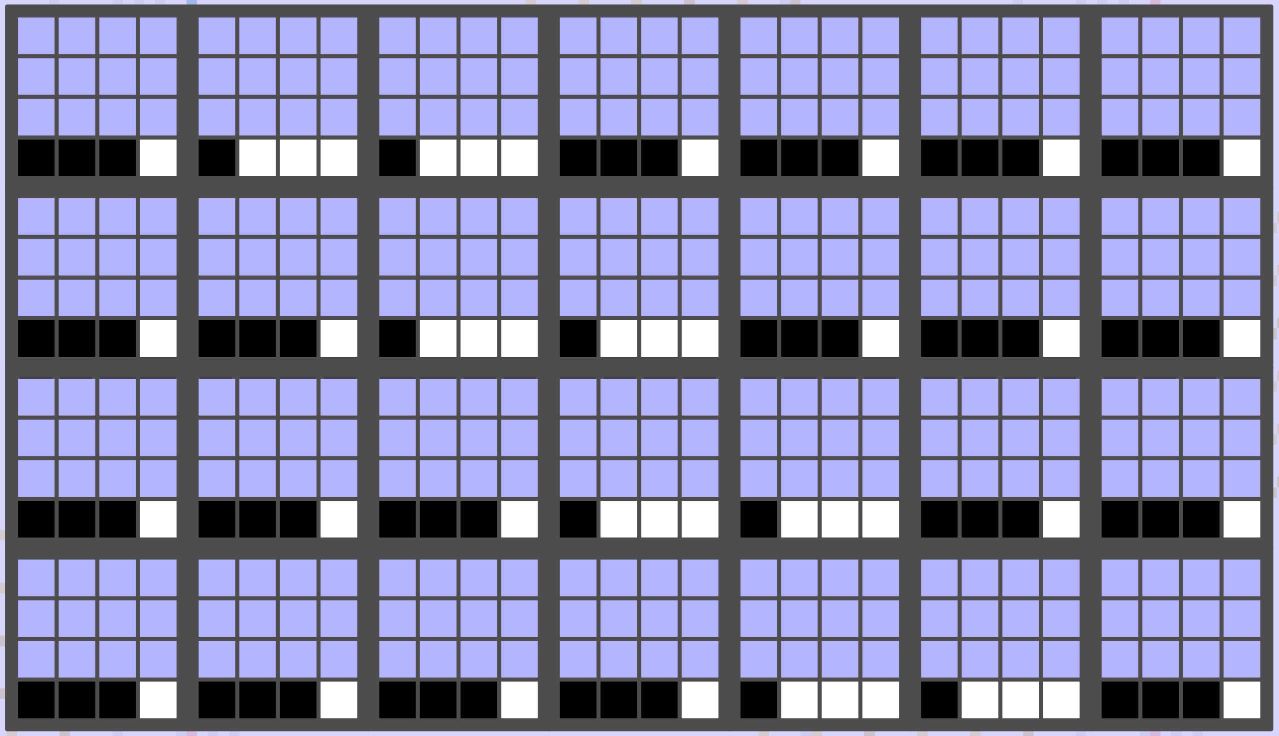

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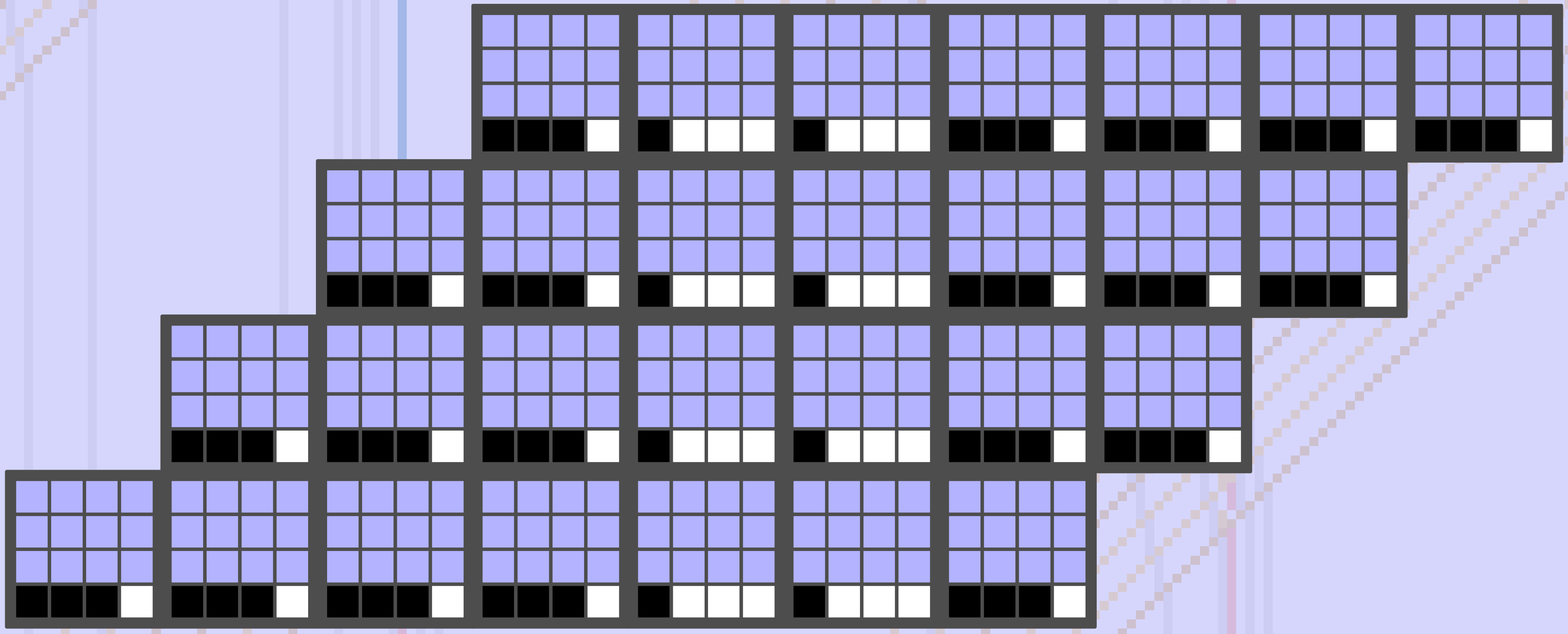

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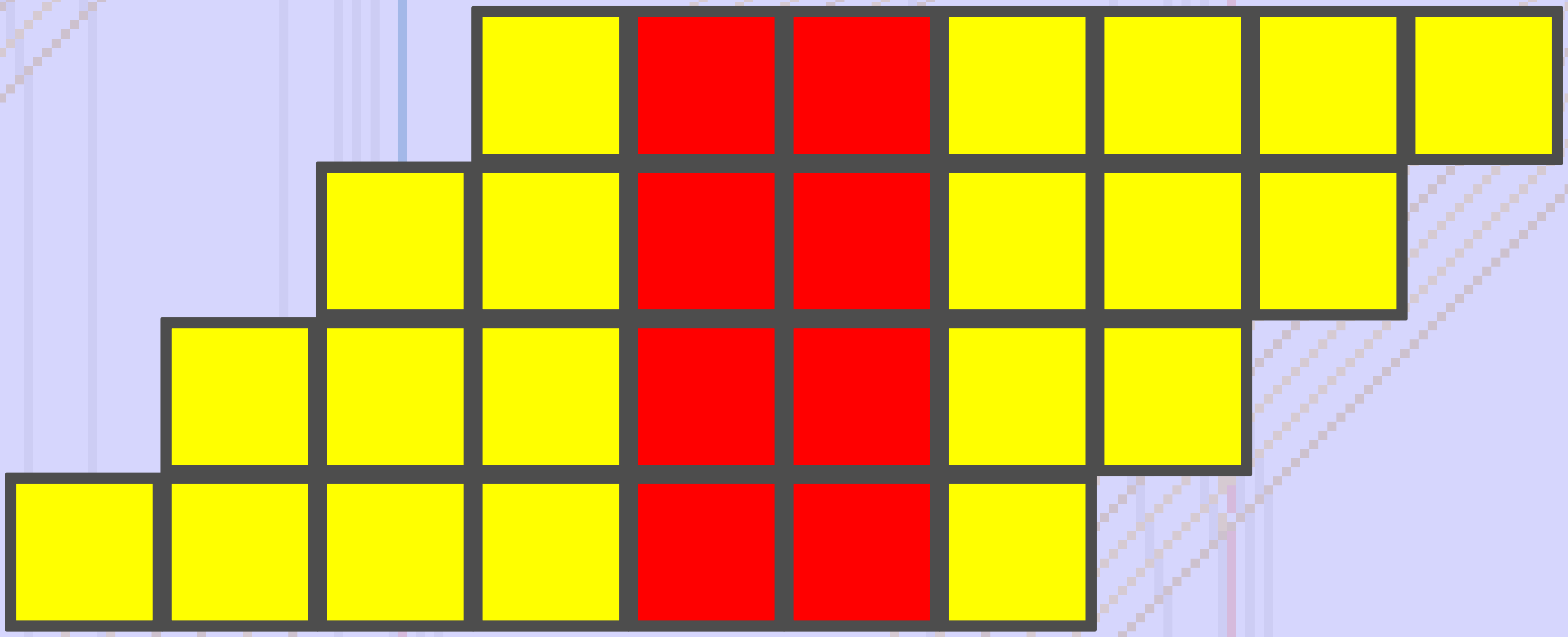

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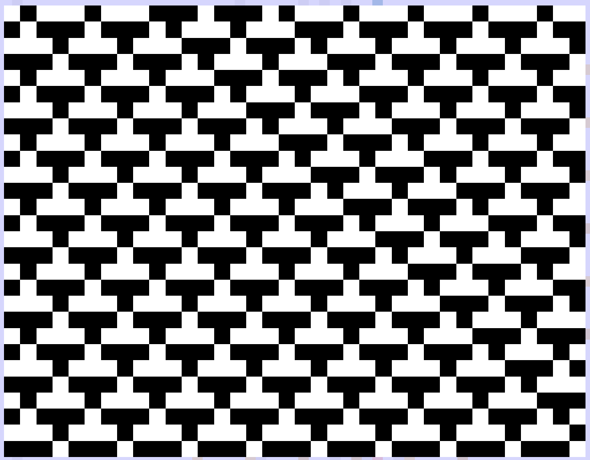
Inducing an Order on CA (3)

Definition. The $\langle m, n, k\rangle$ rescaling of $\mathcal{A}$ is $\mathcal{A}^{\langle m,n,k\rangle}$:

$$G_{\mathcal{A}}^{\langle m,n,k\rangle} = \sigma^k \circ o^m \circ G_{\mathcal{A}}^n \circ o^{-m} .$$



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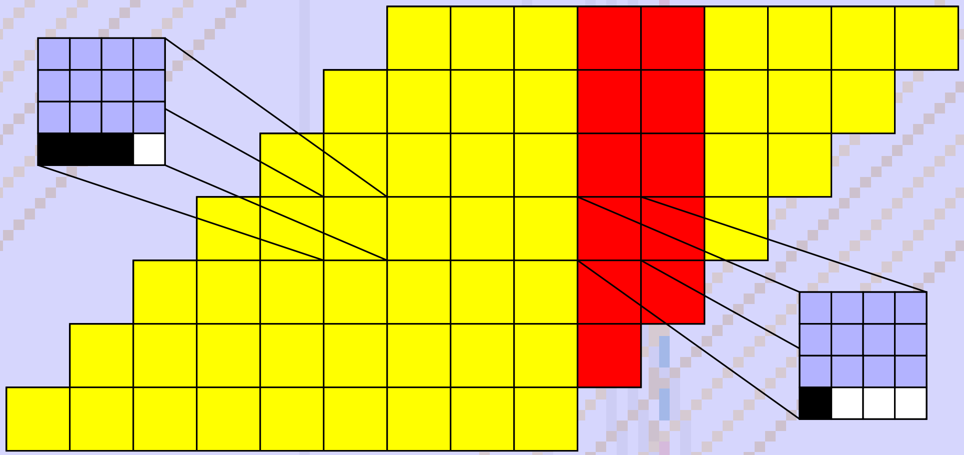

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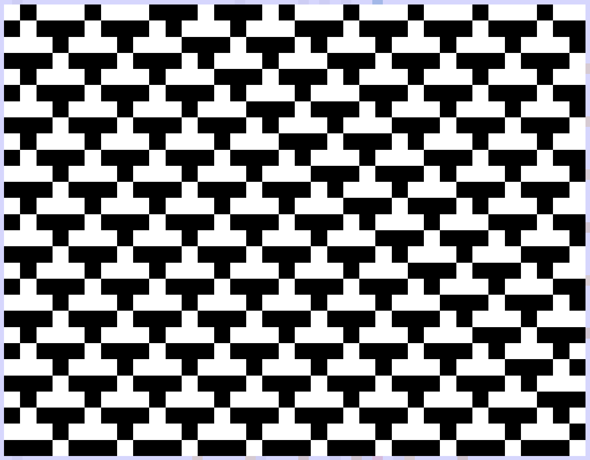

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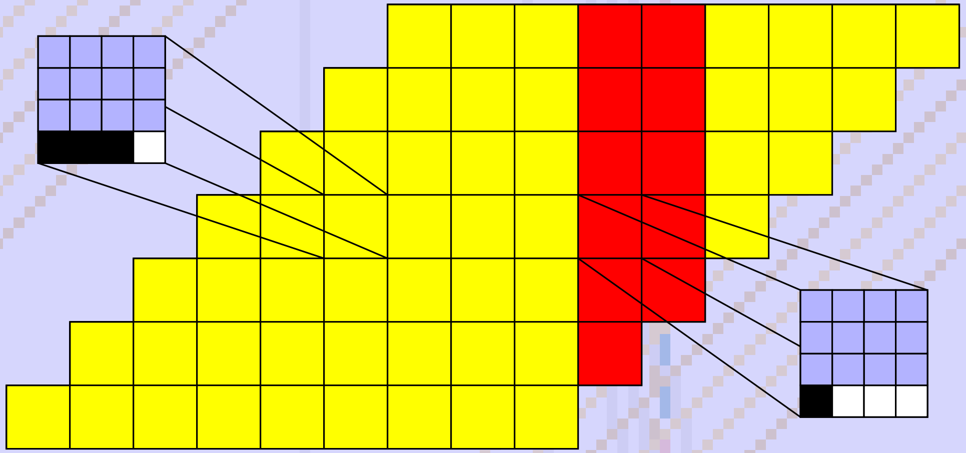

$\mathcal{A}^{\langle 4,4,1 \rangle}$

Definition. $\mathcal{A} \leqslant \mathcal{B}$ if there exist $\langle m, n, k \rangle$ and $\langle m', n', k' \rangle$ such that $\mathcal{A}^{\langle m,n,k \rangle} \subseteq \mathcal{B}^{\langle m',n',k' \rangle}$.

Inducing an Order on CA (4)

Proposition. The relation $\leqslant$ is a quasi-order on CA.

- The induced order admits a maximal element.

Definition. A CA $\mathcal{A}$ is *intrinsically universal* if:

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Proposition. Every CA in the maximal equivalence class of $\leqslant$ is intrinsically universal.

Sketch of the proof

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if and only if
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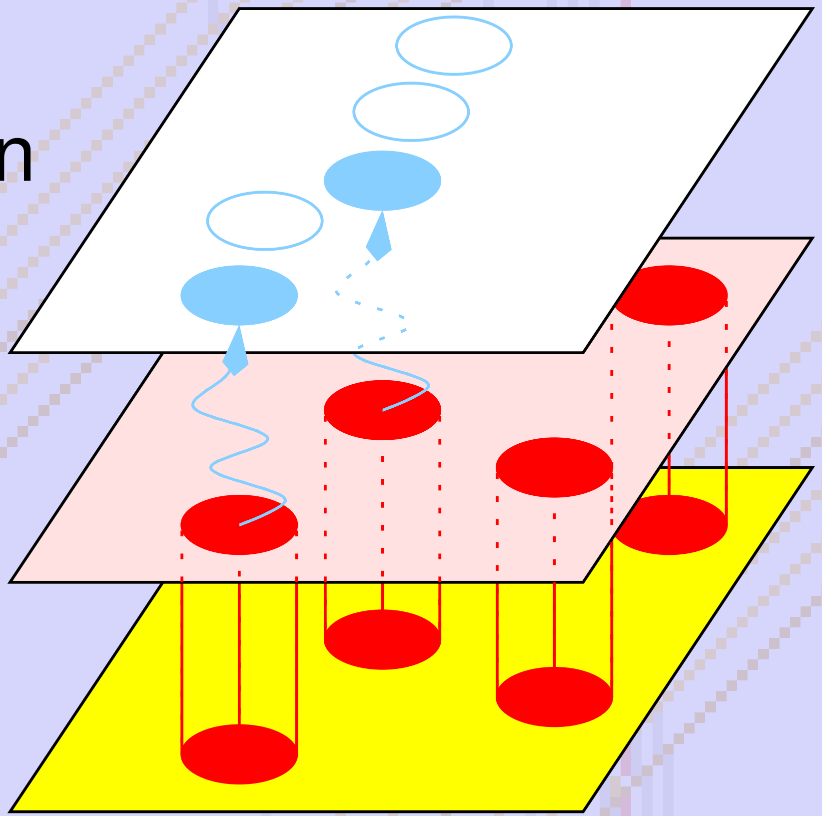
Remark. We suppose $\mathcal{U}$ is universal in the strong meaning and prove that $\mathcal{A} \circledast_s \mathcal{U}$ is universal in the weak meaning.

Introducing Boiler CA

energy consumption

energy diffusion

energy production



$S_{\mathcal{U}}$

$\{\cdot, *\}$

$S_{\mathcal{A}}$

$$\mathcal{A} \circledast_s \mathcal{U} = (\mathbb{Z}, S_{\mathcal{A}} \times \{\cdot, *\} \times S_{\mathcal{U}}, \mathcal{N}_{\mathcal{A}} \cup \{-1, 0\} \cup \mathcal{N}_{\mathcal{U}}, \delta)$$

If $\mathcal{A}$ is not s-nilpotent on periodic

- There exists a space-time pattern $\mathcal{P}$ of $\mathcal{A}$, periodic both in time and space, which is not s-monochromatic.

	?	?	?
	?	?	?
	?	?	?
$s \neq$	●	?	?

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- This pattern can be used to produce energy in a uniform way to the $\mathcal{U}$ layer in such a way that this layer simulates $\mathcal{U}$ behavior up to some slowdown factor.

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- Every periodic configuration of $\mathcal{A}$ evolves to the s-monochromatic configuration in finite time.

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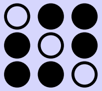
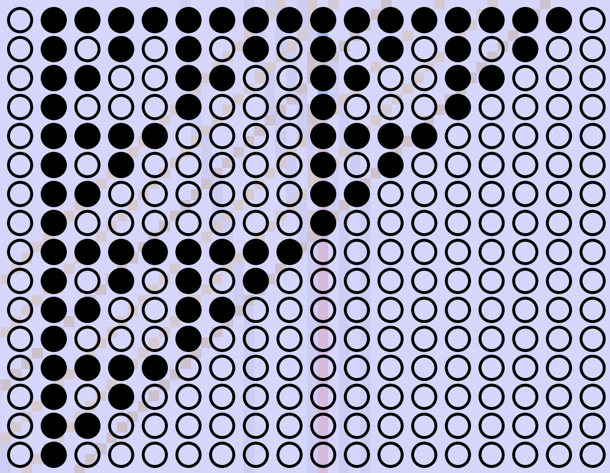
Idea. The $\mathcal{U}$ layer do not get enough energy to compute everything needed...

- In fact, $\mathcal{A} \circledast_s \mathcal{U}$ cannot simulate $(\mathbb{Z}, \{\circ, \bullet\}, \{-1, 0\}, \oplus)$

Two key patterns:  and 

Future Work

- Adapt this “energy driven” proof technic to other properties.

- Can we extend the analogy between computation universality for TM and intrinsic universality for CA to derive some kind of “Rice’s theorem” for non trivial properties on space-time diagrams?

Test Page (+ pdf$\TeX$ & Acrobat issue)

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