Cellular Automata, Tilings, Undecidability

revisiting classics

N. Ollinger
Escape, LIF, Marseille

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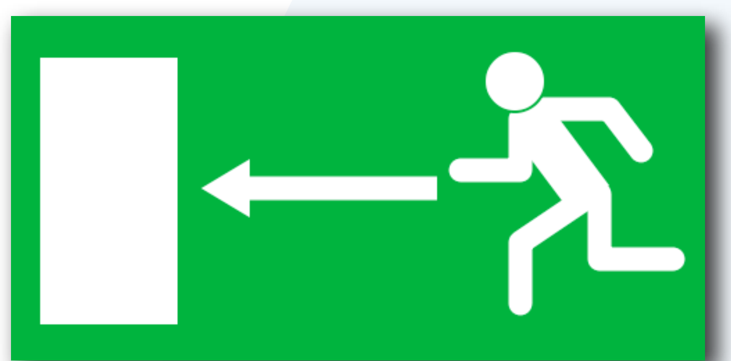

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“au menu”

- A self-contained proof of the undecidability of the nilpotency problem for 1D cellular automata.

[Kari92] **Jarkko Kari.** The Nilpotency Problem of One-Dimensional Cellular Automata. **SIAM J. Comput. 21(3): 571-586 (1992)**

- Same proof skeleton, same technics, new tiling constructions to allow a shorter proof.

§1. Cellular Automata

Definition

Cellular Automata

A 1D *cellular automaton* is a pair (S, f) where:

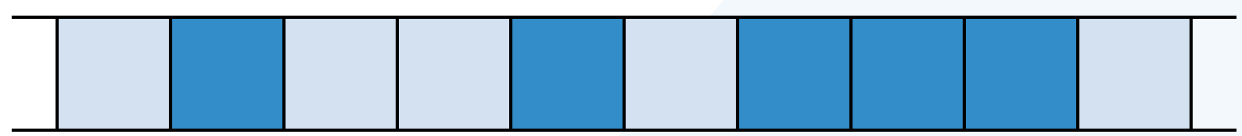
- S is a finite set of states;
- $f : S^3 \longrightarrow S$ is the local rule.

1D

Dynamics

Configuration

A *configuration* is a color map $c \in S^{\mathbb{Z}}$.



Global rule

The *global rule* $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ applies the local rule uniformly and synchronously:

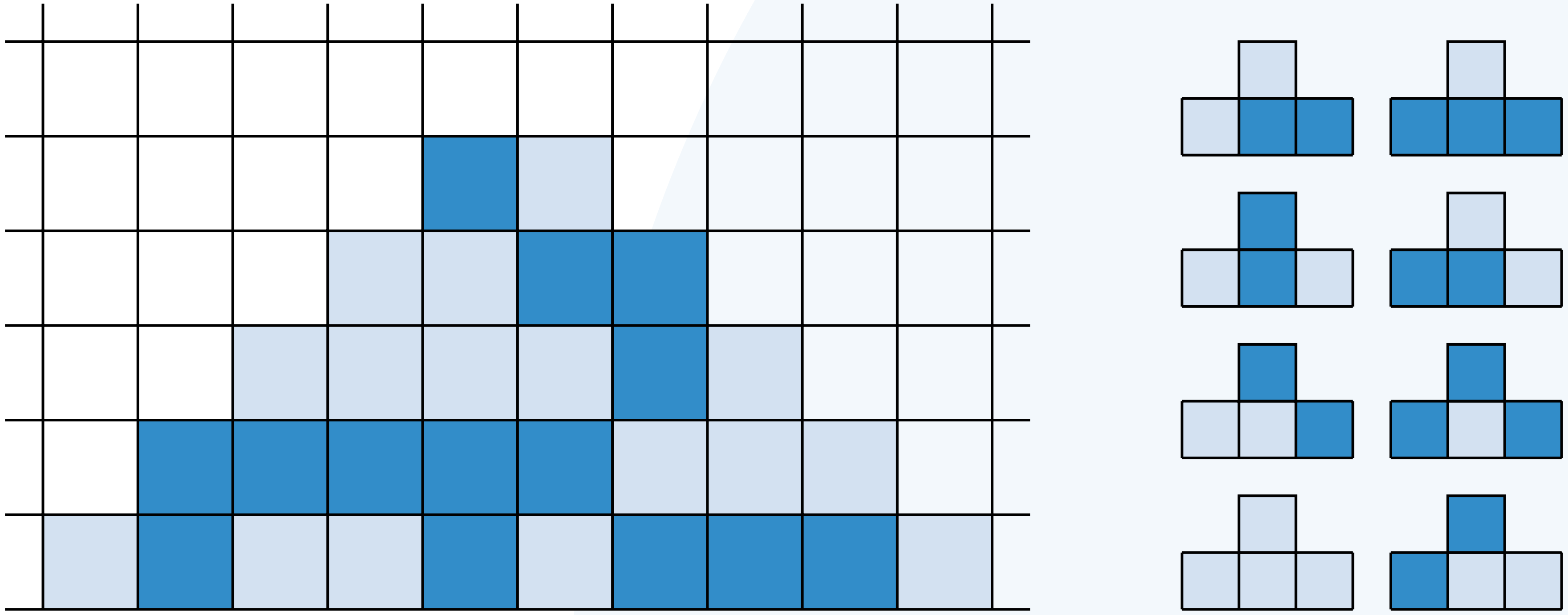
$$\forall c \in S^{\mathbb{Z}},\ \forall p \in \mathbb{Z}, \qquad G(c)_p = f(c_{p-1}, c_p, c_{p+1}).$$

Space-time diagram

Space-time diagram

A *space-time diagram* $\Delta \in S^{\mathbb{N}\times\mathbb{Z}}$ is a graphical representation of an orbit $\mathcal{O}(c)$:

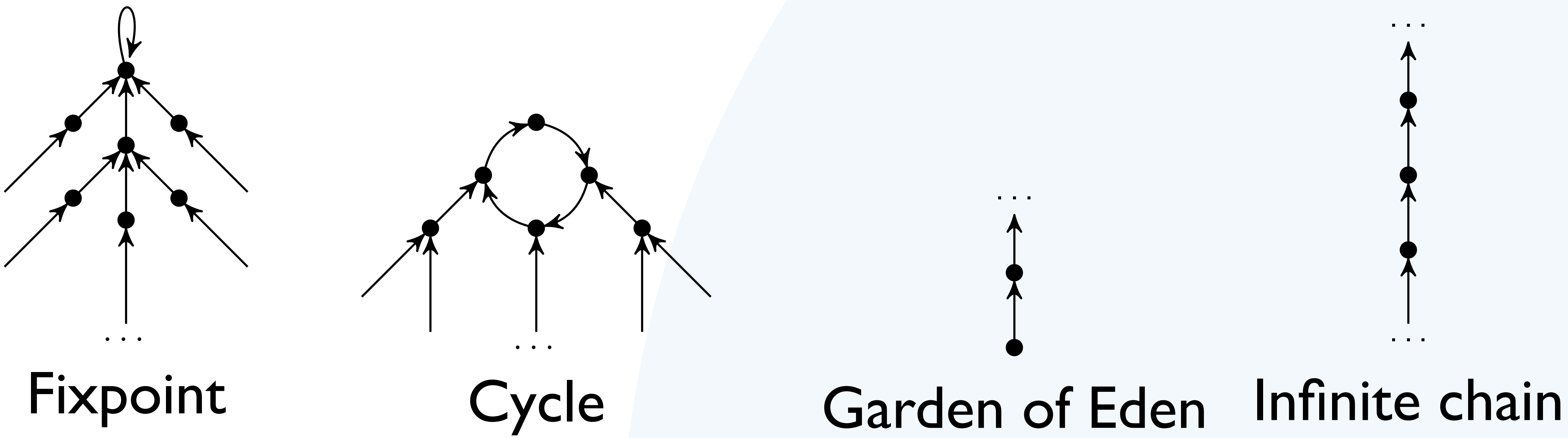
$$\forall t \in \mathbb{N}, \forall p \in \mathbb{Z}, \qquad \Delta_t(p) = G^t(c)_p.$$



Phase space

Phase space

The phase space of a CA $(S^{\mathbb{Z}}, G)$ is the graph with vertices the configurations and an edge from c to c' iff $G(c) = c'$.



Limit set

Limit Set

The limit set of a CA is the set of configurations that may appear at any time step:

$$\Omega = \bigcap_{t \in \mathbb{N}} \Omega^{(t)} \quad \text{where} \quad \Omega^{(t)} = G^t\left(S^{\mathbb{Z}}\right)$$

- With Cantor topology, by compacity, the limit set is a non-empty subshift (implies compact).
- The limit set is exactly the set of configurations from the extended space time diagrams $\Delta \in S^{\mathbb{Z}^2}$ with time in $\mathbb{Z}$ (*bi-infinite orbits*).

Nilpotency

Nilpotency

A CA (S, f) with a quiescent state s (meaning $f(s, s, s) = s$) is nilpotent if each configuration $c \in S^{\mathbb{Z}}$ converges in finite time to the s-monochromatic configuration ($\exists t \in \mathbb{N}, G^t(c) = {}^{\omega}s^{\omega}$).

Proposition

1. A CA is nilpotent iff there exists some t such that G^t is a constant map;
2. A CA is nilpotent iff its limit set is a singleton.

Deciding nilpotency

Nilpotency problem (Nil1D)

Input. a CA (S, f)
Question. Is it a nilpotent CA?

- If it is nilpotent, find t such that G^t is constant;
- If it is **not** nilpotent, what can we enumerate?
- As a special case, if it is not nilpotent and admits a second periodic configuration in Ω, one can enumerate periodic configurations.

[Kari92] The problem is undecidable.
Proof by reduction to the Halting Problem.

The reduction

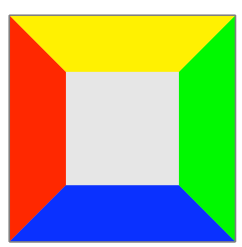
- Build a recursive family (A_i) of CA such that A_i is nilpotent iff the Turing Machine φ_i halts starting from the empty word.
- We know that such CA will admit only one periodic configuration in their limit set.
- **Challenge.** TM computation (= TM heads) everywhere in every configuration.
- **Hint.** consider tilings!

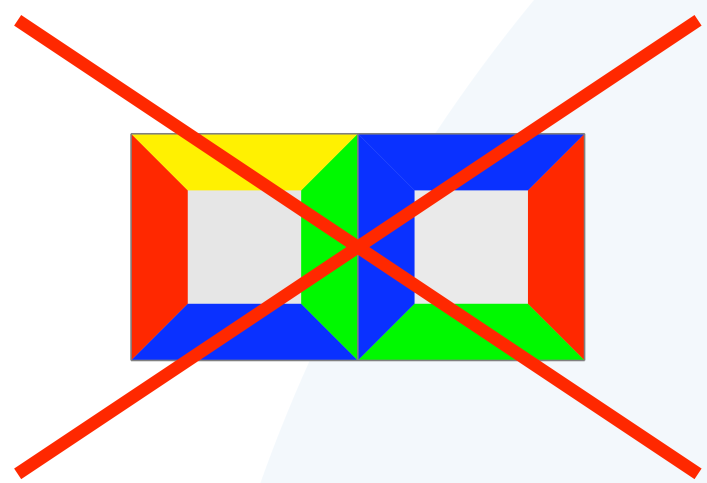
§2. Tilings

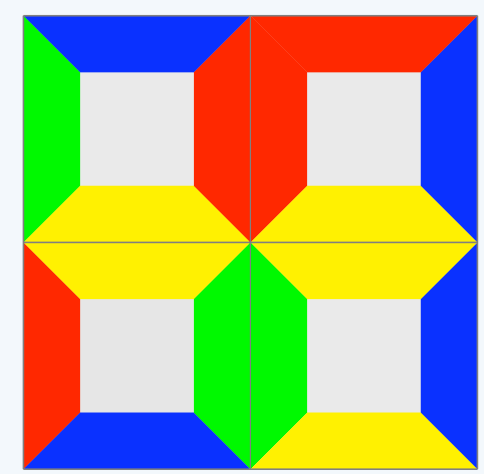
Wang tiles

Wang tiles

A set of Wang tiles is a finite set $\mathcal{T} \subseteq C^4$ where C is a finite set of colors. A Wang tile is a square with colored edges.

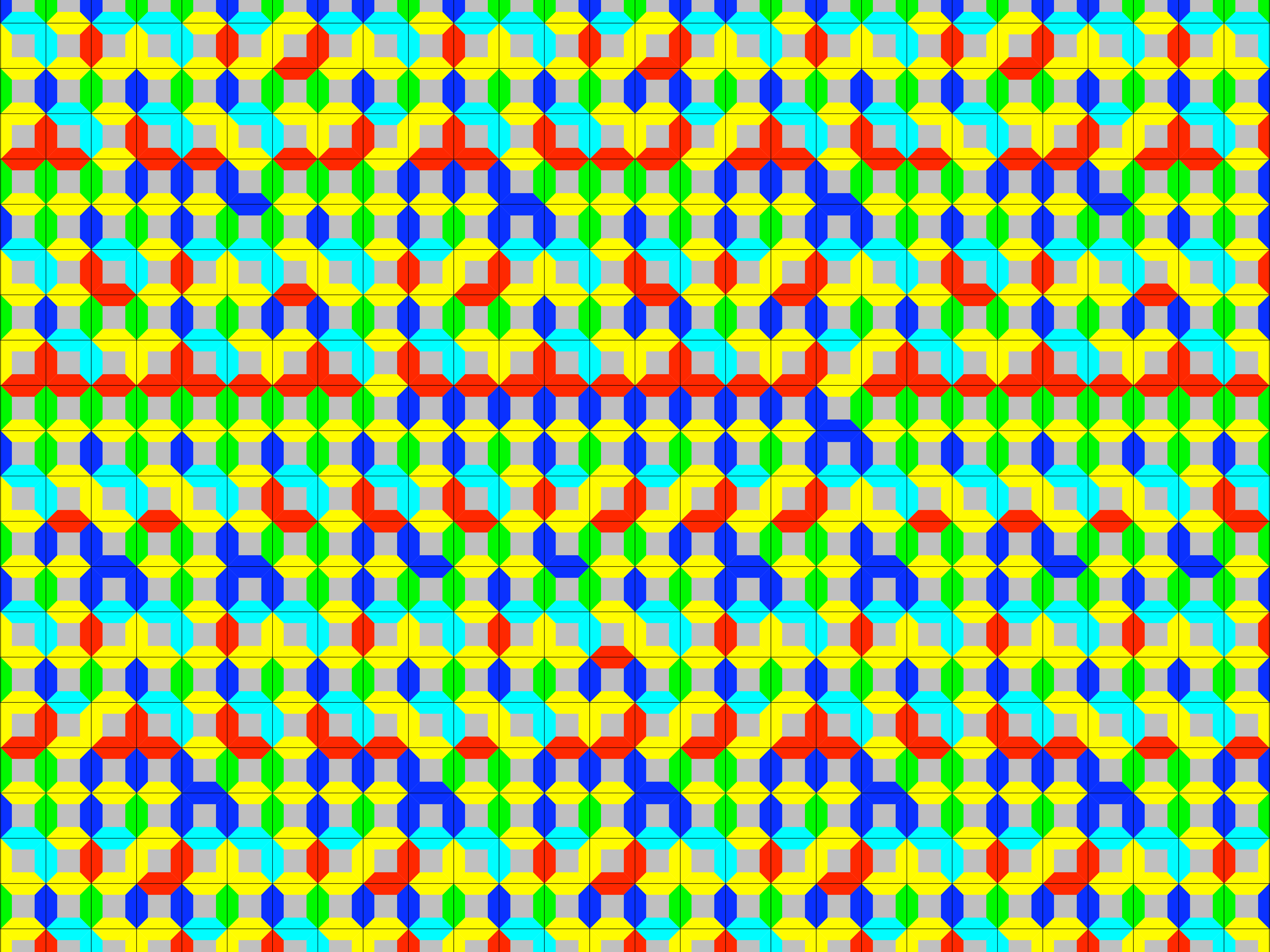






Matching rule

Two Wang tiles put side by side match if their common edge share a same color on both tiles.



Tiling the plane

Tiling

A tiling of the plane by a set of Wang Tiles $\mathcal{T}$ is a map $\tau \in \mathcal{T}^{\mathbb{Z}^2}$ such that any two neighbor tiles match on their common edge.

- With Cantor topology, by compacity, the set of tilings is compact.
- A tile set admits a tiling iff one can tile a finite square of each size (meaning filling $\mathcal{T}^{[0,n]^2}$ with matching edges for each $n \in \mathbb{N}$).
- If a tile set admits no tiling, there is a finite square which cannot be tiled.

Tiling the plane

no rotation
no symmetry

Tiling

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Tiling, patterns, periodicity

- If $\mathcal{T}$ does not tile the plane, find a square of size n that cannot be tiled;
- If $\mathcal{T}$ tiles the plane, what can we enumerate?
- As a special case, if it admits a periodic tiling, one can enumerate periodic tilings.

Aperiodic set of tiles

A set of Wang tiles is aperiodic if it can tile the plane but admits no periodic tiling.

Tiling, patterns, periodicity

- If $\mathcal{T}$ does not tile the plane, find a s
- If $\mathcal{T}$ tiles the plane, what can we enum
- As a special case, if it admits a periodic tiling, one can enumerate periodic tilings.

With Wang tiles periodic implies biperiodic

Aperiodic set of tiles

A set of Wang tiles is aperiodic if it can tile the plane but admits no periodic tiling.

Domino Problem

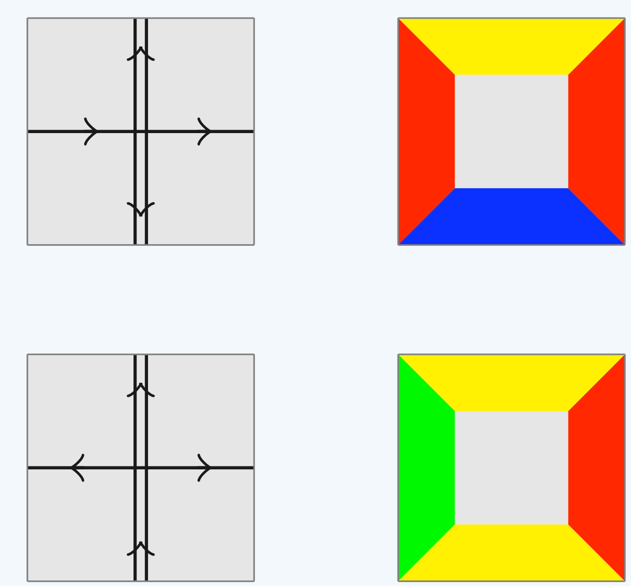
Domino Problem

Input. a set of Wang tiles $\mathcal{T}$
Question. does it tile the plane?

- Wang was interested into the decidability of this problem;
- Conjectured that no aperiodic tile set exist (so DP decidable);
- Berger 1965 *The Undecidability of the Domino Problem*.

More tiles

- Wang tile are convenient for proofs because very local (horizontal and vertical constraints);
- Different kinds of tiles may be more convenient for constructions:
  - Tiles with notches instead of colors;
  - Tiles with arrows;
  - Polygonal tiles with rational coordinates;
  - Local rule as for CA;
- All these models are equivalent with respect to DP;
- It's just geometrical syntactic sugar (rotations, etc).



Constructions

- Two big directions for constructions since Berger:
  1. Understanding aperiodicity;
  2. Understanding DP undecidability.
- Two orthogonal way of doing it:
  1. Minimize the number of tiles;
  2. Minimize the length of the proof.
- In this talk, we will go (2) (2).

Aperiodic tile sets

- Running for as few Wang tiles as possible:

20426	Berger, 1965
104	Berger, shortly
92	Knuth, 1966
40	Laüchli, 1966
56	Robinson, 1967
35	Robinson, 1971
34	Penrose, 1973
32	Robinson, 1973
24	Robinson, 1977
16	Ammann, 1978
13	Culik and Kari, 1995

- See Grünbaum and Shephard *Tilings and Patterns*

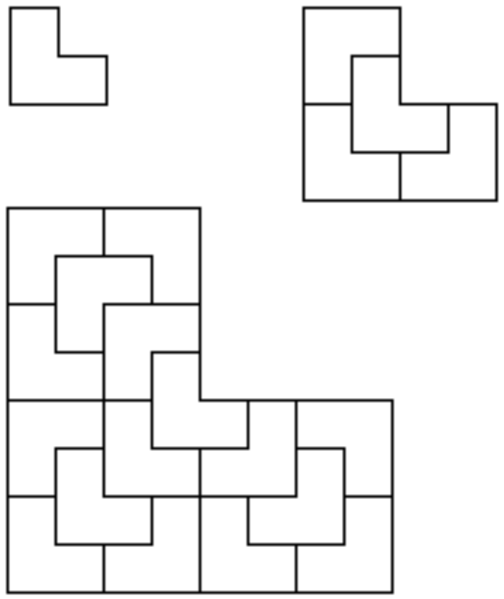
Simulating TMs

- Simplifying proof length:

Berger	1965, original proof
Robinson	1971, simplifying using explicit substitutions
Culik-Kari	1995, completely different tools
Durand-Levin-Shen	200x, new construction, Robinson-style

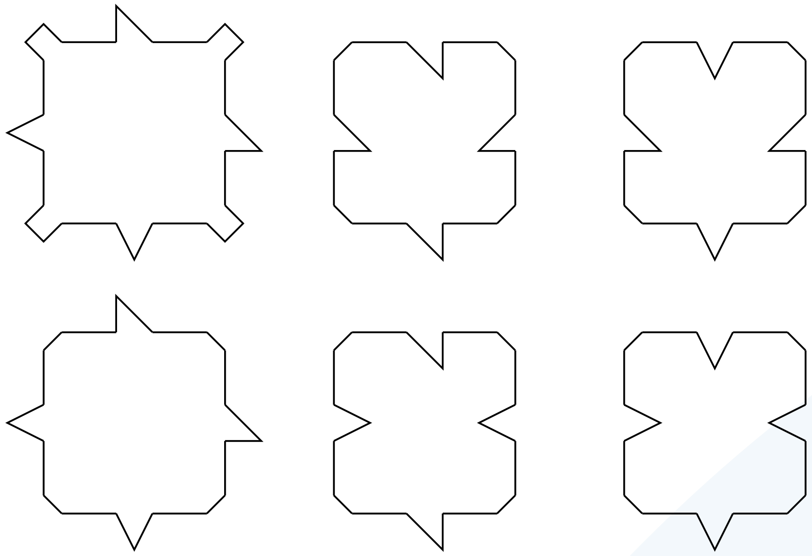
- Only the first two discuss TM simulation.
- In this talk, we apply DLS-alike technics to the whole proof.

Substitutions

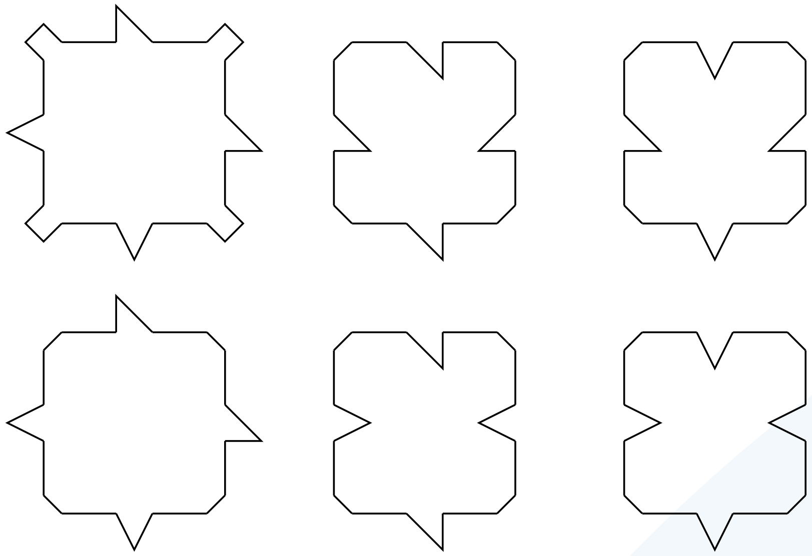


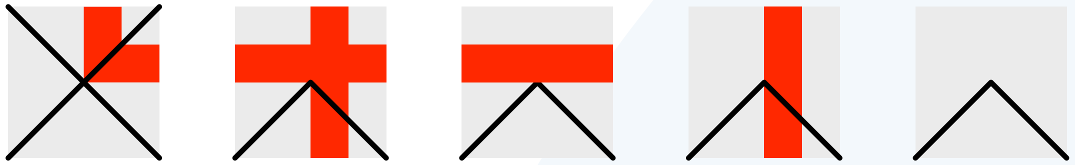
- Substitutions easily define aperiodic plane color maps.
- Substitutions more or less correspond to automata defined on binary addressing of the plane cells.
- The main difficulty is to enforce the substitution with local rules, *i.e.* with unary addressing automata.

Robinson 1971

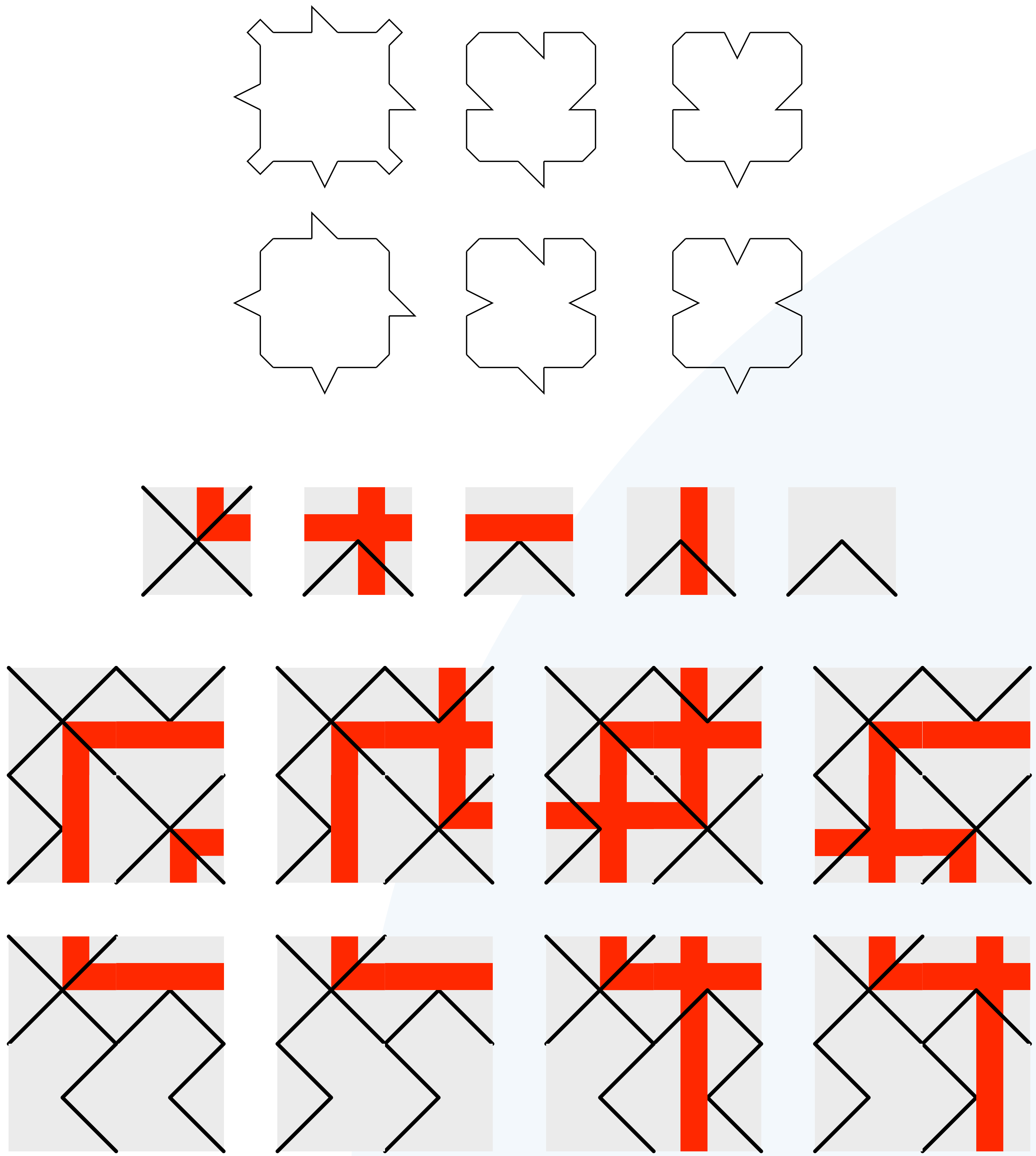


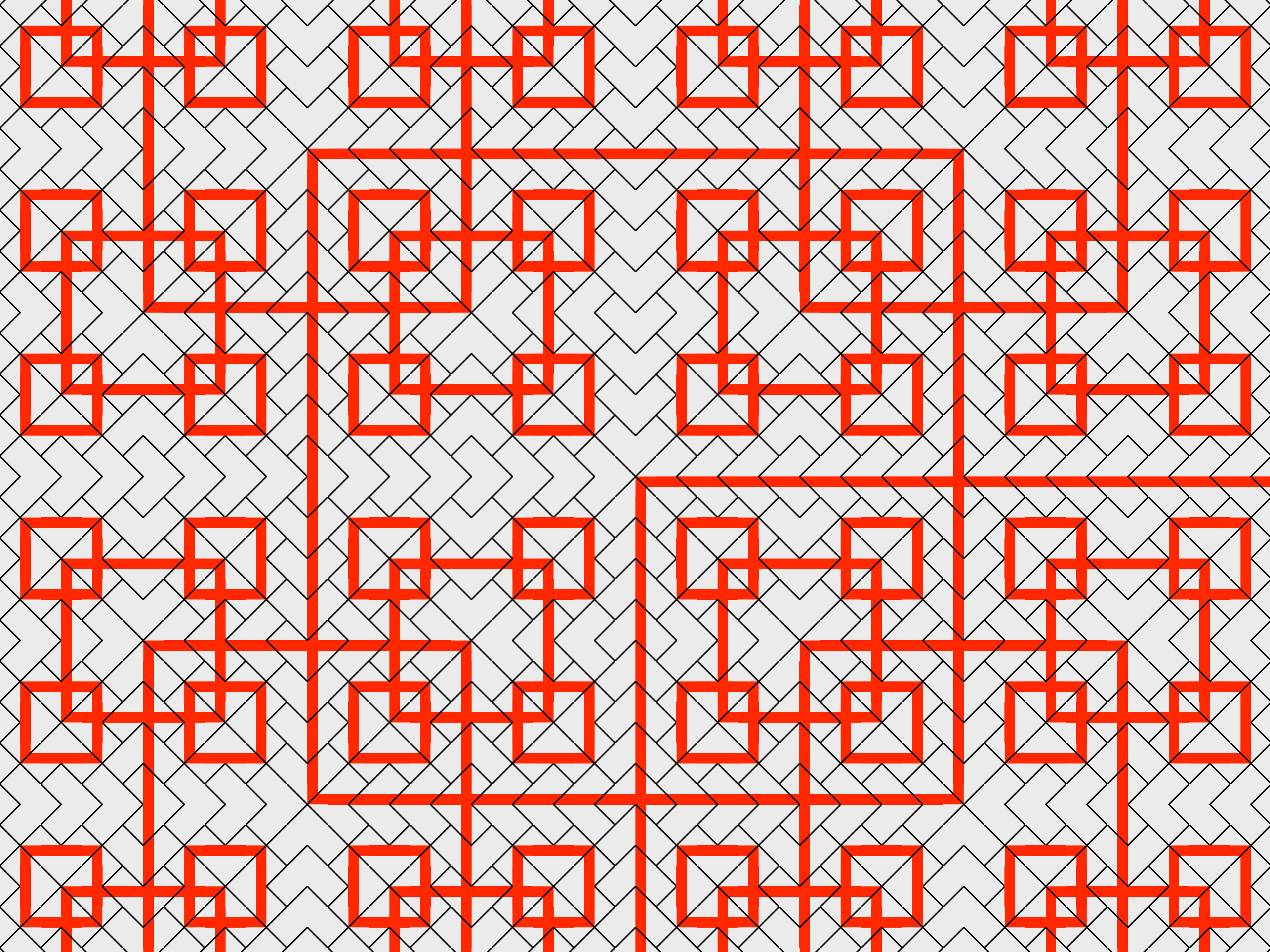
Robinson 1971





Robinson 1971





Kari’s proof

- Start with a slight modification of Robinson aperiodic tile set;
- Add computation area for Turing machines in the spirit Robinson technic of recursive computing areas of all sizes;
- Modify the tile set to enforce NW-determinism.

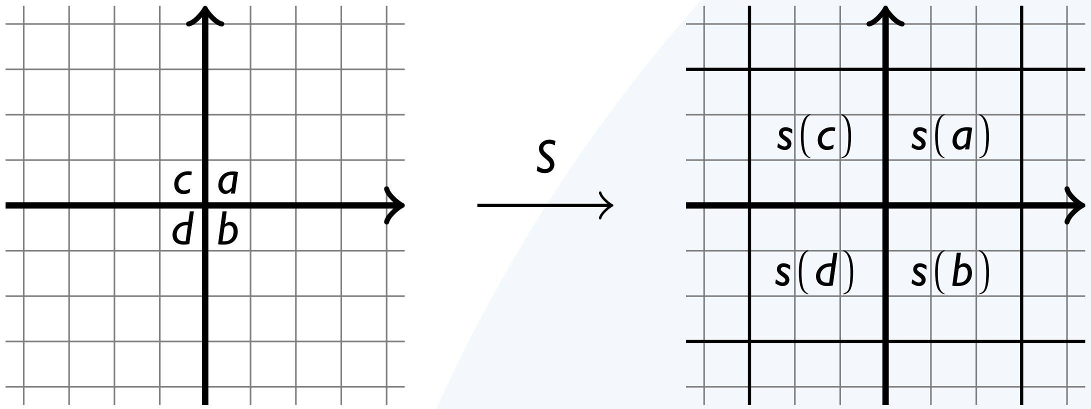
- It definitely works…
- …but it is painful.
- and you need to master Robinson construction first.

§3. Let’s prove it

Substitution systems

Substitution de Moore

Une substitution de Moore est une application $s : \Sigma \longrightarrow \Sigma^9$ où Σ est un alphabet fini.



Fonction globale

La fonction globale $S : \Sigma^{\mathbb{Z}^2} \longrightarrow \Sigma^{\mathbb{Z}^2}$ d'une substitution s vérifie :

$$\forall c \in \Sigma^{\mathbb{Z}^2}, \forall i \in \mathbb{Z}^2, \forall x, y \in \{0, 1, 2\}, \quad S(c)_{3i+(x,y)} = s(c_i)_{3y+x}$$

Limit set

Translation de k : $\forall i, k, c, \quad \sigma_k(c)_i = c_{i-k}$

S est continue pour Cantor et $\forall k, \quad S \circ \sigma_k = \sigma_{3k} \circ S$

On pose $\Lambda^{(n)} = \left[S^n\left(\Sigma^{\mathbb{Z}^2}\right)\right]_\sigma$ (clotûre par σ_k, 9^n suffisent)

Ensemble limite

L'ensemble limite d'une substitution s est le shift non vide Λ_s définit par :

$$\Lambda_s = \bigcap_{n \in \mathbb{N}} \Lambda^{(n)}$$

History

Historique

L'ensemble limite Λ_s est l'ensemble des configurations c qui possèdent un historique $(c_t, k_t) \in \left(\Sigma^{\mathbb{Z}^2} \times \{0, 1, 2\}^2\right)^{\mathbb{N}}$ vérifiant :

$$\begin{cases} c_0 = c \\ c_t = \sigma_{k_t} \circ S(c_{t+1}) \quad \forall t \in \mathbb{N} \end{cases}$$

Rq. par compacité, pour reconstruire un quart de plan minimum, on peut se contenter des $\left((c_t)_0, k_t\right)$.

Injectivity

Injectivité

La fonction globale S est injective si et seulement si s est injective.
Dans ce cas toute configuration possède au plus un historique.

Proposition

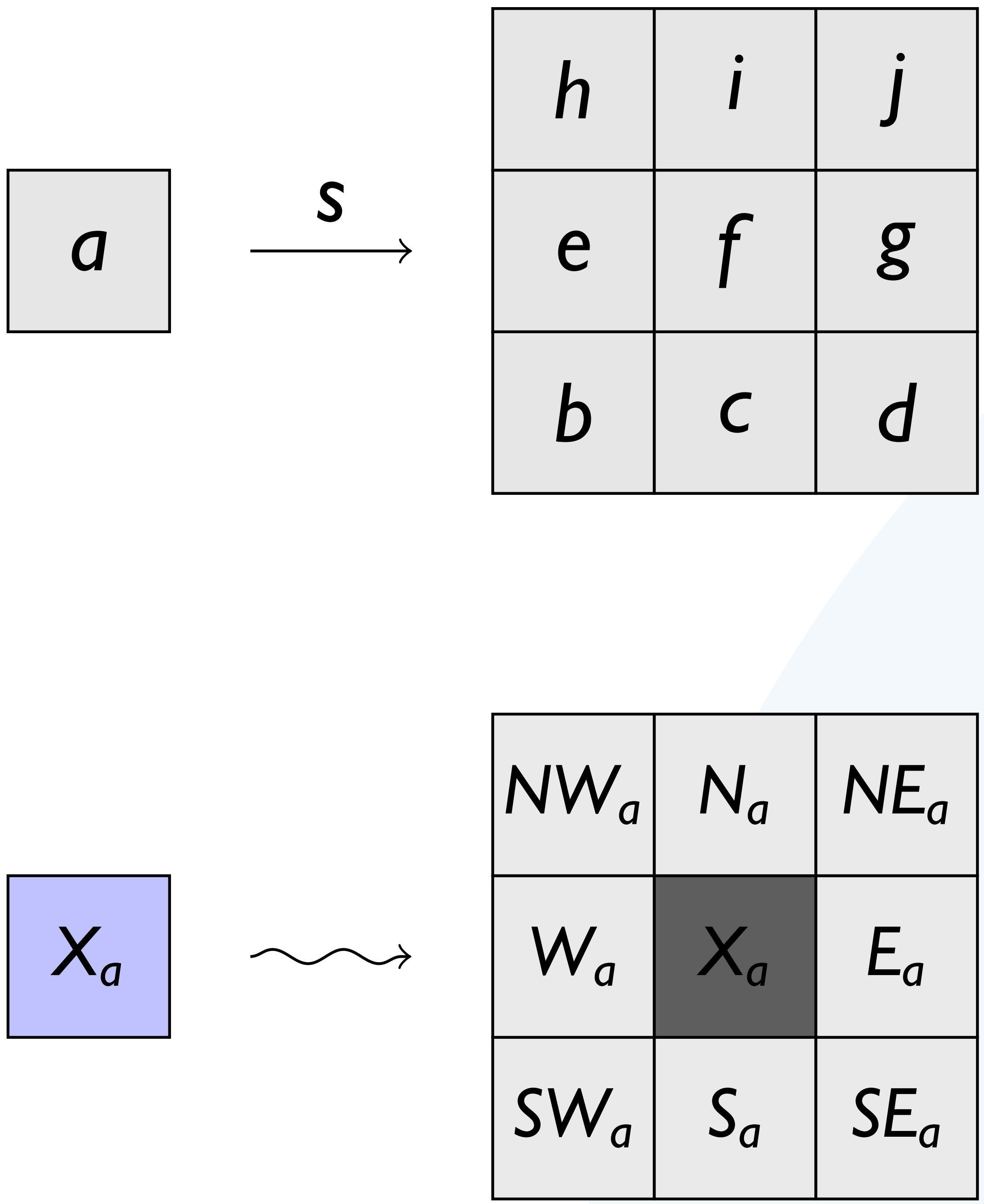
Proposition

Toute substitution s peut être transformée en une substitution injective s' et une fonction de coloriage $\pi : \Sigma' \to \Sigma$ telles que :

$$\pi(\Lambda_{s'}) = \Lambda_s.$$

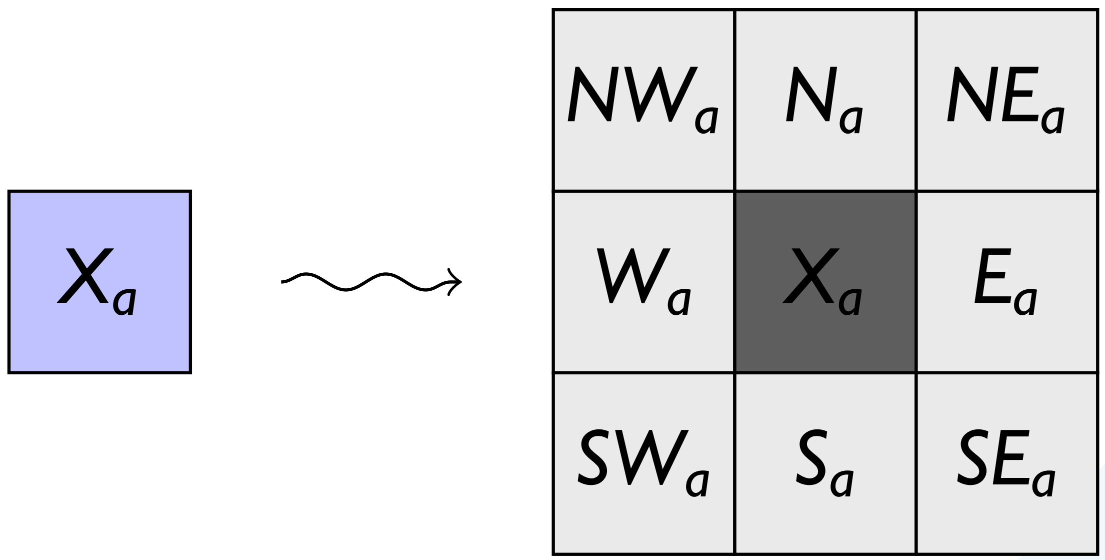
De plus on peut choisir s' pour lequel $\Lambda_{s'}$ est l'ensemble des pavages d'un jeu de tuiles de Wang apériodique, déterministe selon chaque diagonale (de rayon 2).

Encoding (step 1)

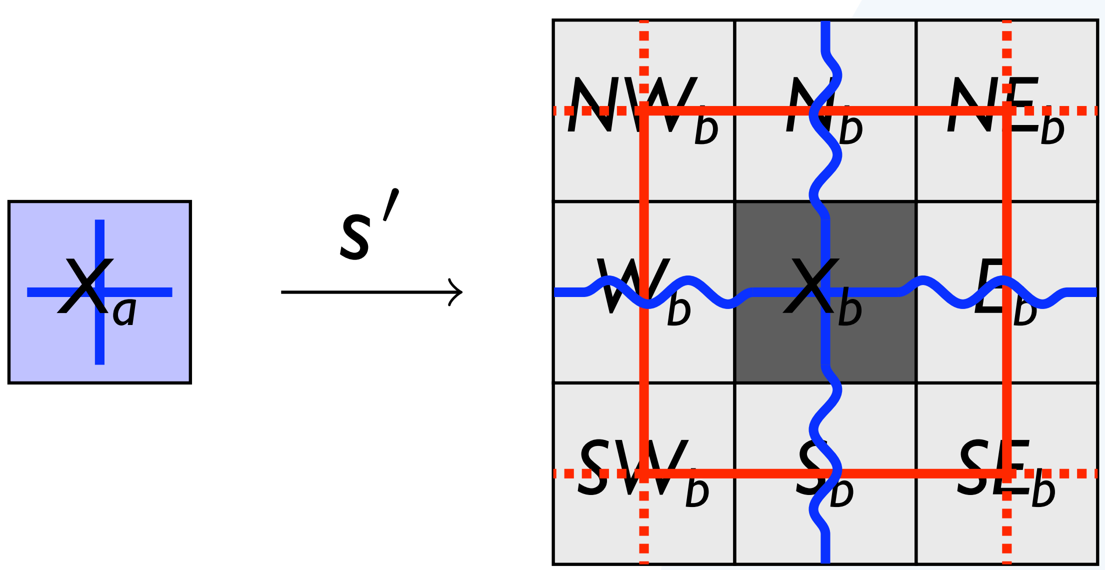


$$\pi(Y_a^{\blacksquare}) = s(a)_{Y\blacksquare}$$

Encoding (step 2)



$$\pi(Y_a^{\blacksquare}) = s(a)_{Y\blacksquare}$$

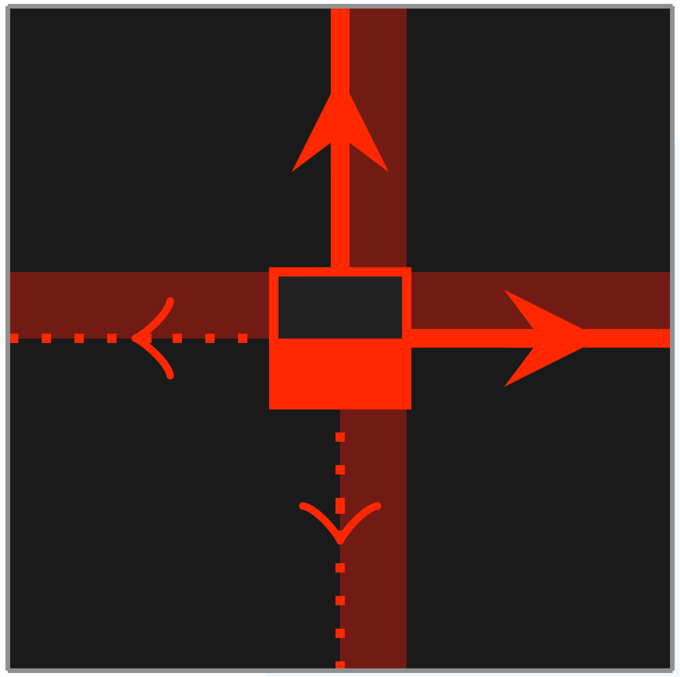


tag $X_a^{\blacksquare}$

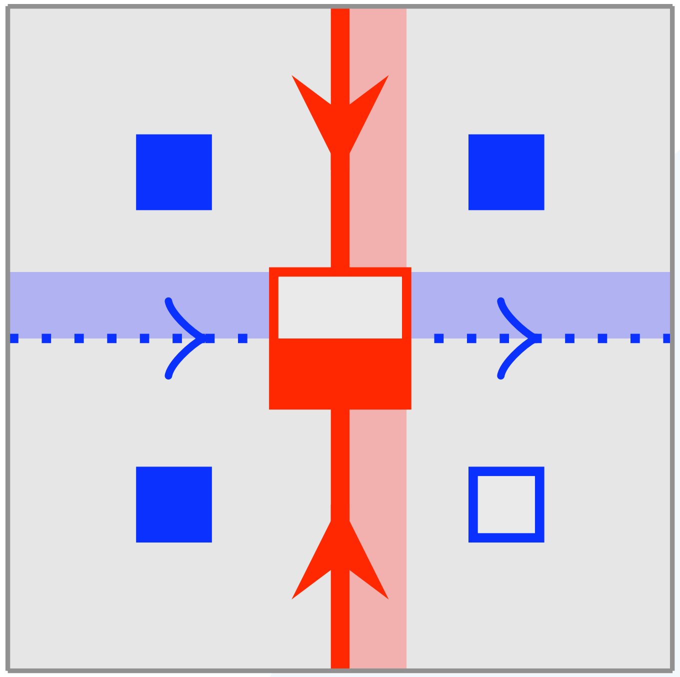
copy tag

$$s(a)_{X\blacksquare} = b$$

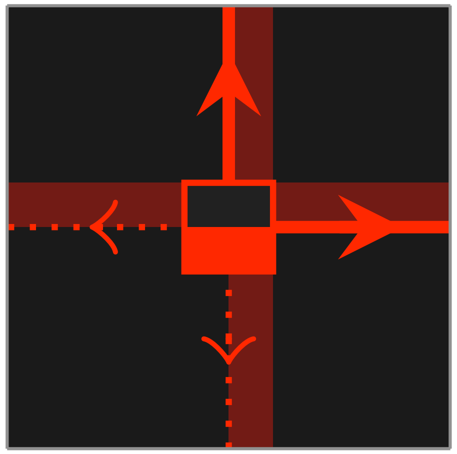
Corner tile

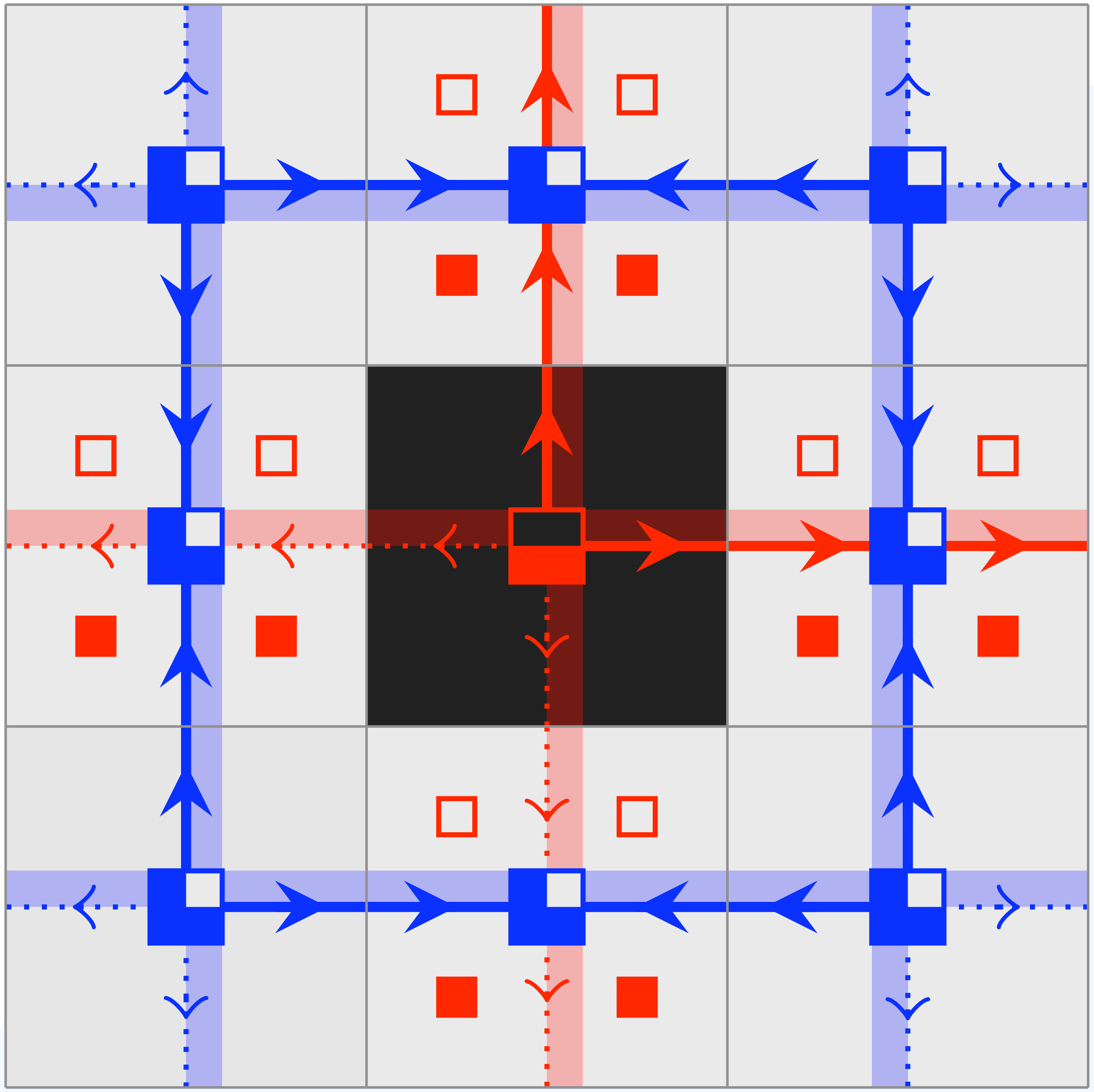


Border tile

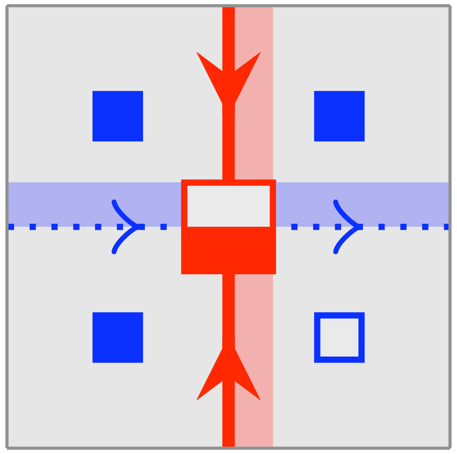


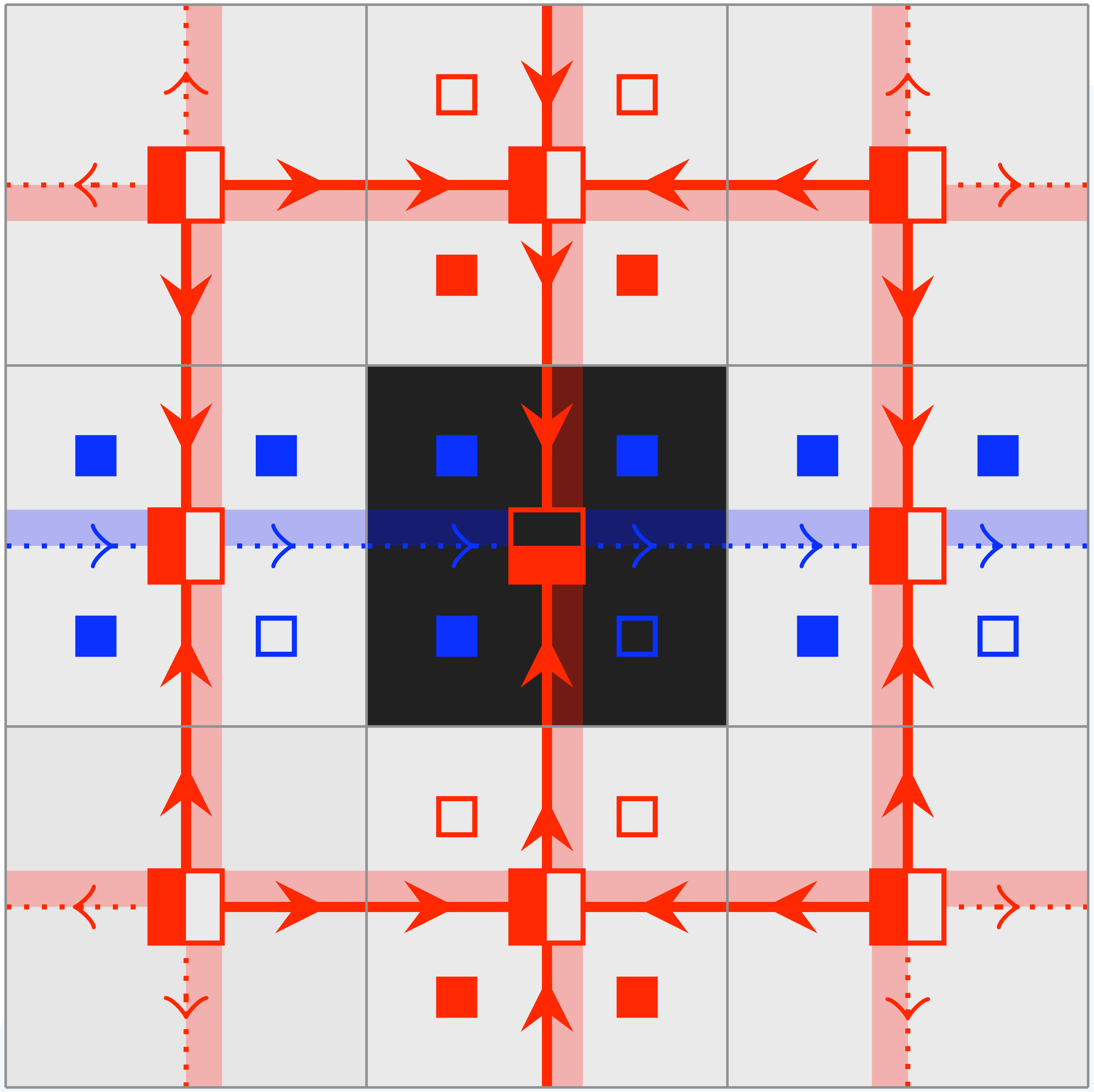
Substitution rule

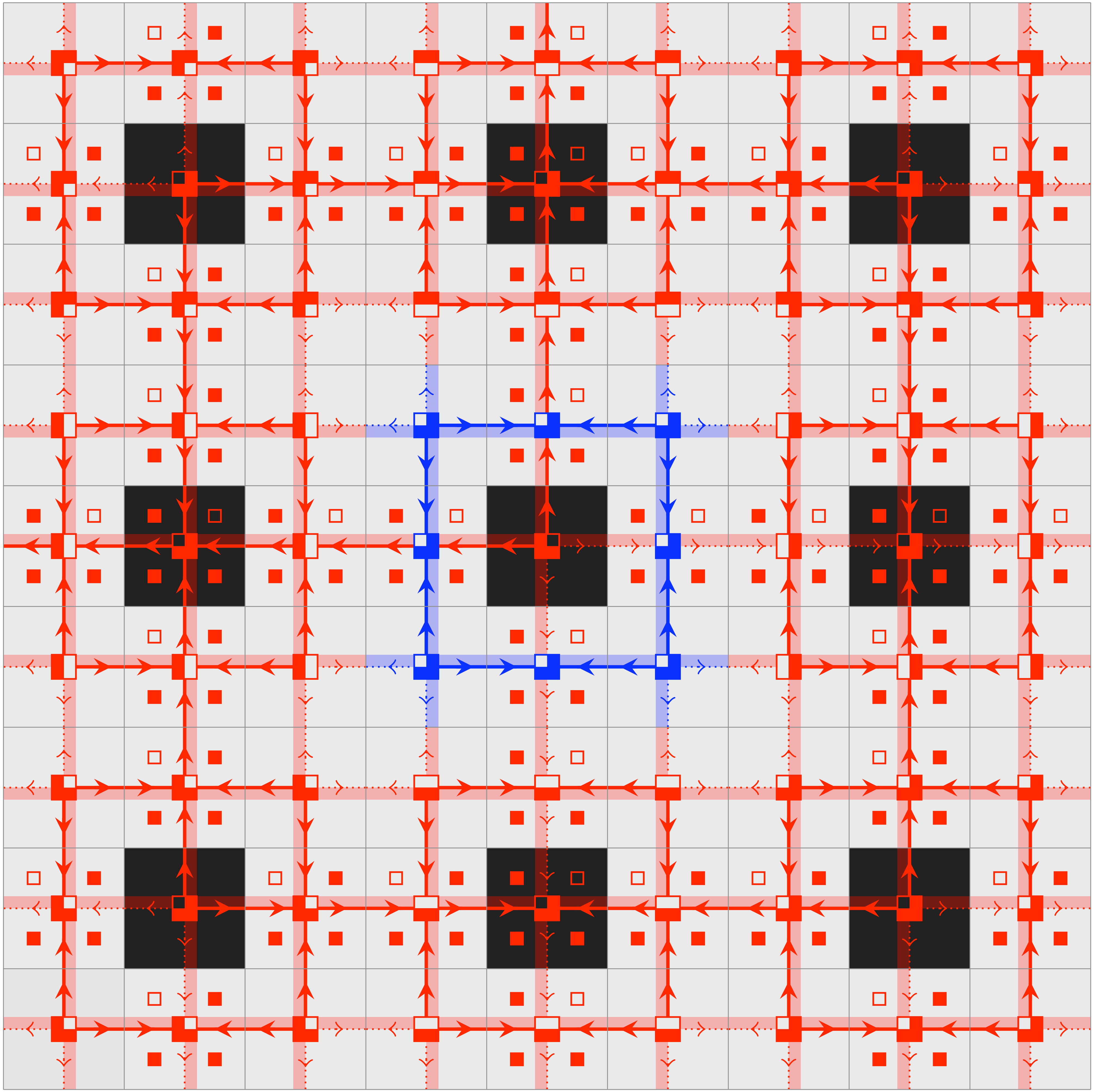


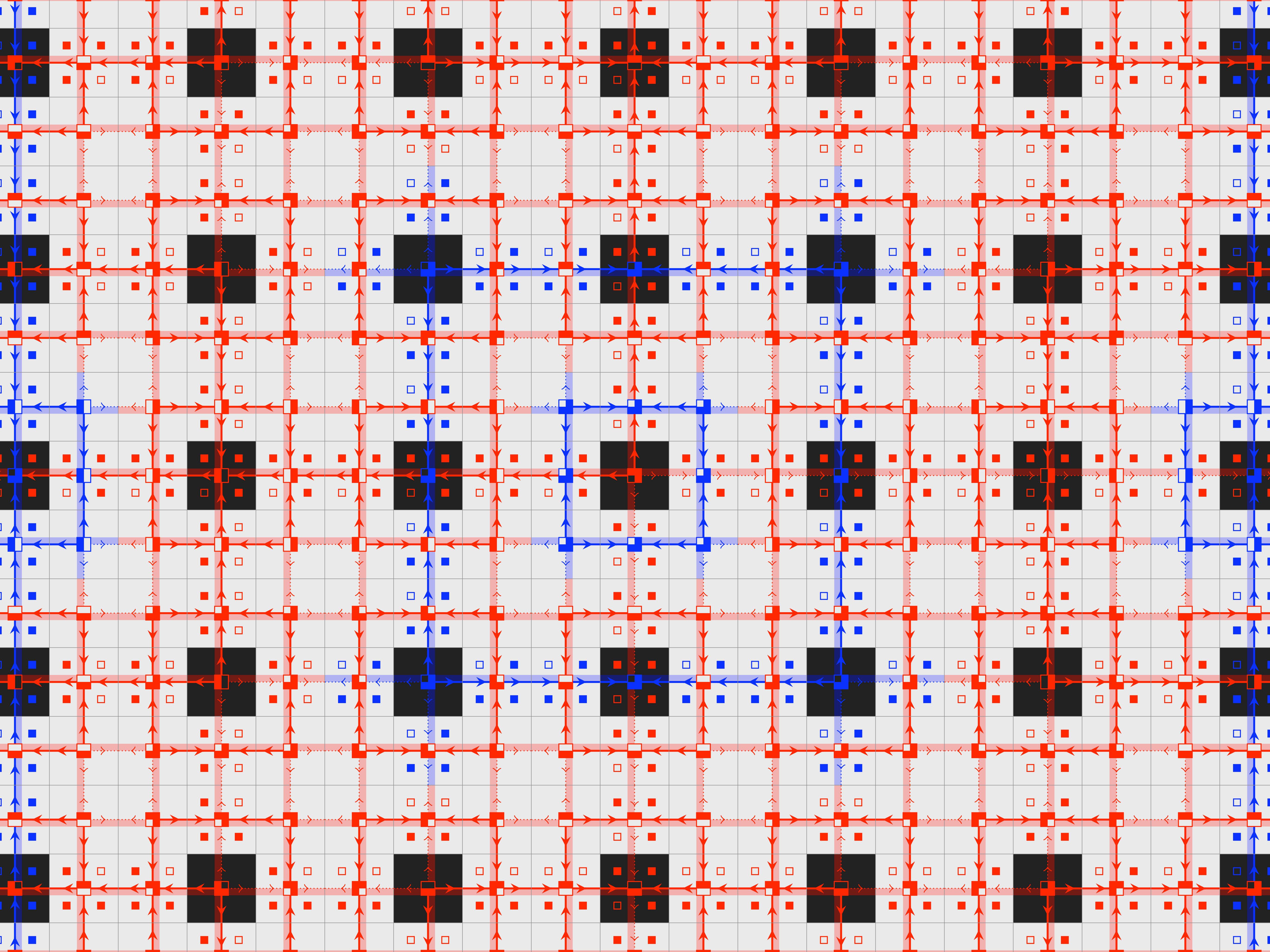


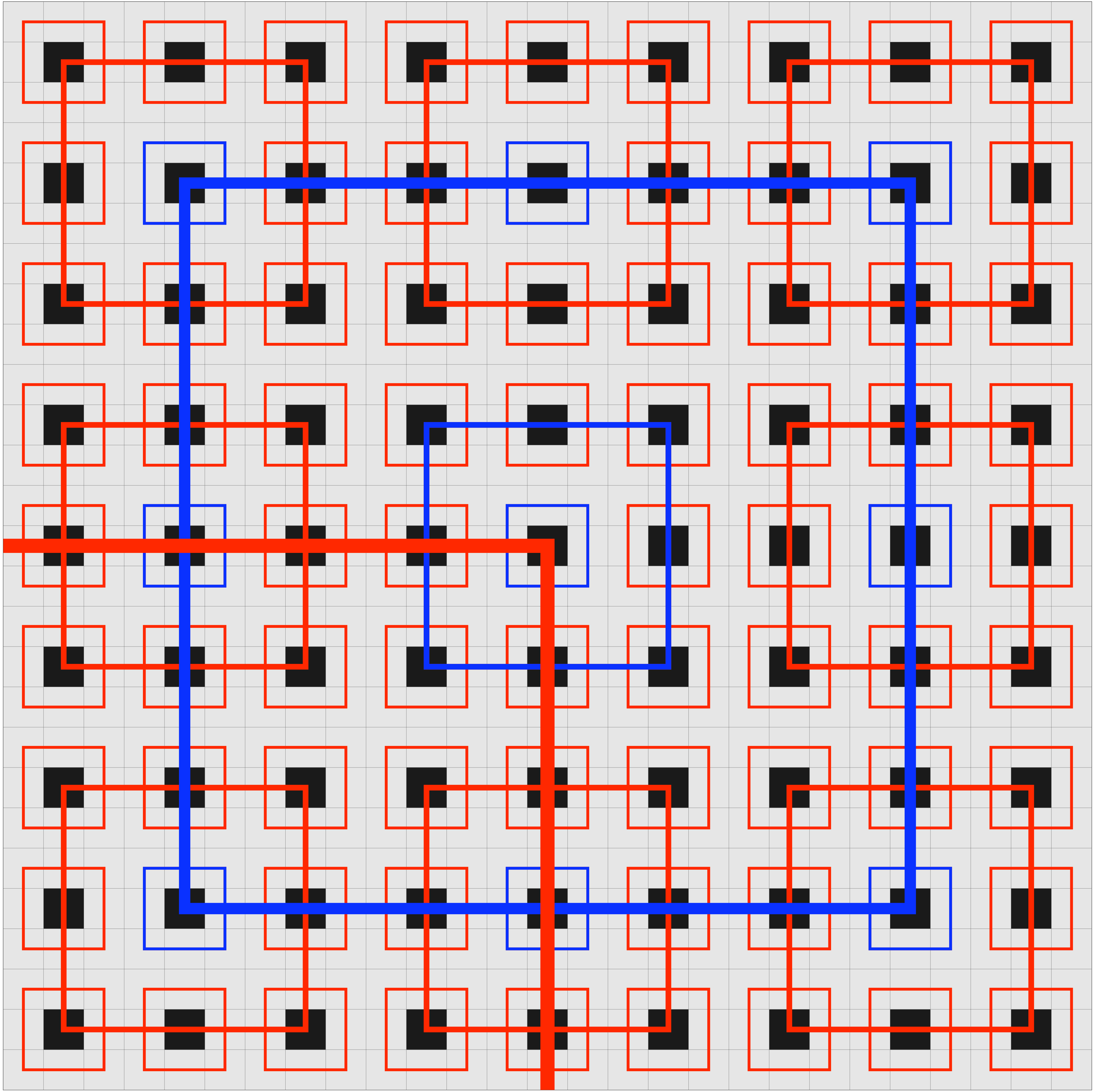
Substitution rule











Local rule: *grid*

Forcer 3x3

Local rule: *threads*

Sur les fils : cohérence partout

Pointillé face pointillé bien orienté : on applique *grid*

Local rule: *history*

cohérence étiquette/fils traversés

Lemma 1

Tout pavage valide se découpe en grille 3x3 d'image par la substitution

Grid + Threads + History

Lemma 2

L'image inverse d'un pavage est un pavage

grid: vient de threads
threads: on a juste enlevé des fils
history: on a juste enlevé des fils

Corollary

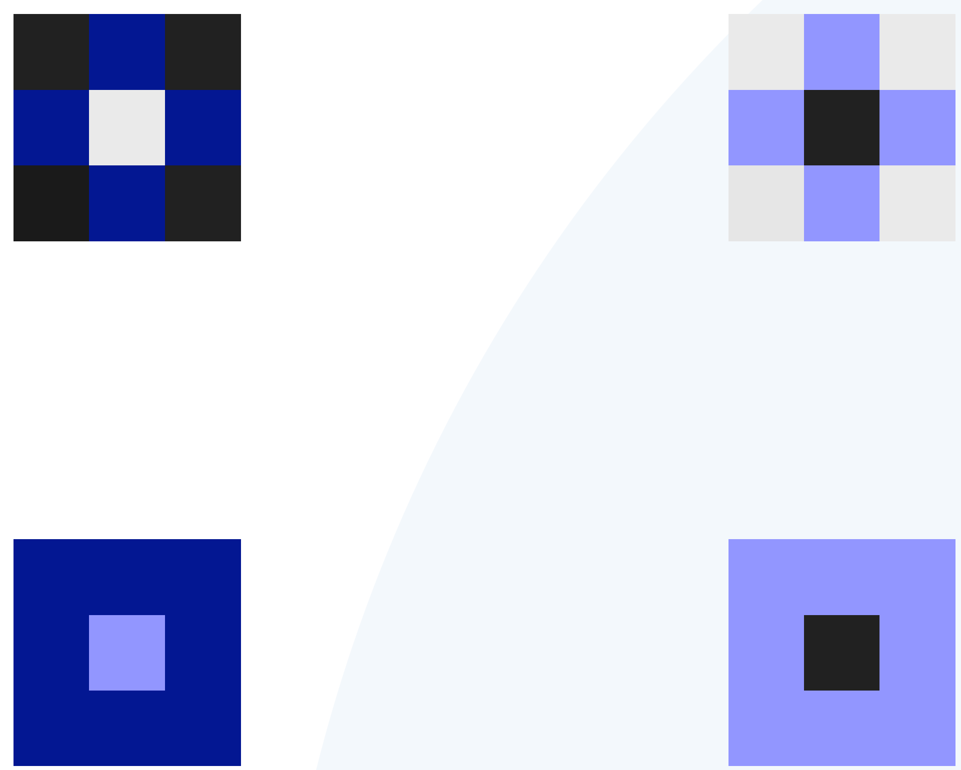
All the valid tilings are aperiodic

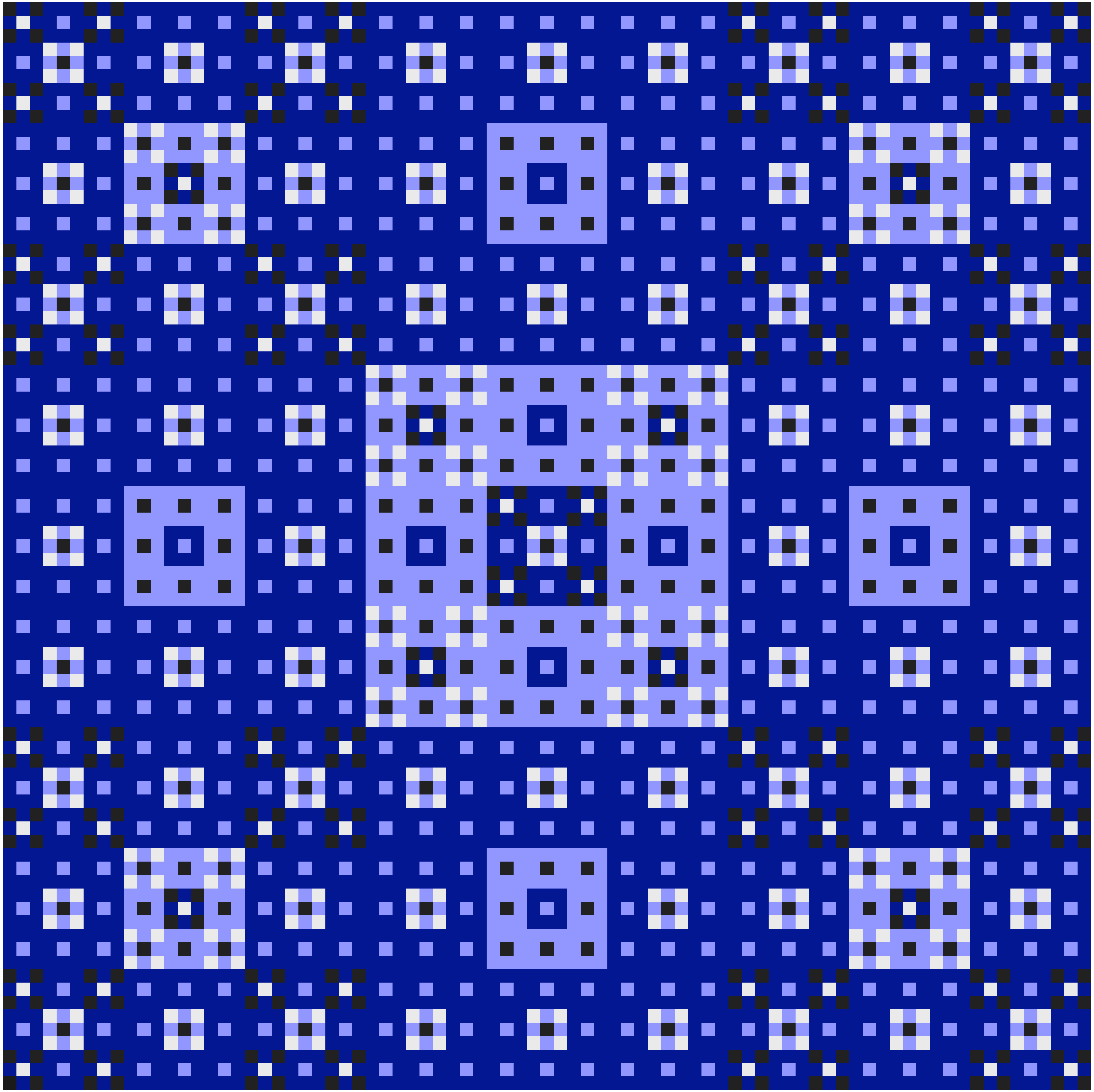
Valid tilings = limit set

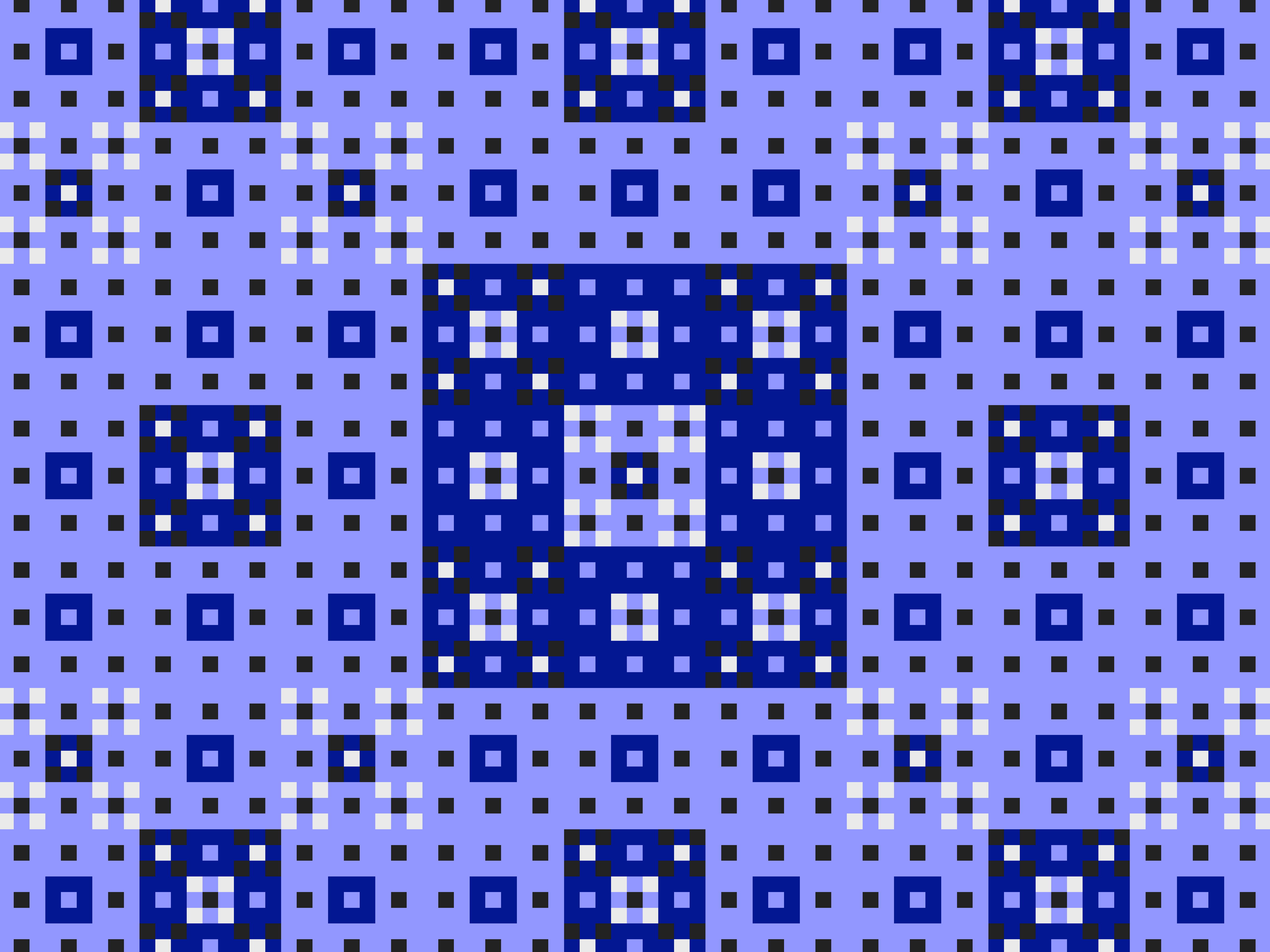
Determinism

expliquer...

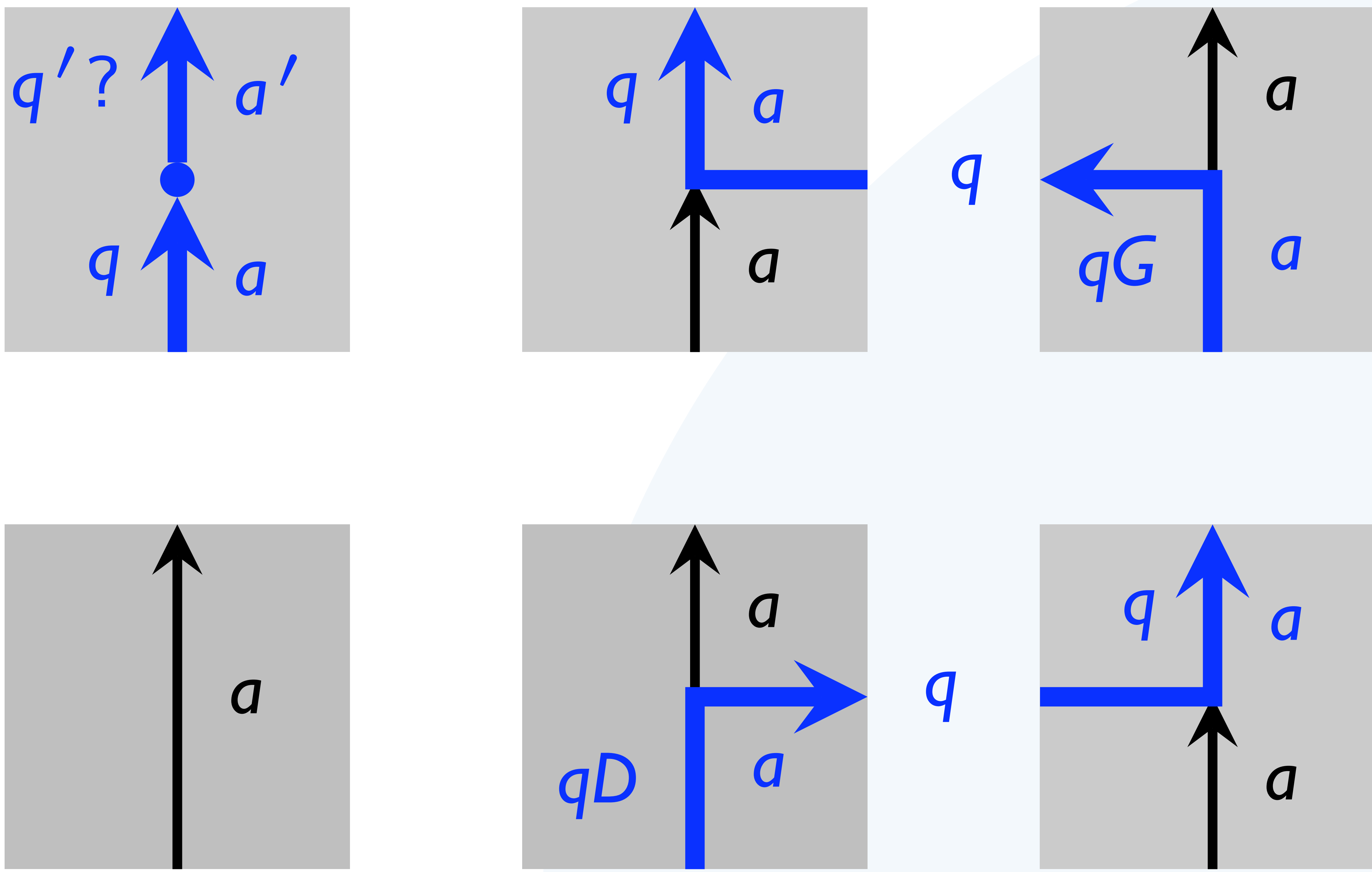
Shadows for TM



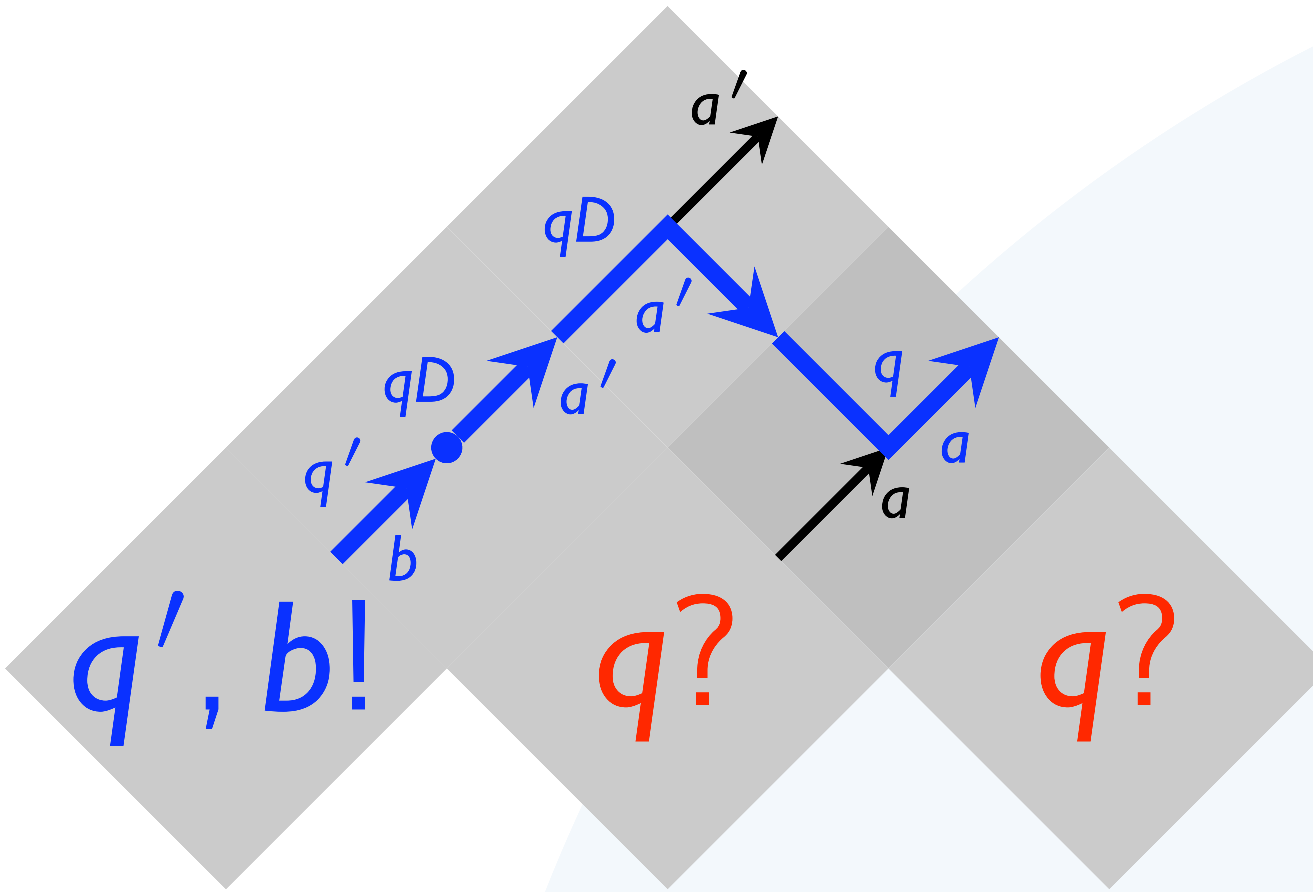




Slow TM simulation



Determinism



Conclusion

Nil1D est indécidable

Les autres preuves classiques d'indécidabilité s'adaptent aussi très bien à ces pavages : Surj 2D, Inj 2D, Nil Pér, etc