104 tiles

N. Ollinger (Escape, LIF)
in Frac d'automne 2007

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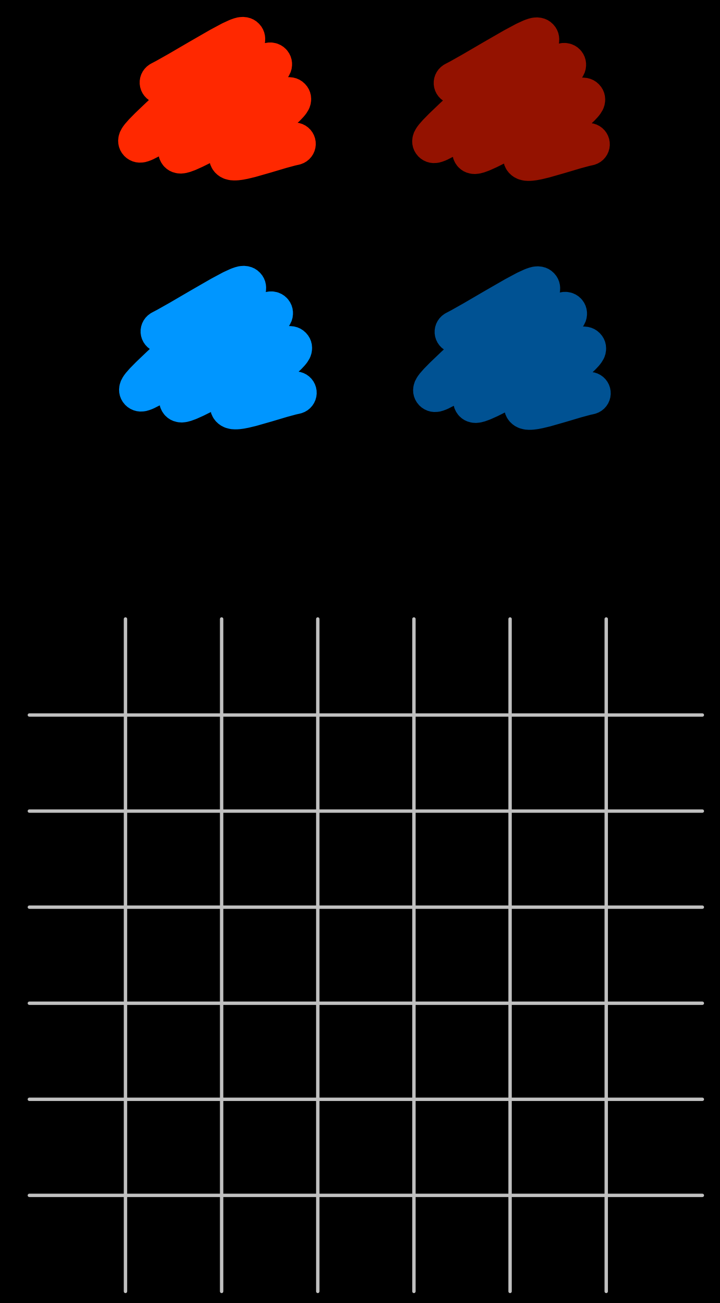
Foreword

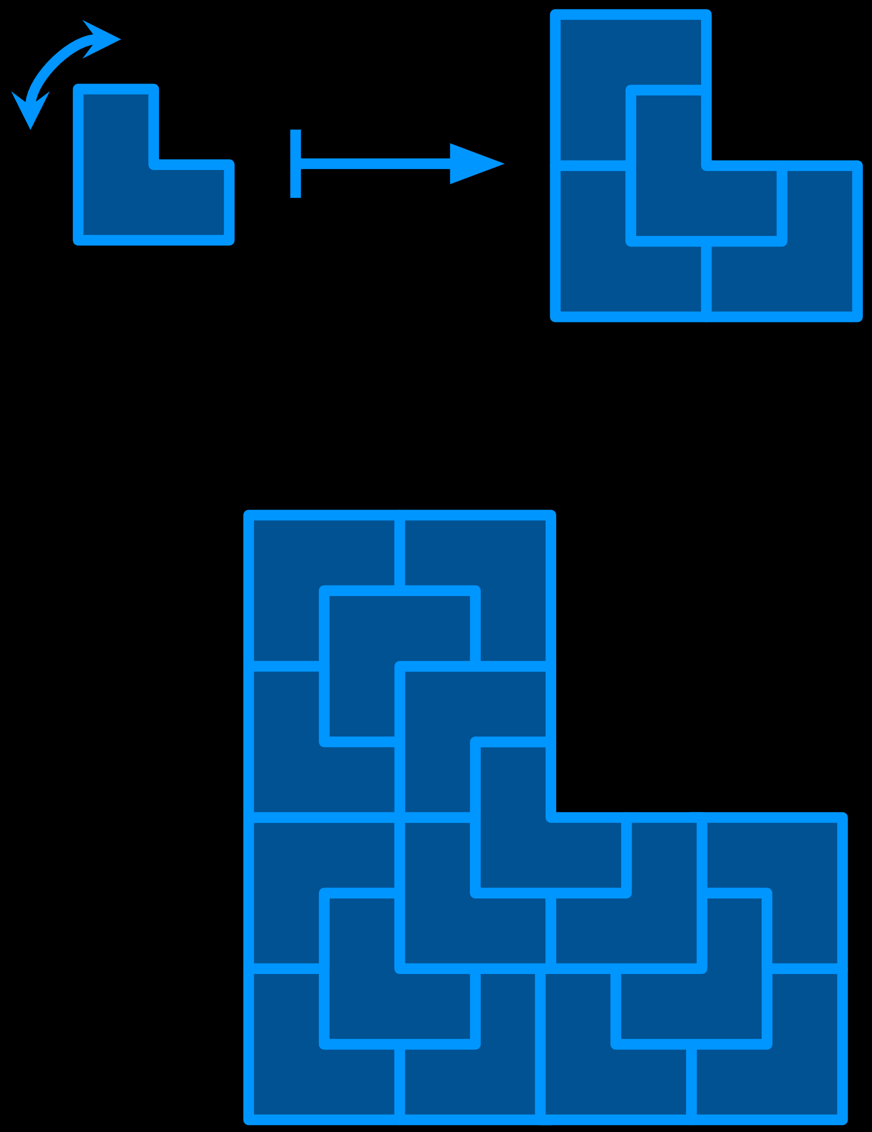
- Yet another presentation on the undecidability of the Domino Problem **[Berger 64]**
- Formal stuff (a few) stand on white background.
- Pictures (a lot of) stand on black background.
- This talk is very geometrical: we draw pictures on the Euclidian plane.



2×2 Substitutions

Substitutions

- Geometric substitutions provide a convenient recursive way to define aperiodic colorings of the plane.
- Subtle geometrical arguments are required: discuss dissection, inflation, scaling factor, etc.
- The classical L (or chair) substitution.

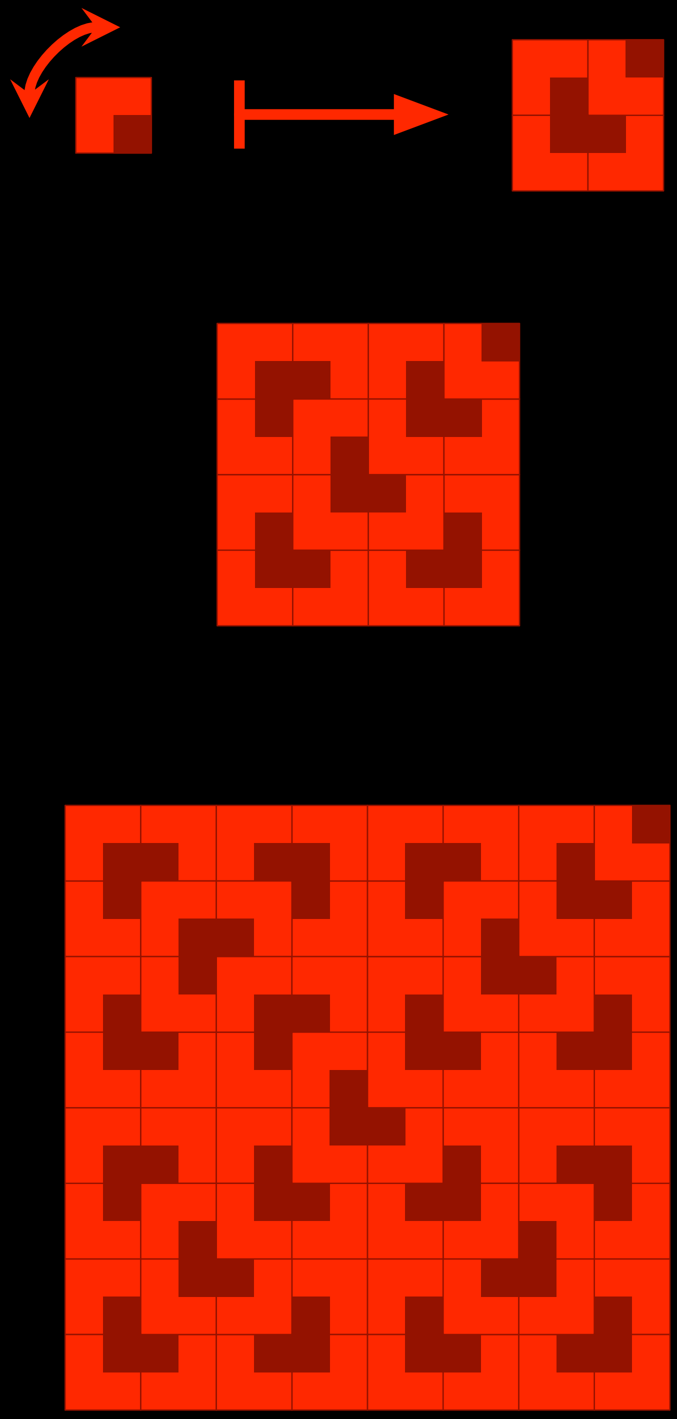


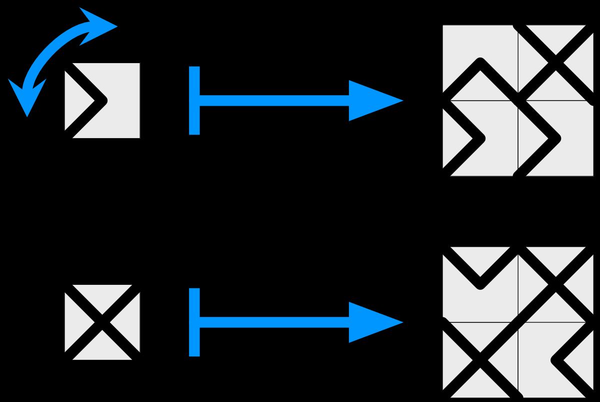
2×2 Substitutions

- A 2×2 substitution maps a finite alphabet to 2×2 squares of letters on that alphabet.
- The substitution is iterated to generate bigger squares.

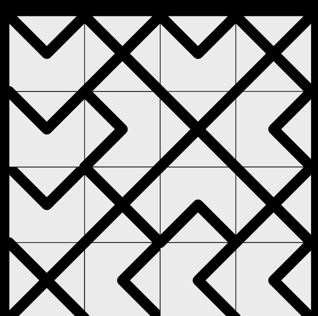
$$s : \Sigma \longrightarrow \Sigma^{\boxplus}$$

$$S \circ \sigma_u = \sigma_{2u} \circ S$$

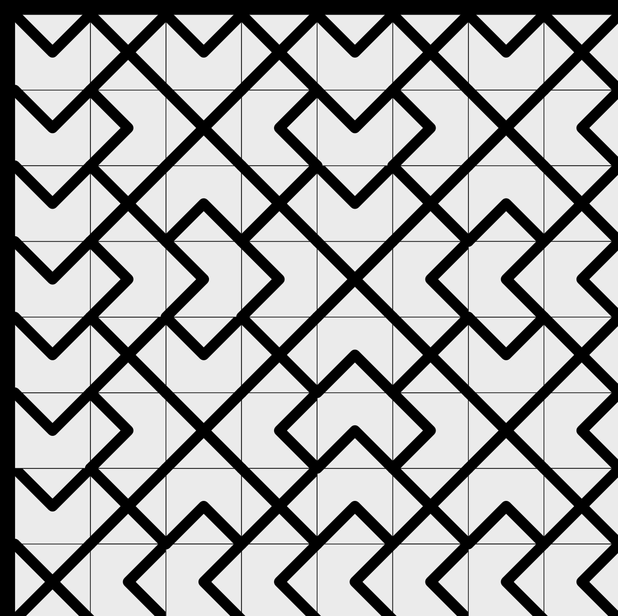




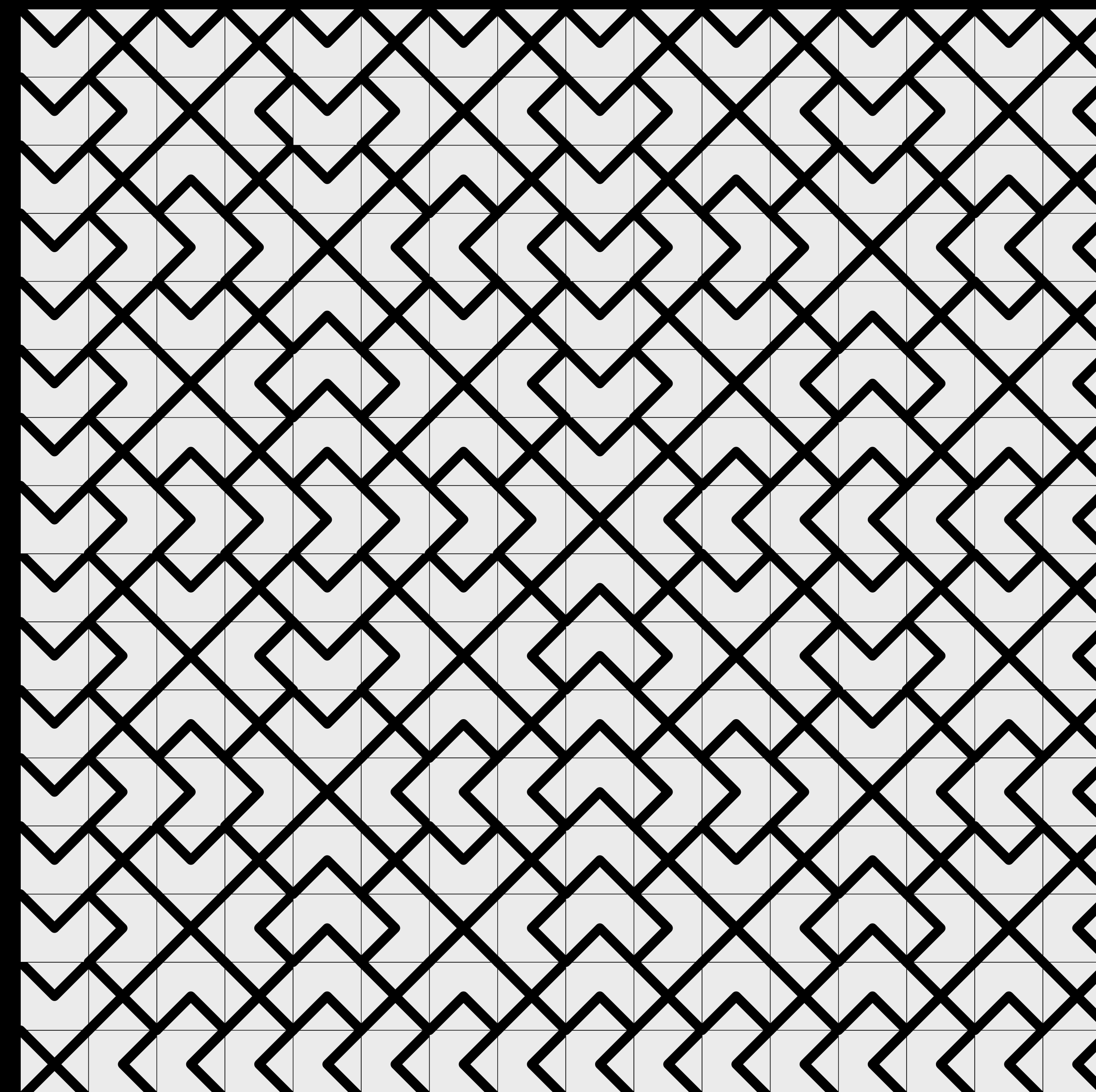
(5 letters)



$s^2(\boxtimes)$



$s^3(\boxtimes)$



$s^4(\boxtimes)$

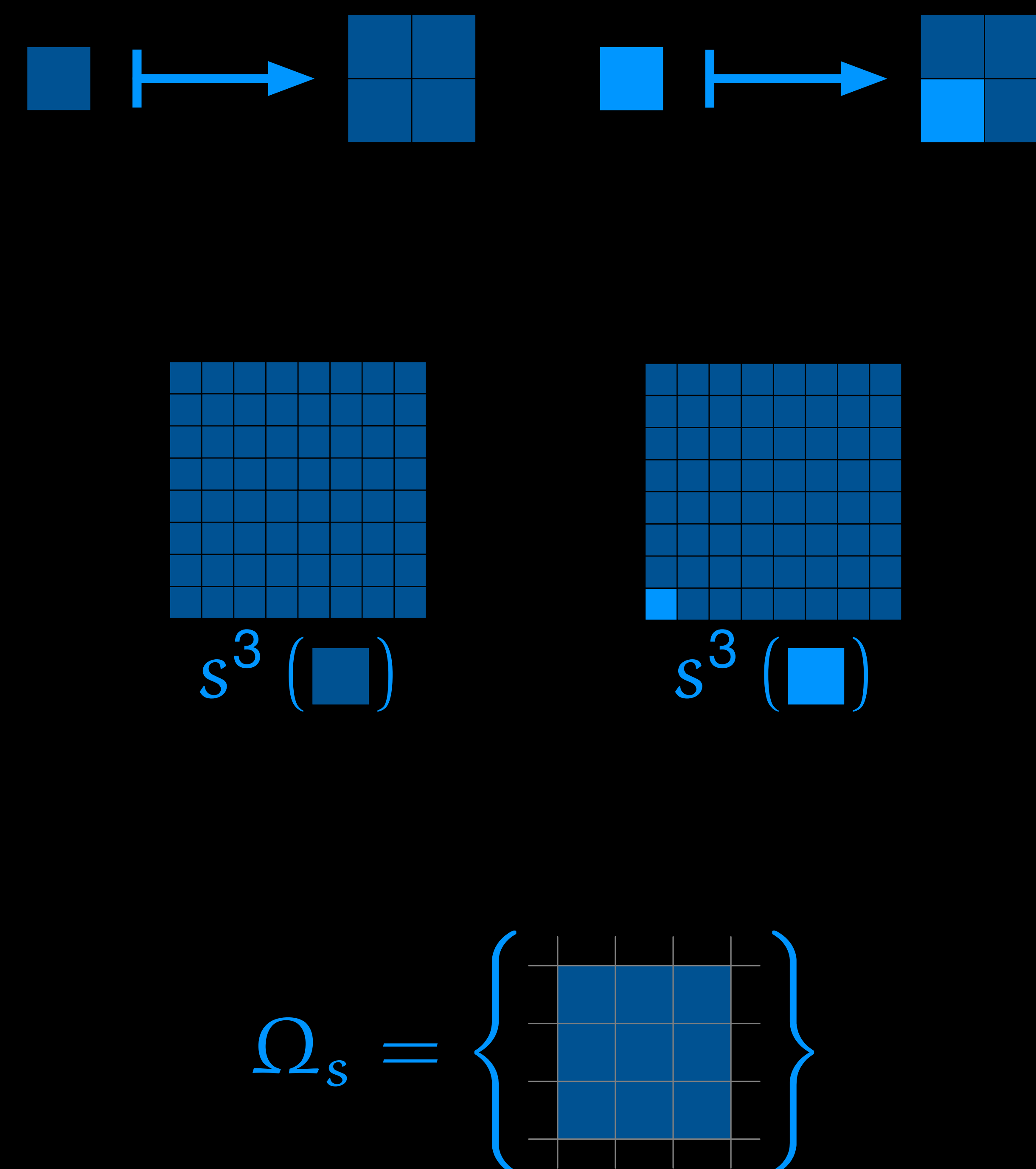
Another L-style substitution

Pattern closure?

- What is a coloring of the plane generated by a substitution?
- One possibility is to consider a *closure* of generated patterns up to translations.
- Difficult to check...

$C \in \Omega_s$ if

$$\forall P \prec_{\mathcal{F}} C, \quad \exists a, k, \quad P \prec s^k(a)$$

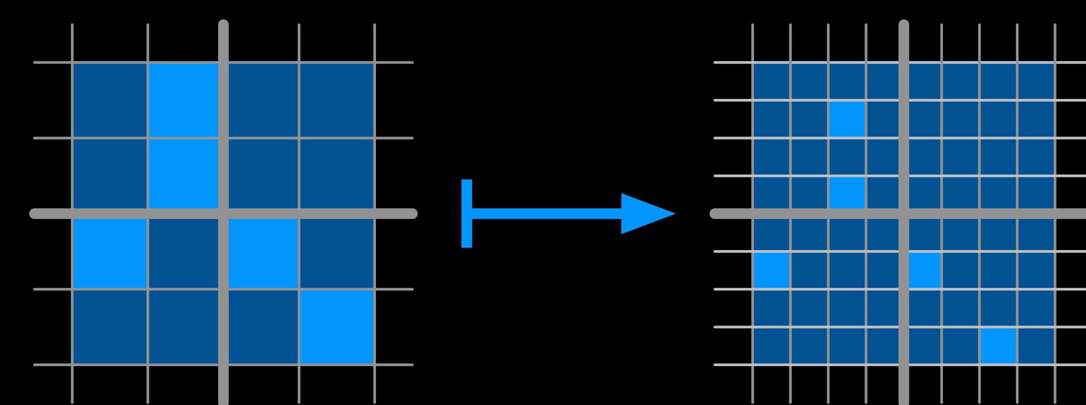


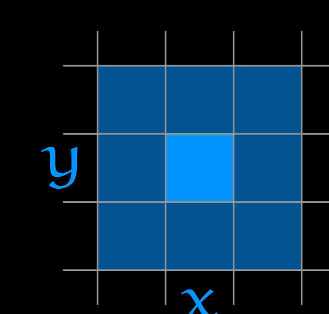
Limit set!

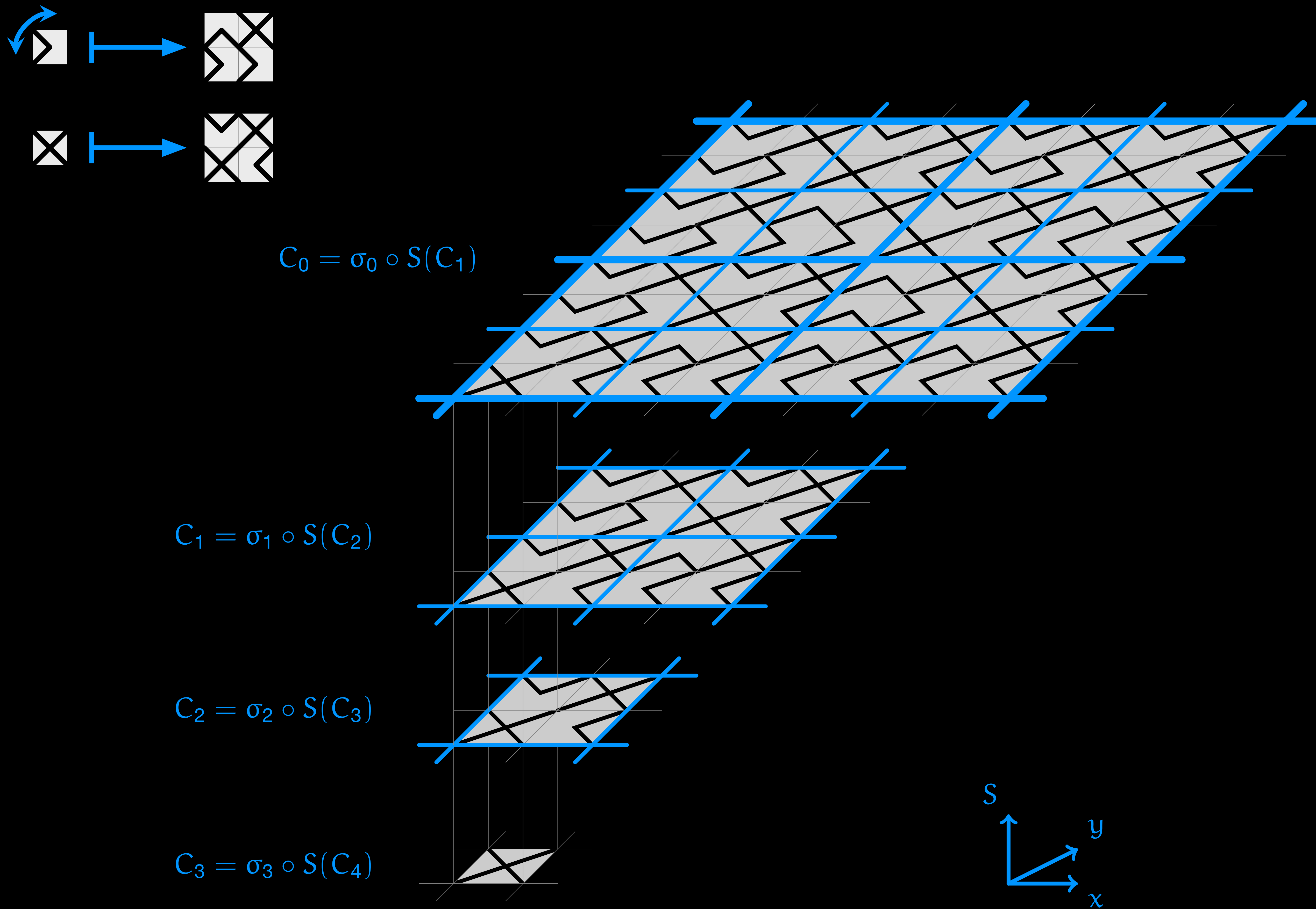
- Extend the substitution to a global map inflating around 0.
- Take the limit set of iterations of this map closed up to translations.
- More colorings!

$\Lambda_s = \bigcap_n \Lambda_s^{(n)}$ where

$$\Lambda_s^{(0)} = \Sigma^{\mathbb{Z}^2}$$
$$\Lambda_s^{(n+1)} = \left\langle S(\Lambda_S^{(n)}) \right\rangle_\sigma$$



$\Lambda_s = \Omega_s \cup \Bigg\{$  $\Bigg\}_{x,y \in \mathbb{Z}^2}$



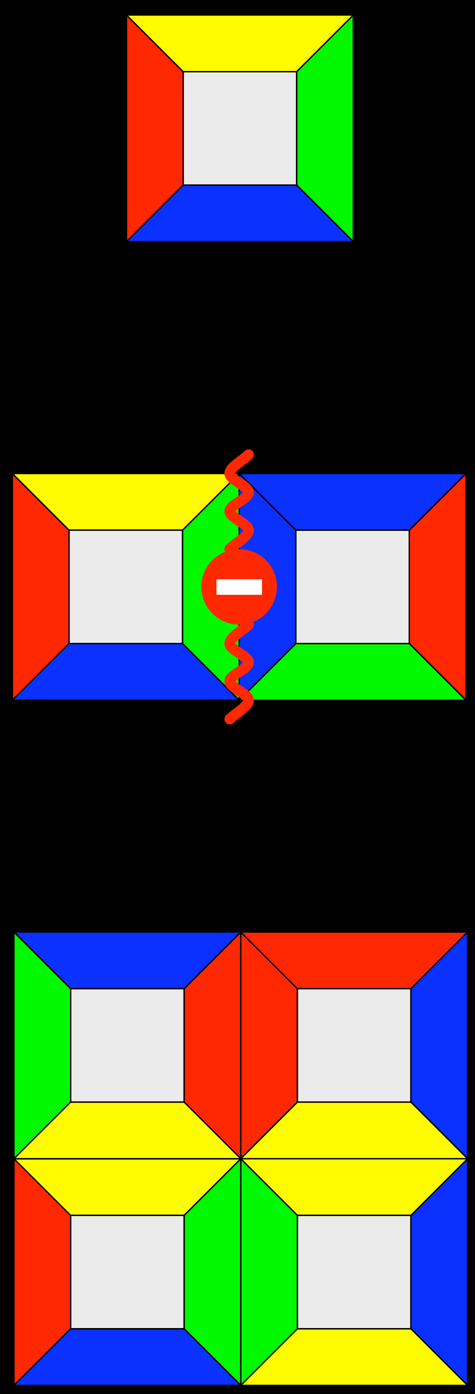
History of a coloring in the limit set

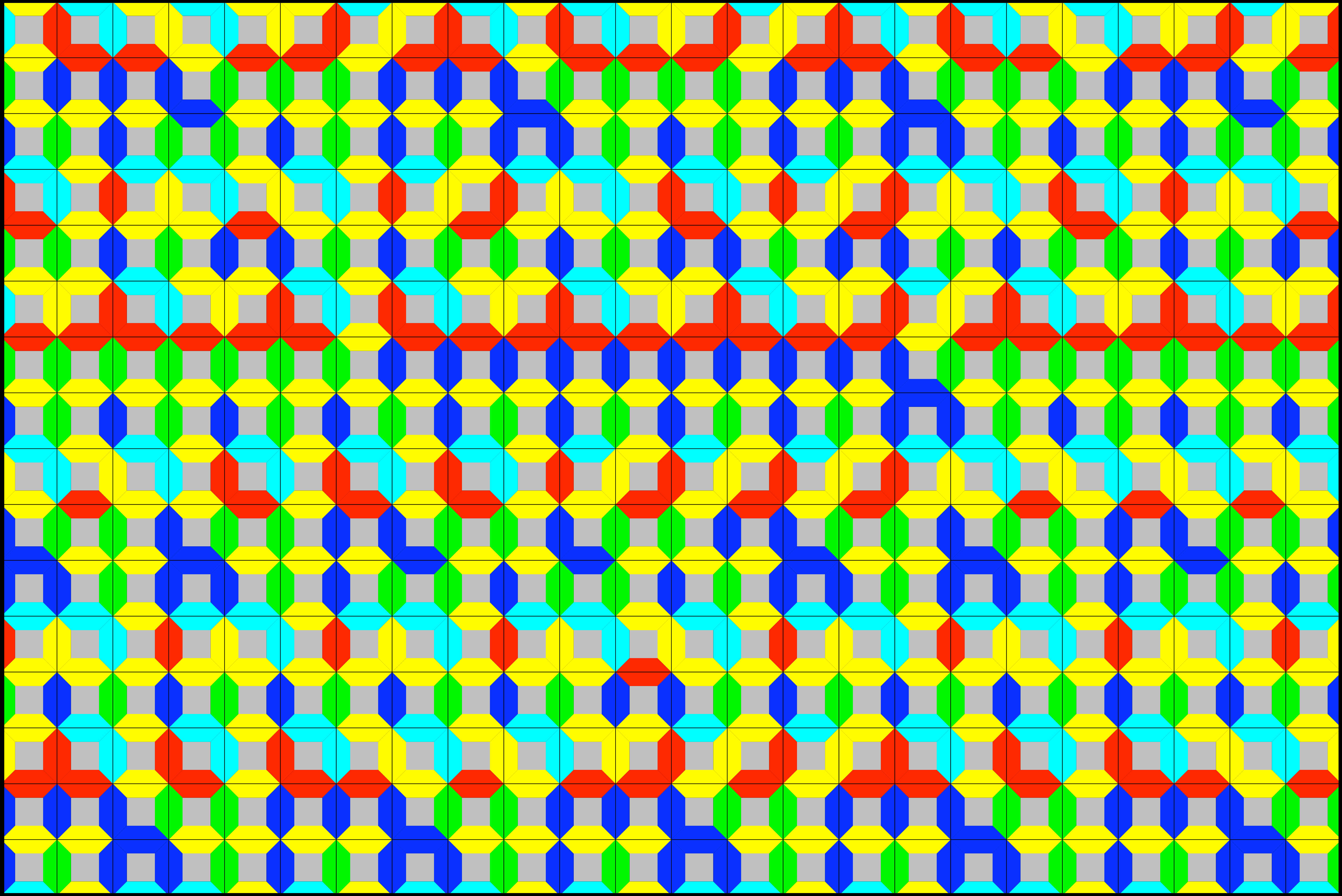
Tiling with Wang tiles

Wang tiles

- A Wang tile is a unit square with colored edges.
- A tileset is a finite set of Wang tiles.
- Tiling is done with translations only (no rotation) by matching colors along edges.

$$\tau \subseteq C^4$$

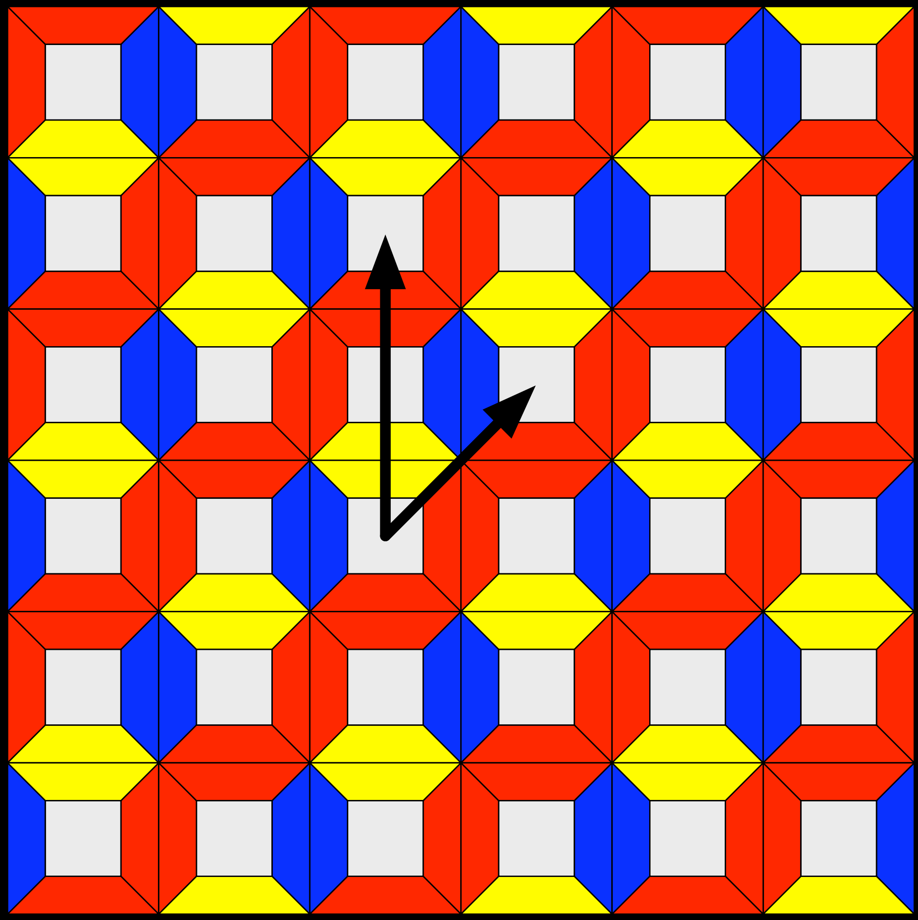




Tiling the whole plane with Wang tiles

Periodicity

- A tiling is periodic if it admits a periodicity vector.
- A tiling is biperiodic if it admits two non colinear periodicity vectors.
- If a tileset admits a periodic tiling then it admits a biperiodic tiling.
- A tileset is aperiodic if it admits no periodic tiling.



The Domino Problem

- **The Domino Problem:** given a tileset, decide if it admits a tiling.
- A long time ago, Wang conjectured there exists no aperiodic tileset and thus the Domino Problem is recursive.
- In 1964, his PhD student Berger proved that the problem is not recursive. To achieve this goal he encoded Turing machine computations inside aperiodic tilings.
- A whole talk could be devoted to the history of aperiodicity, the quest for small aperiodic tileset, the different proofs and technics to prove the non recursivity of the domino problem... **Not today!**

Au menu

- Define a particular set of 104 tiles;
- Prove in details its aperiodicity;
- Briefly explain how to complete the Domino Problem construction.
- This is a mix of different tools and ideas from:

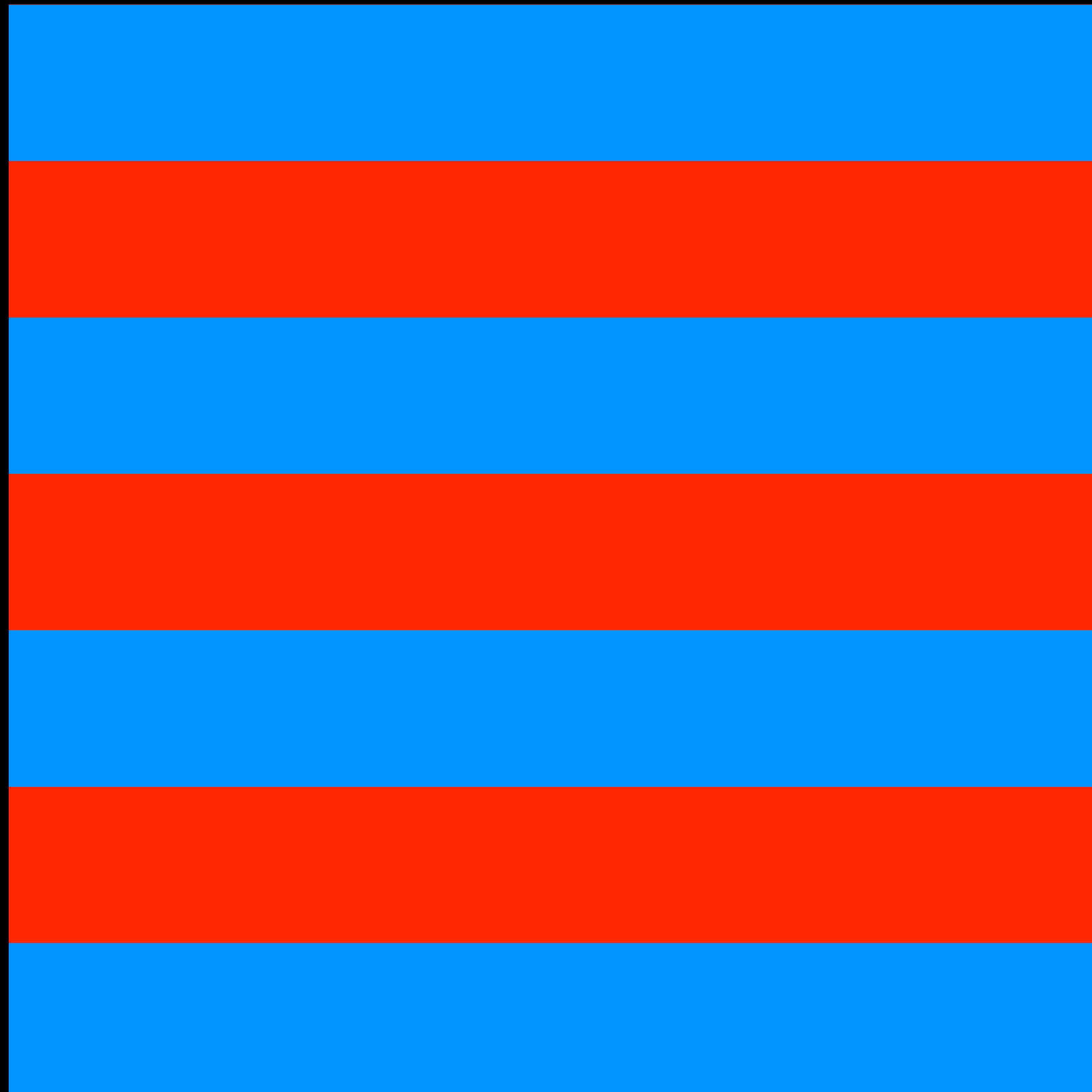
[Berger 64] *The Undecidability of the Domino Problem*

[Robinson 71] *Undecidability and nonperiodicity for tilings of the plane*

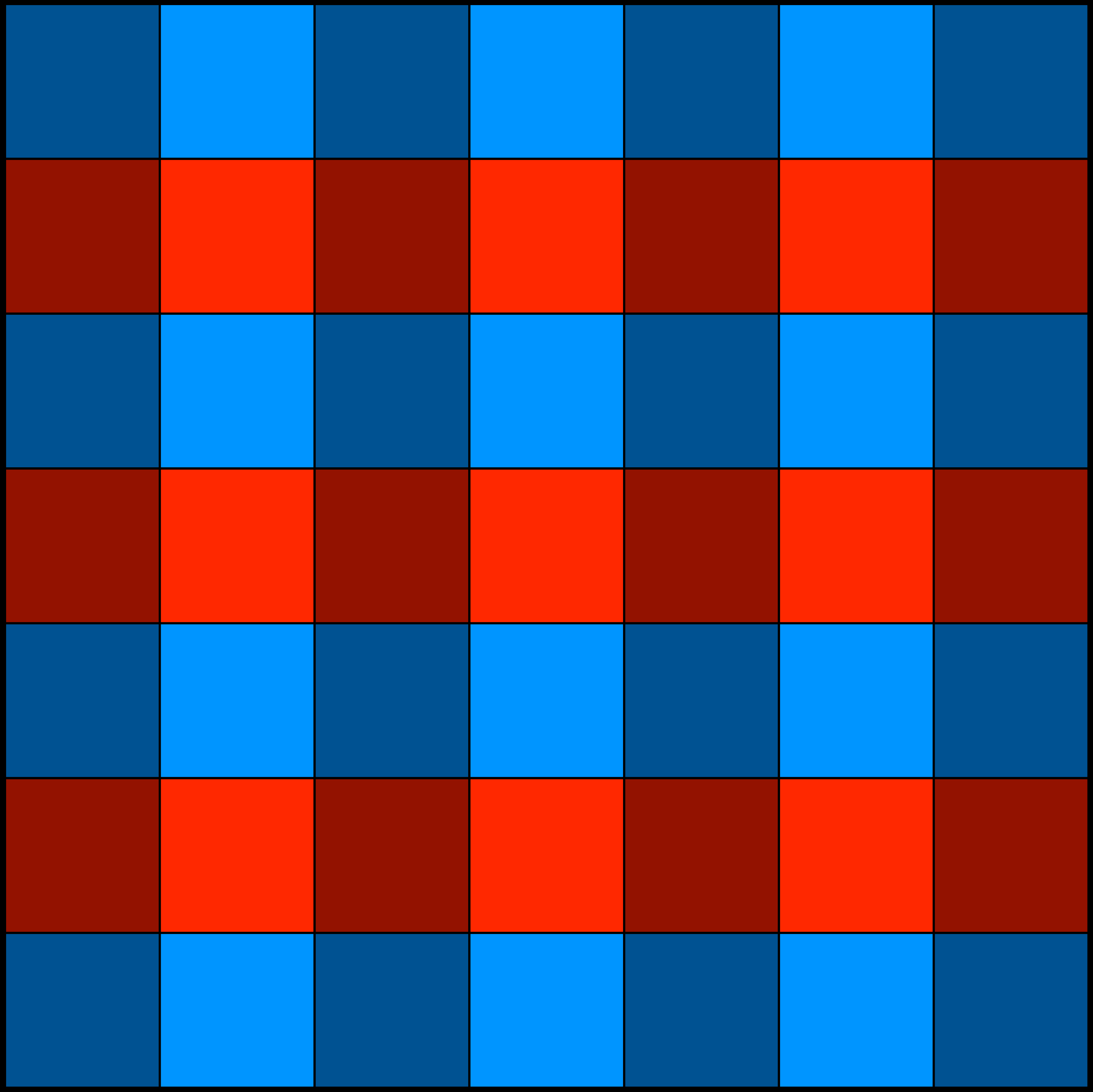
[Grünbaum Shephard 89] *Tilings and Patterns, an introduction*

[Durand Levin Shen 05] *Local rules and global order, or aperiodic tilings*

A set of 104 tiles



Encoding an infinite number of grids



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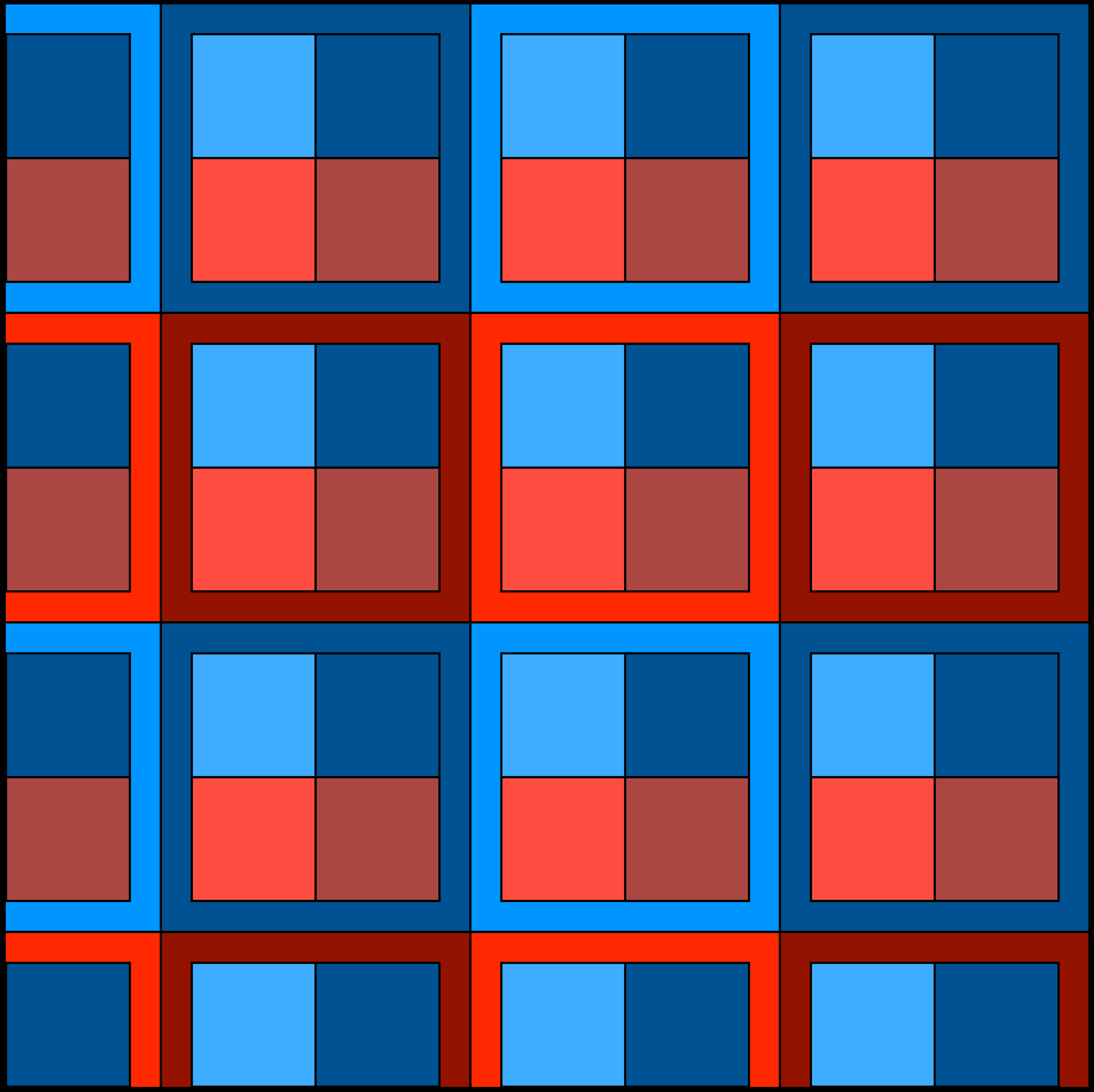

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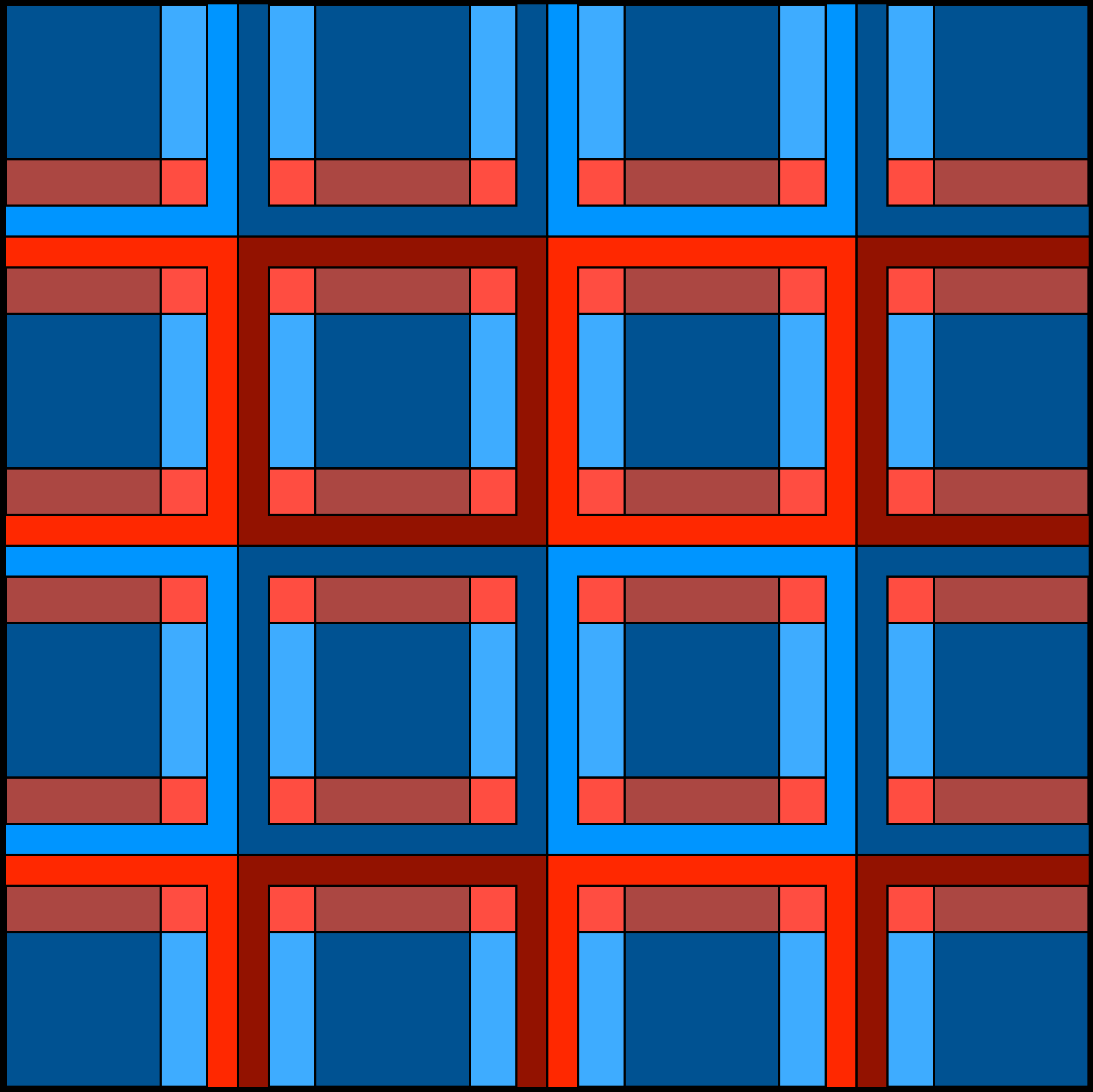

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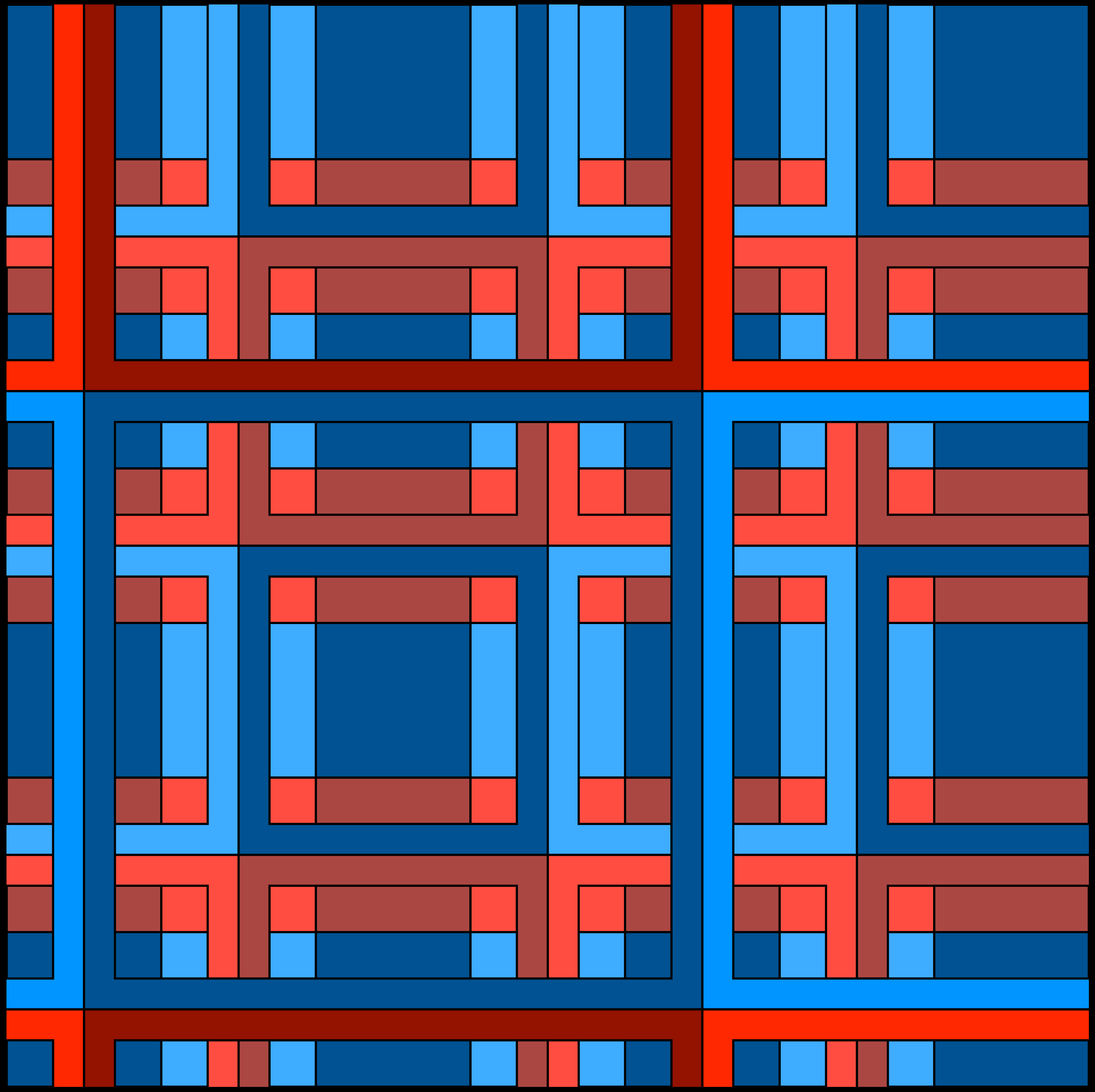

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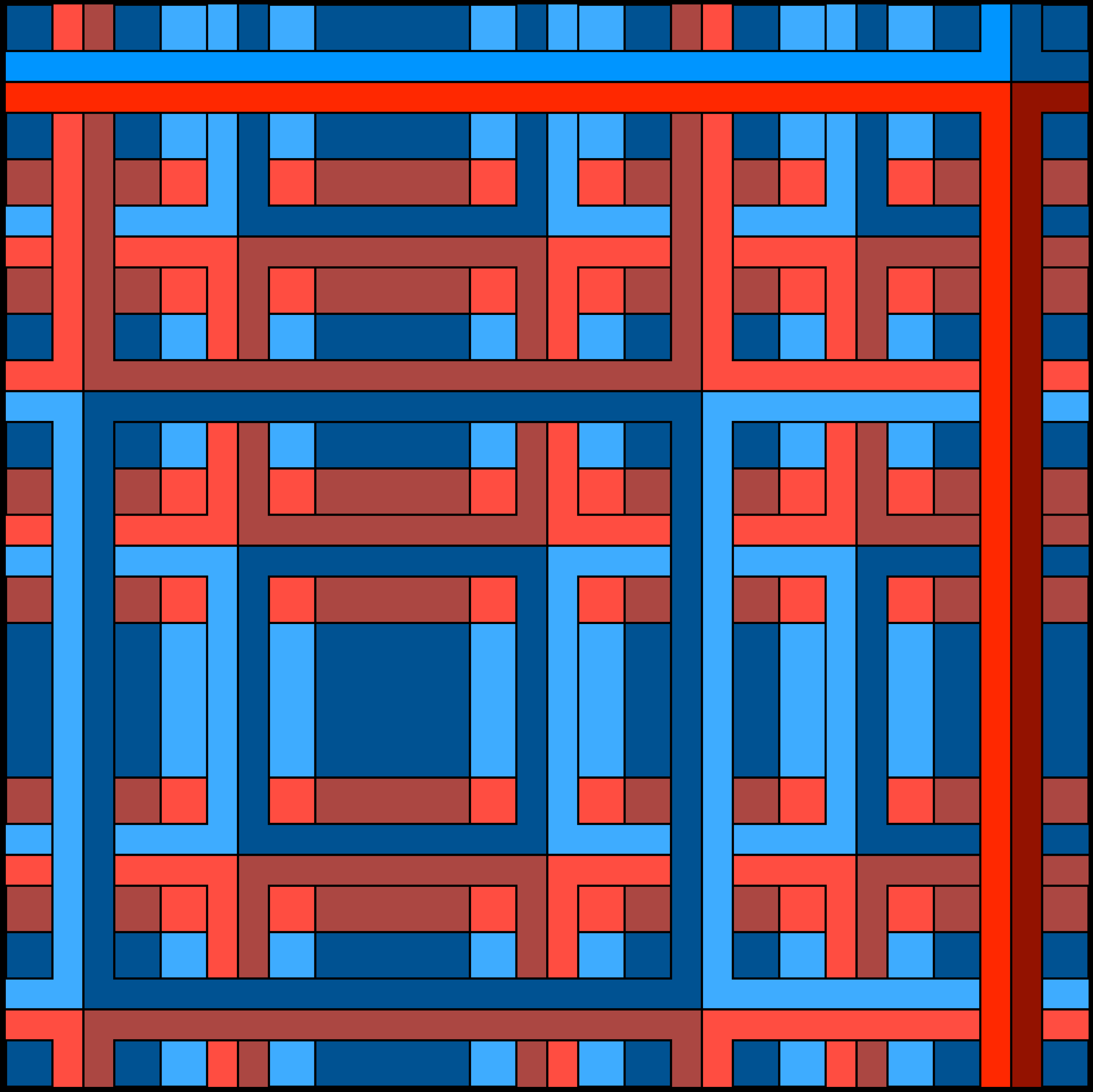

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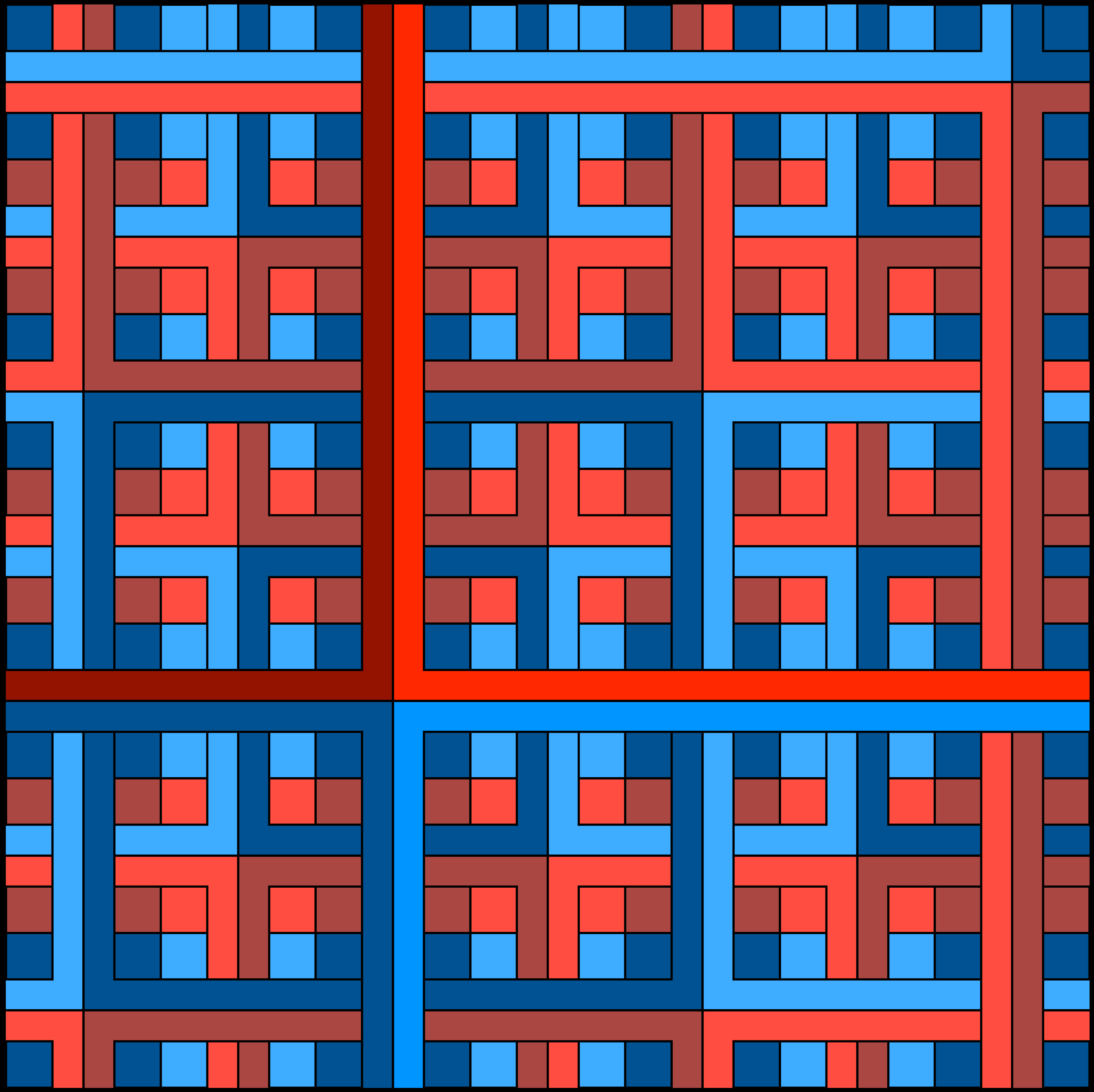

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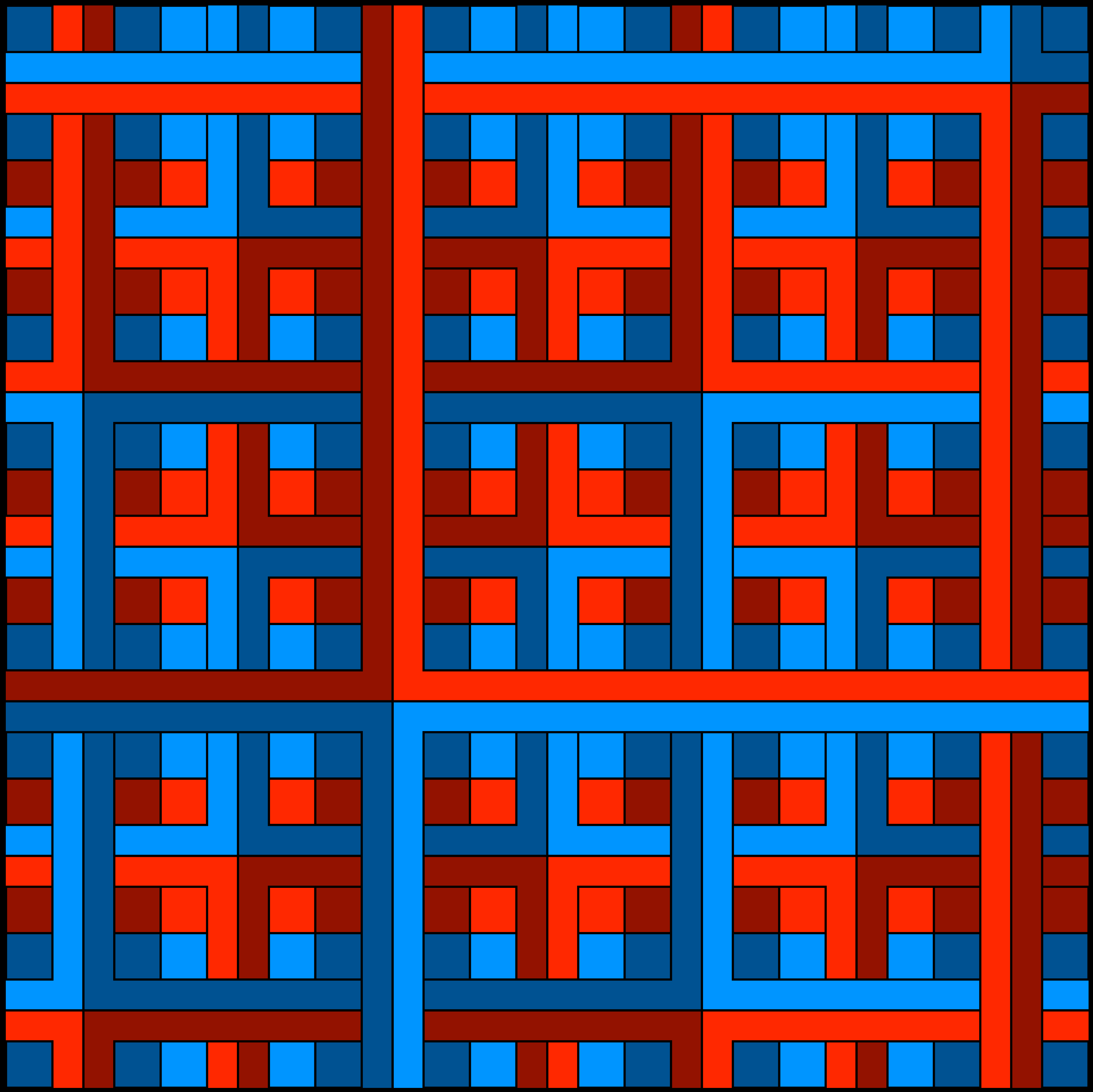
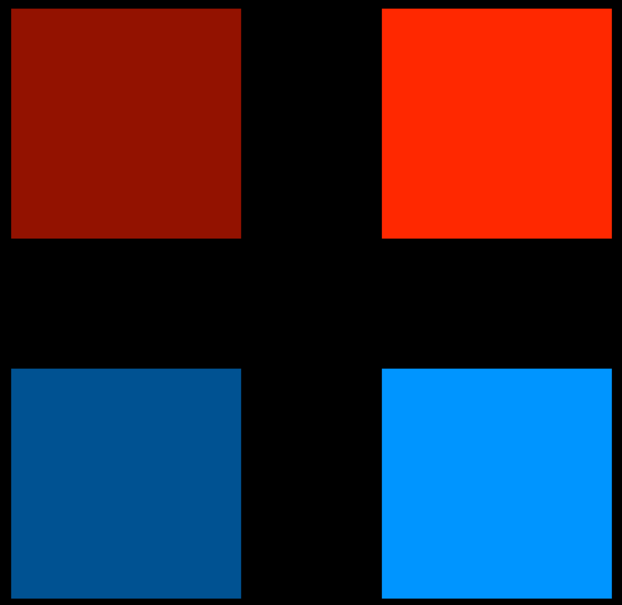
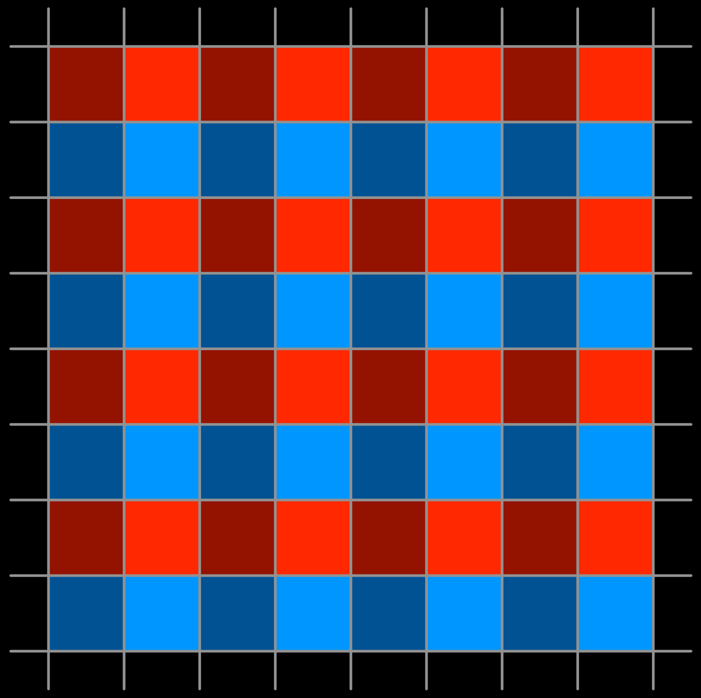


Encoding an infinite number of grids

Layer 1. initialization

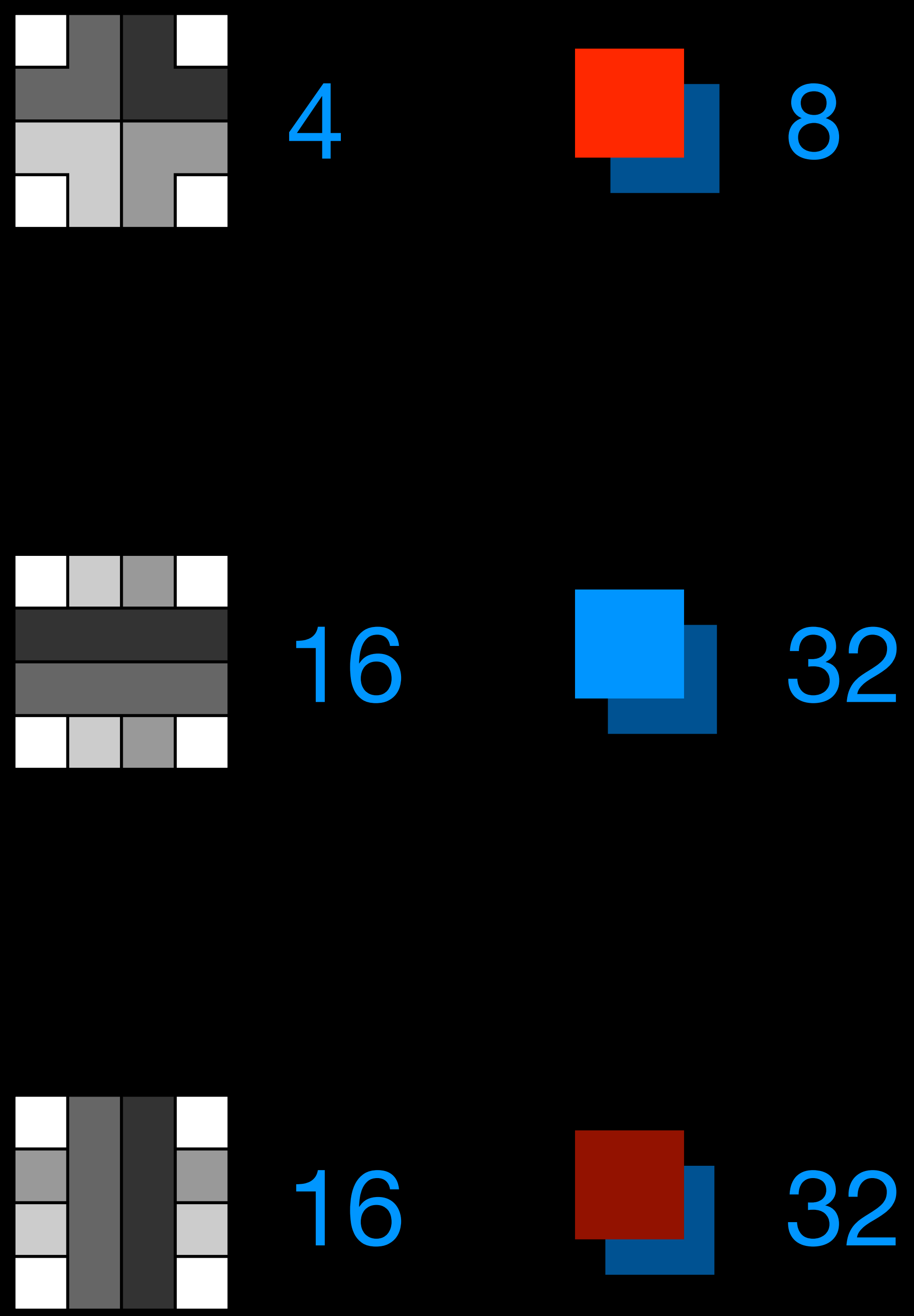
- 4 tiles to group tiles 2×2.
- Alternate red/blue vertically.
- Alternate light/dark horizontally.
- Simple and very constrained matching rule.





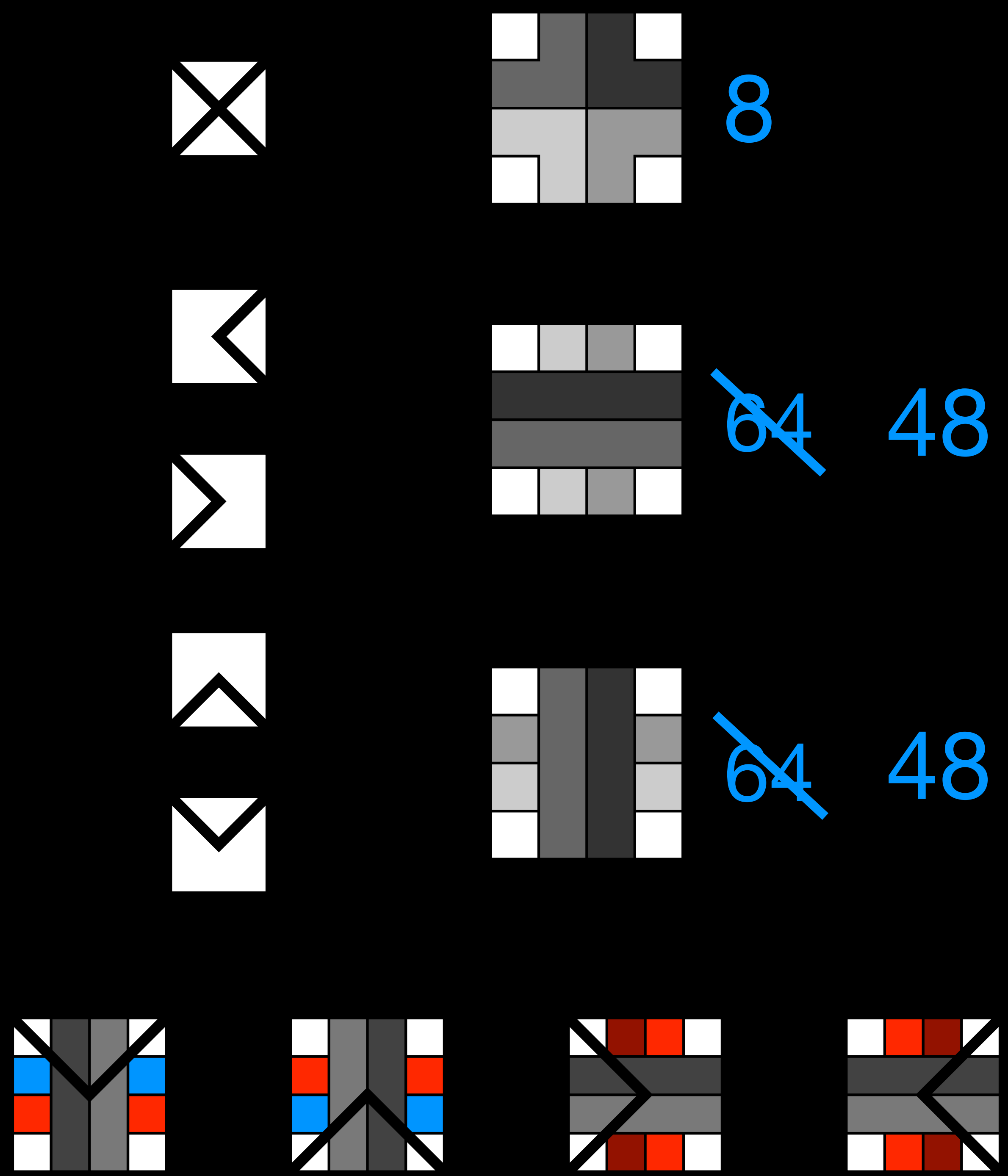
Layer 2. grids

- Tiles carry wires, two on each edge.
- Three kinds of tiles: X, H, V.
- Pairs of wires should be compatible.
- Matching rule: wire colors should match.
- Restrict layer 1 by kind.



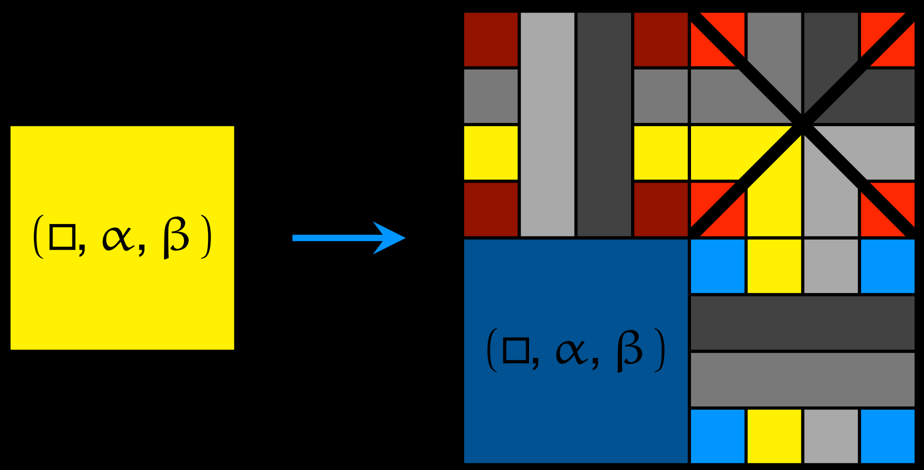
Layer 3. square-*ifier*

- One more bit of information per edge to force squares.
- 5 different tiles with arrows.
- Matching rule: arrows should match along edges.
- Restrict arrows by kind.
- **Important:** red wire is a one-way lane.

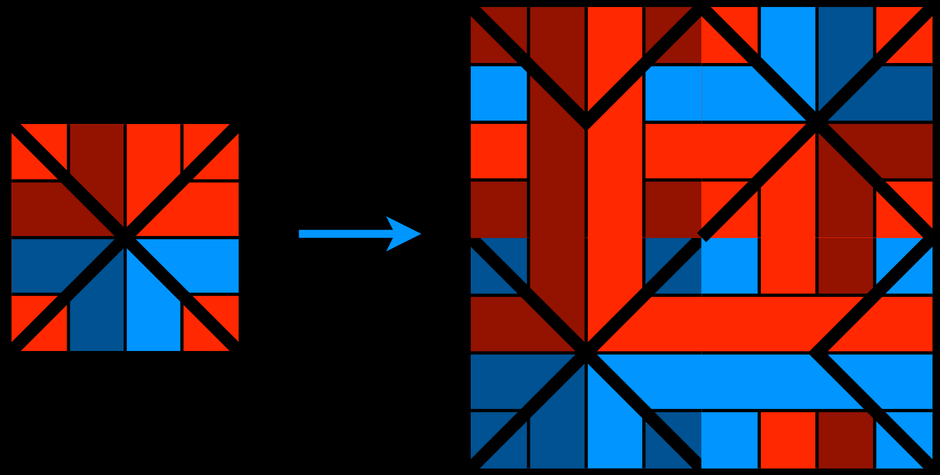


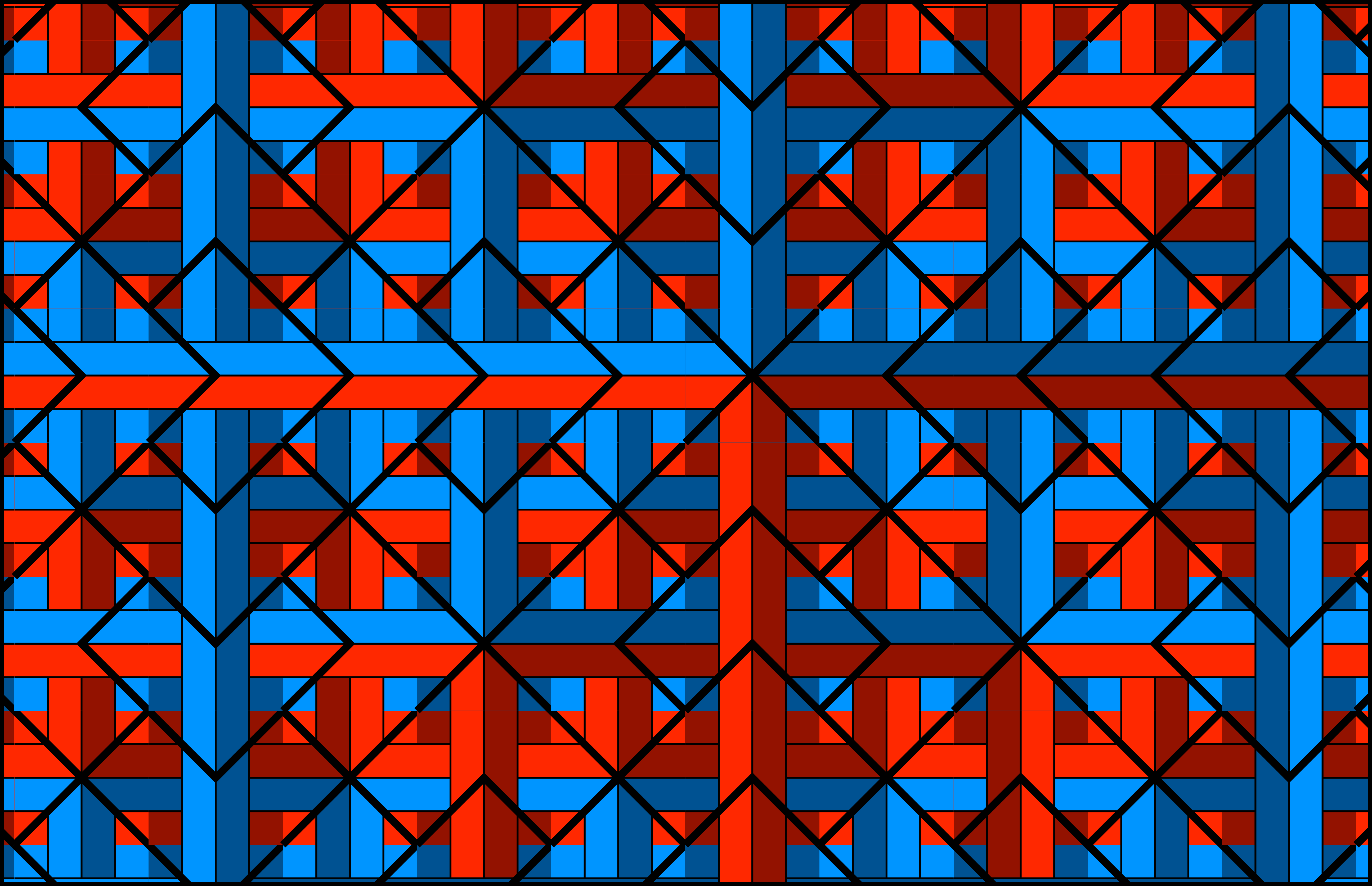
Canonical substitution

- Copy the tile in the SW corner but for layer 1.
- Put the only possible X in NE that carry layer 1 of the original tile on SW wire.
- Propagate wires colors.
- Let H et V tile propagate layer 3 arrows.
- The substitution is injective.

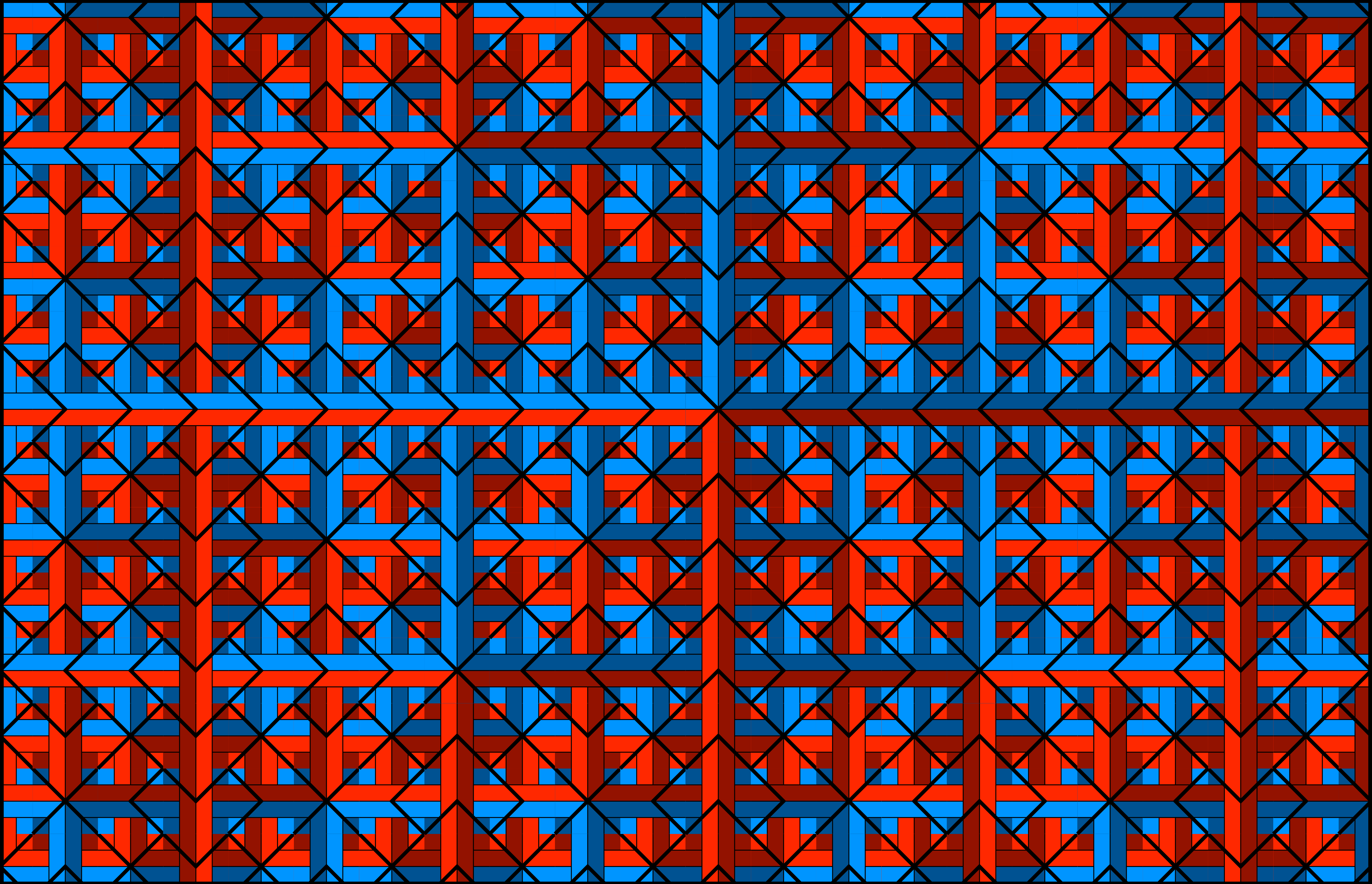


example:





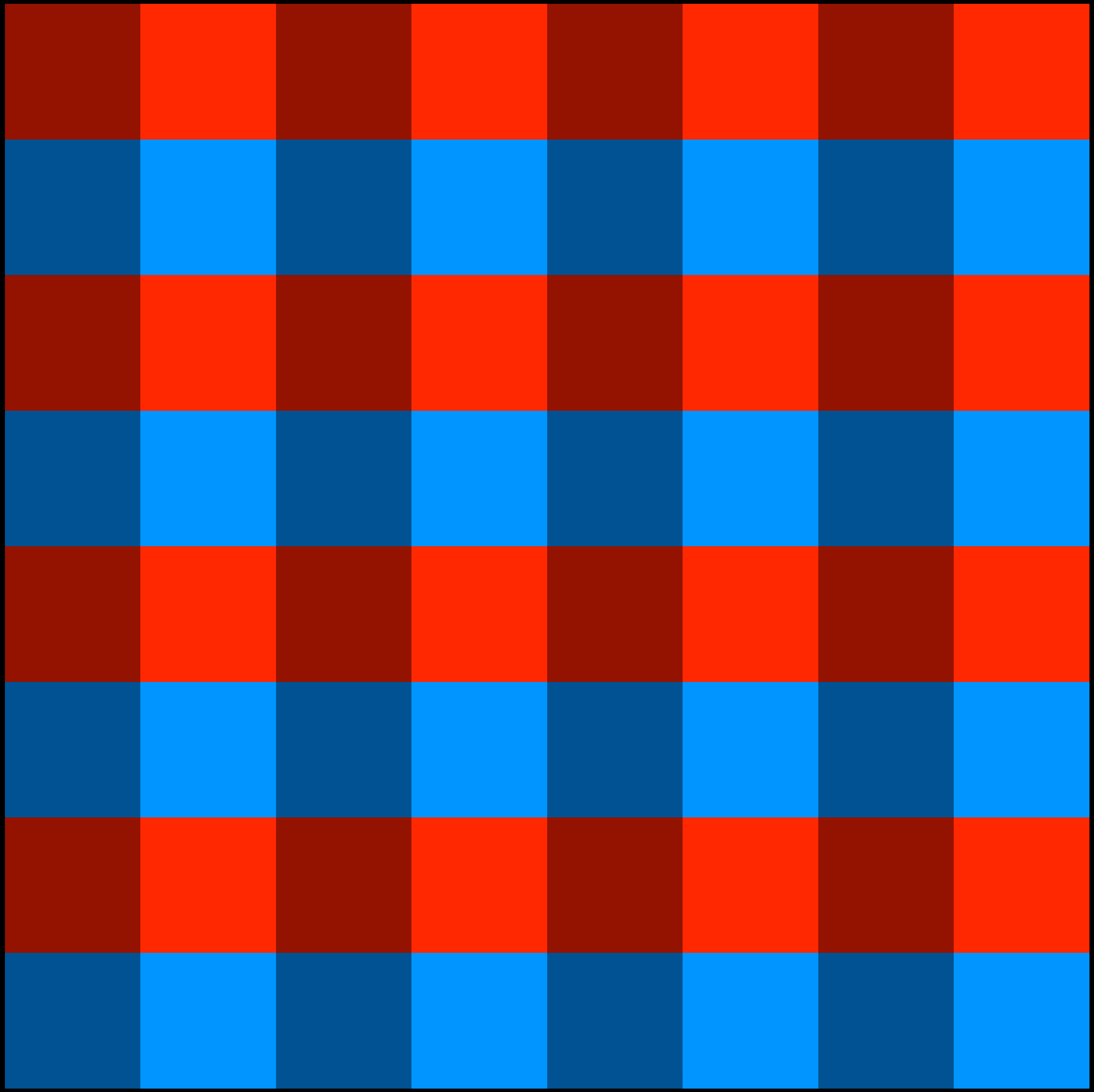
The tileset can tile the whole plane...



...extend *ad lib*!

This set of tiles is aperiodic

Lemma 1. each tiling from this set is an image of the canonical substitution.



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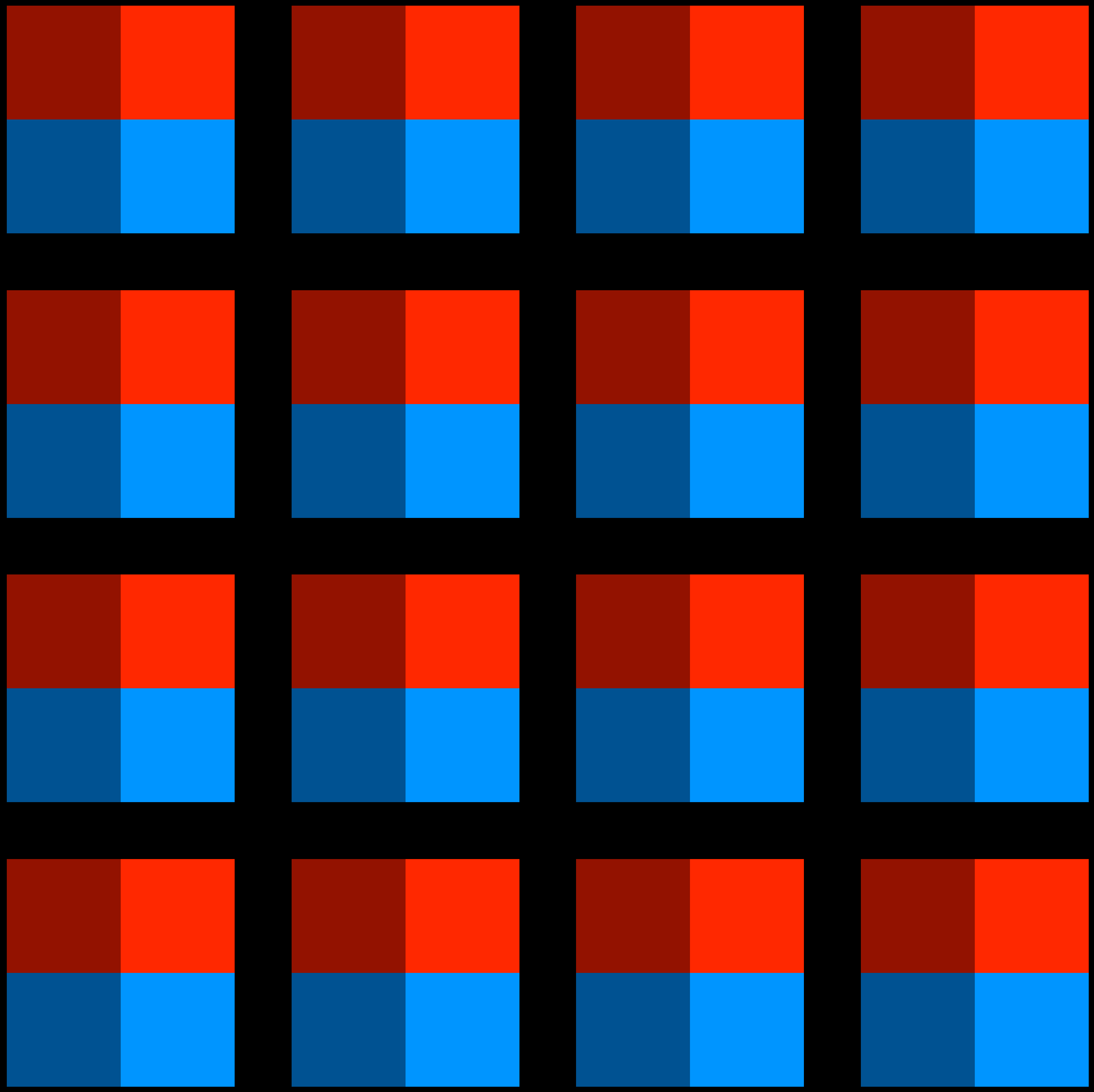

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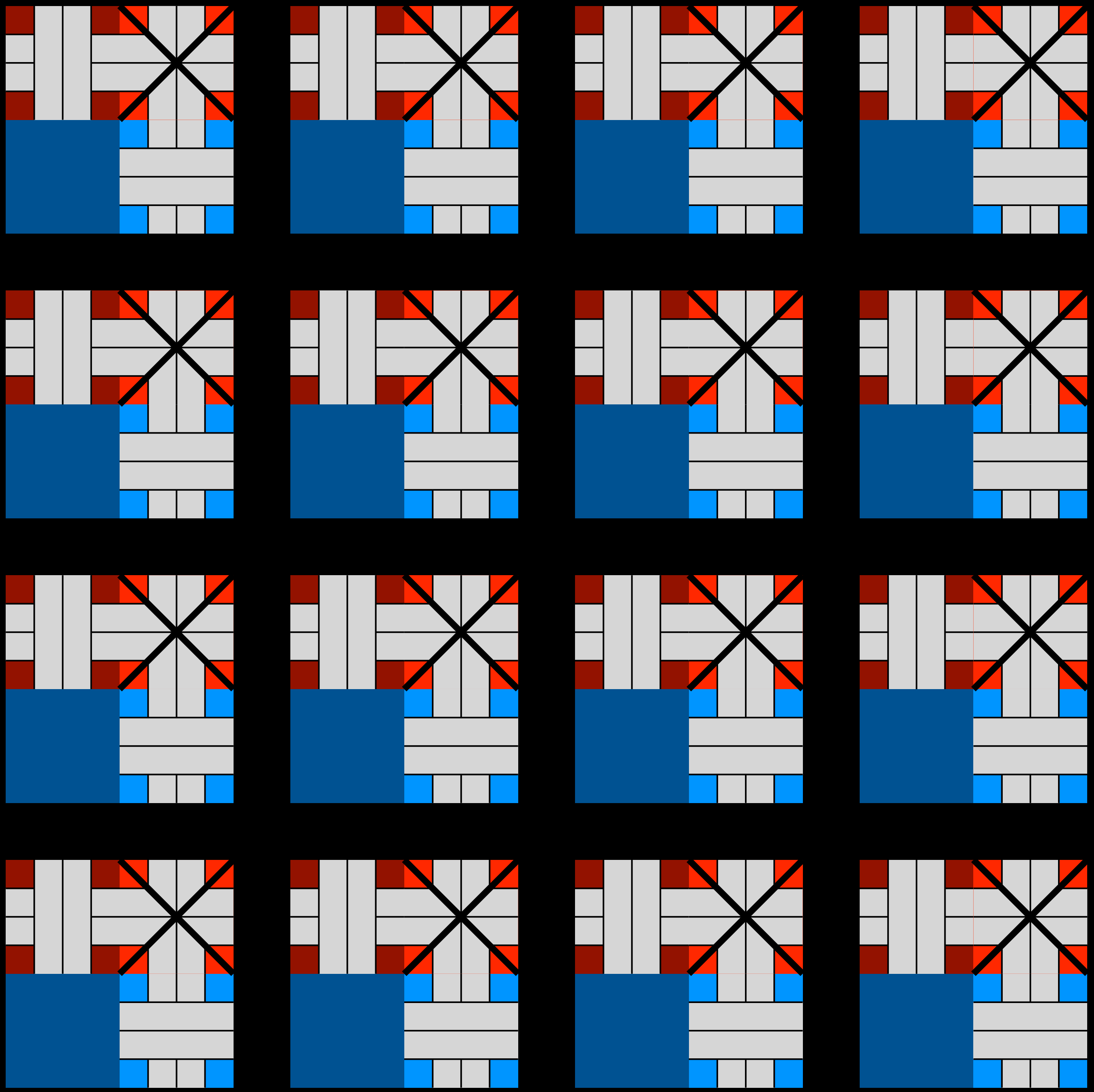

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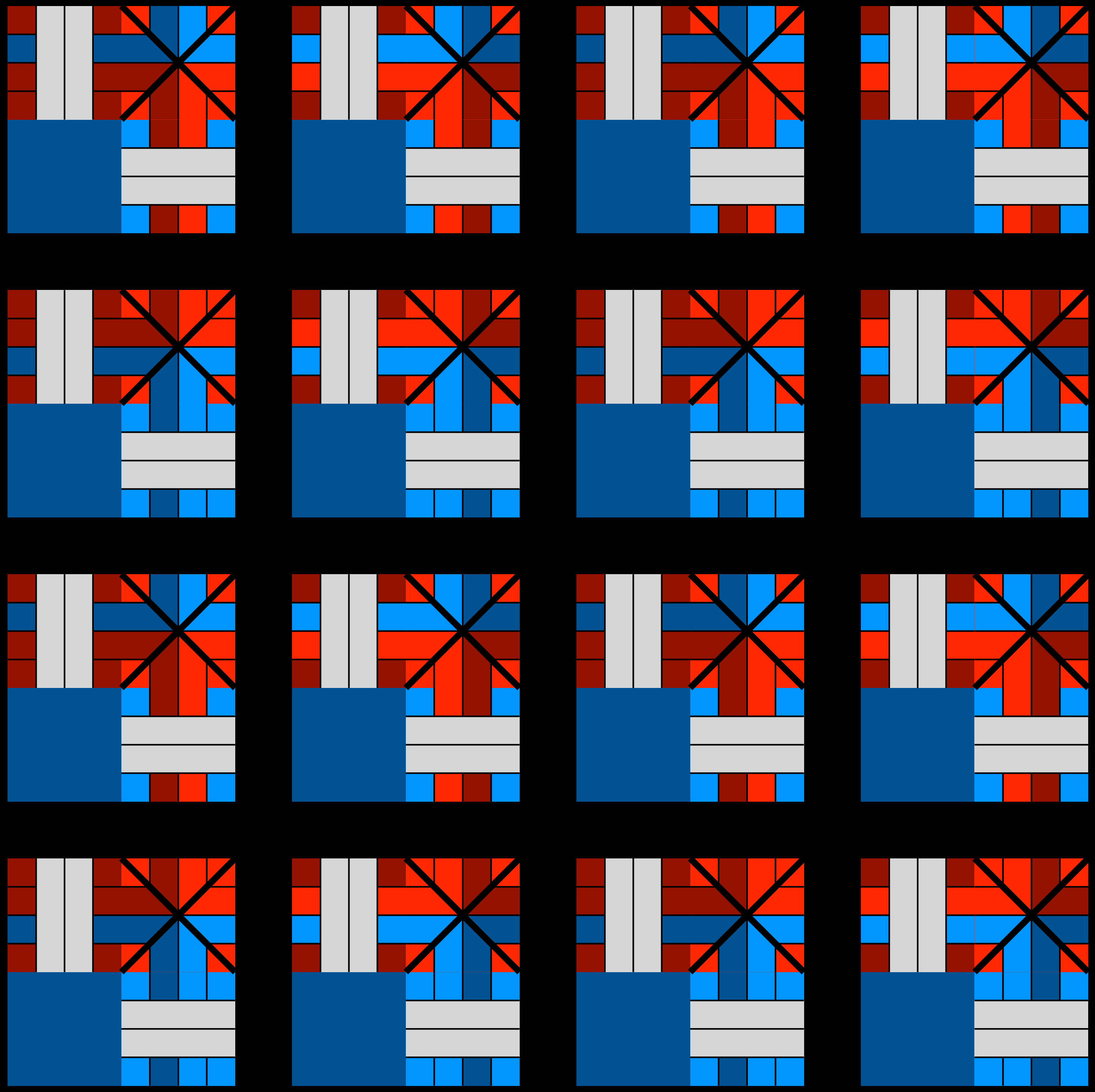

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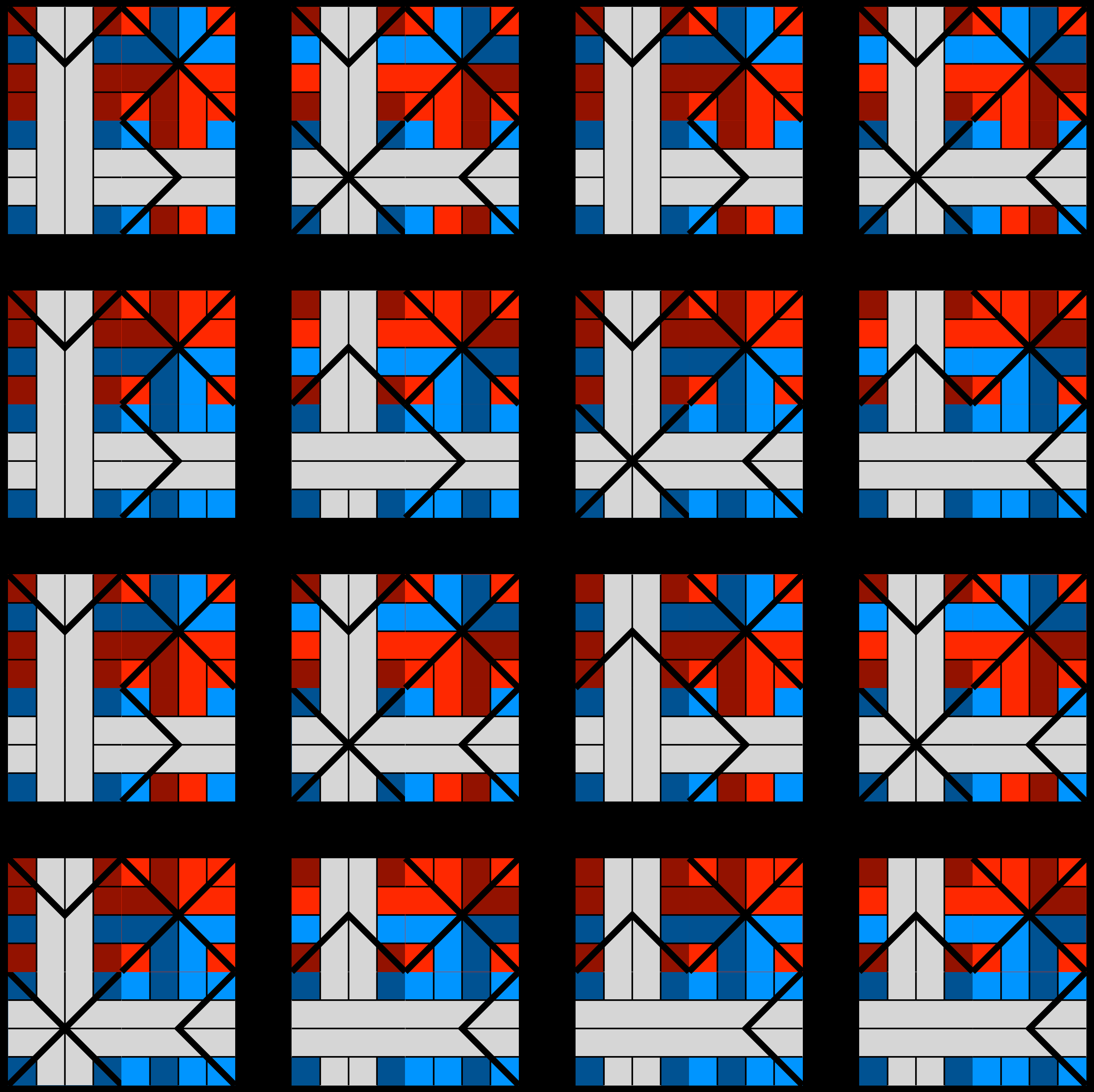

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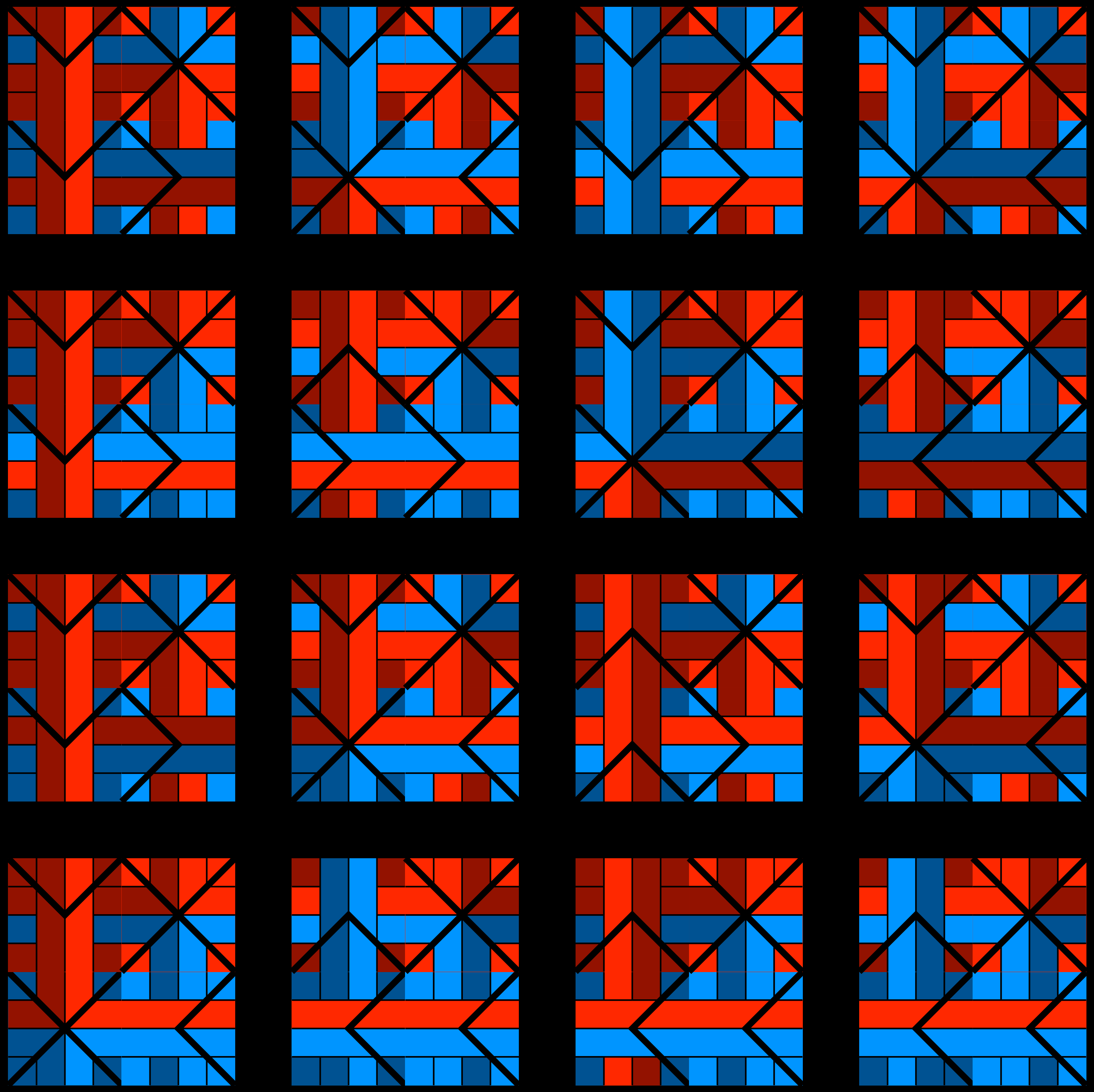

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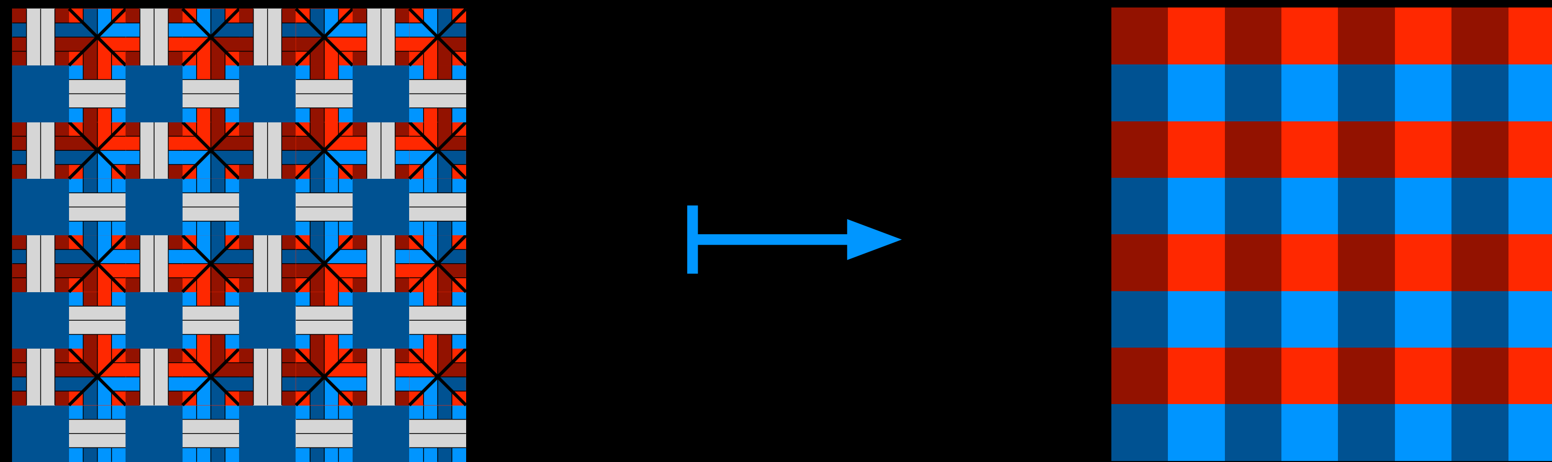
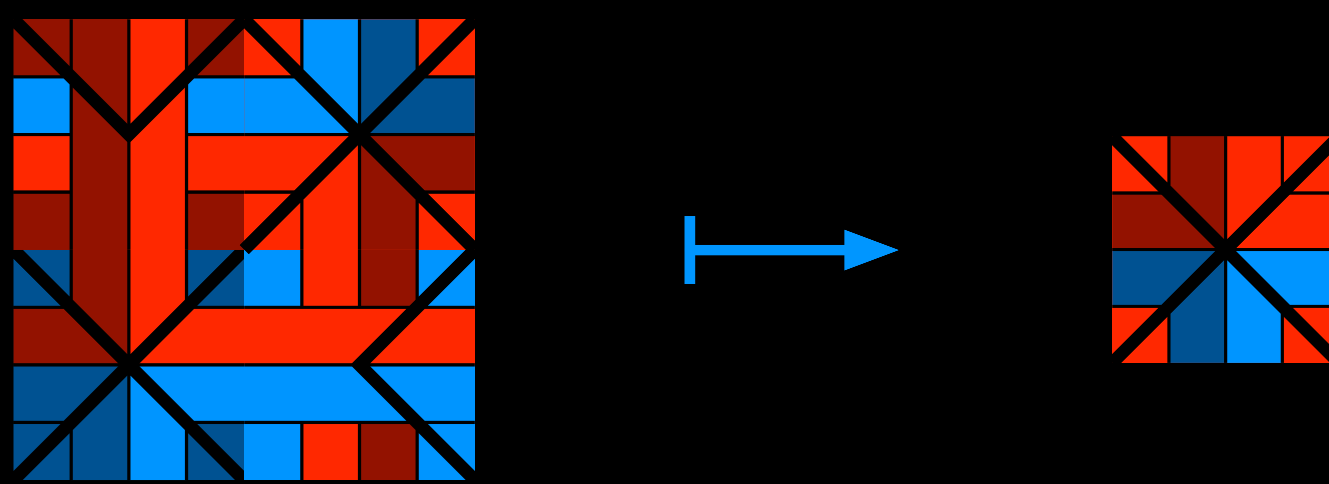


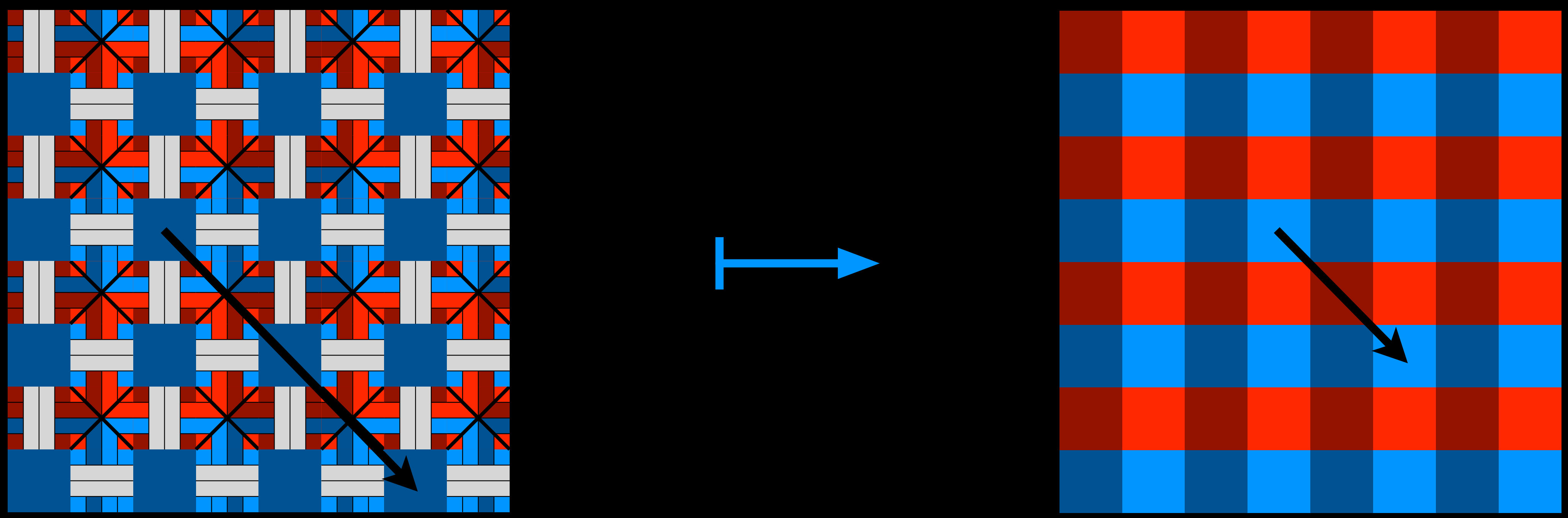
Layer 1

we just remove wires and squarifiers that propagate



Layers 2 & 3

Lemma 2. the preimage of each tiling is a (valid) tiling.

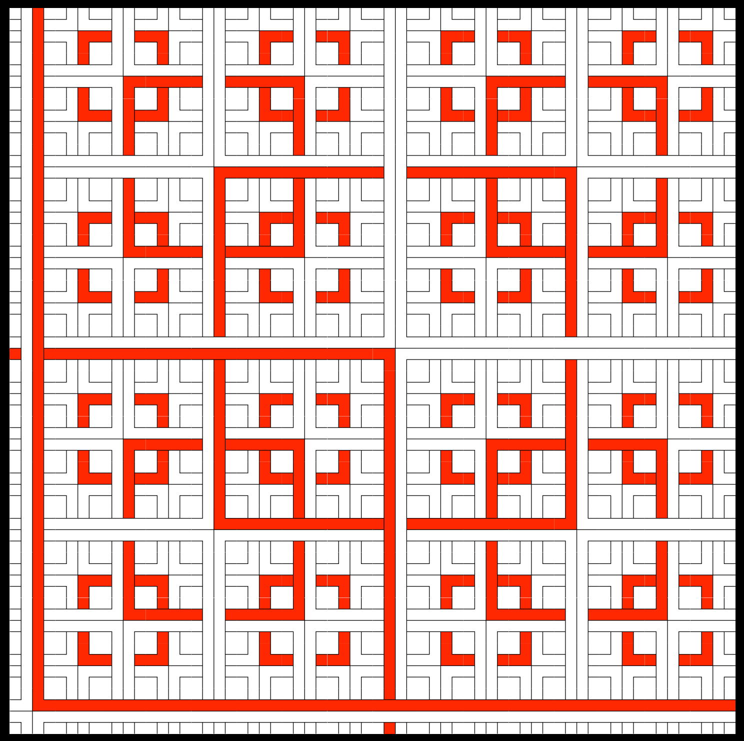


Lemma 3. a periodic tiling of period z has a preimage of period z/2.

Back to the Domino Problem

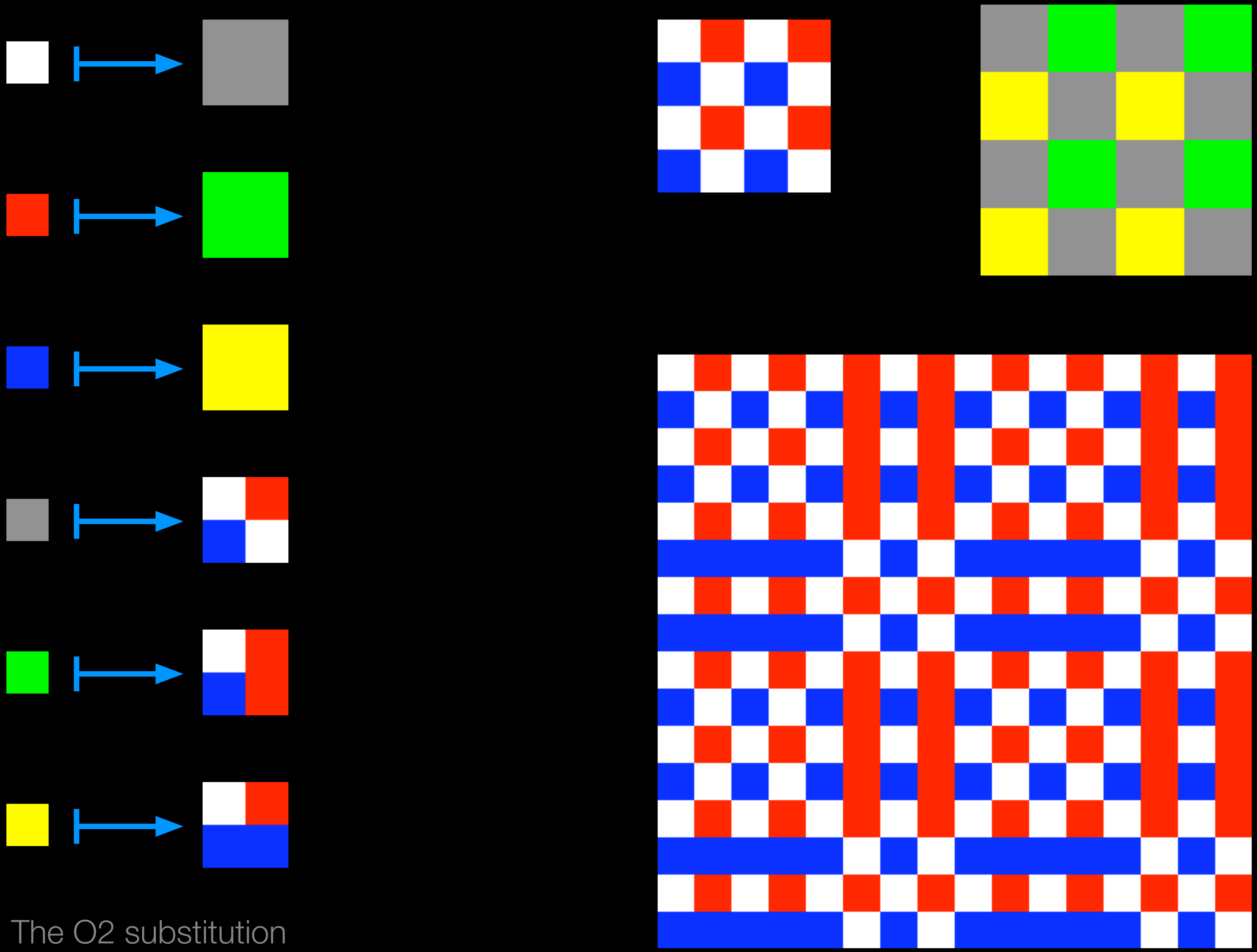
Encode substitutions

- Select one color of wires (say red): the sequence of squares encode a tree.
- Put a letter on each square.
- Apply the substitution rule to go from one square to the next one.
- Up to projection the substitution limit set is encoded in the set of tilings.

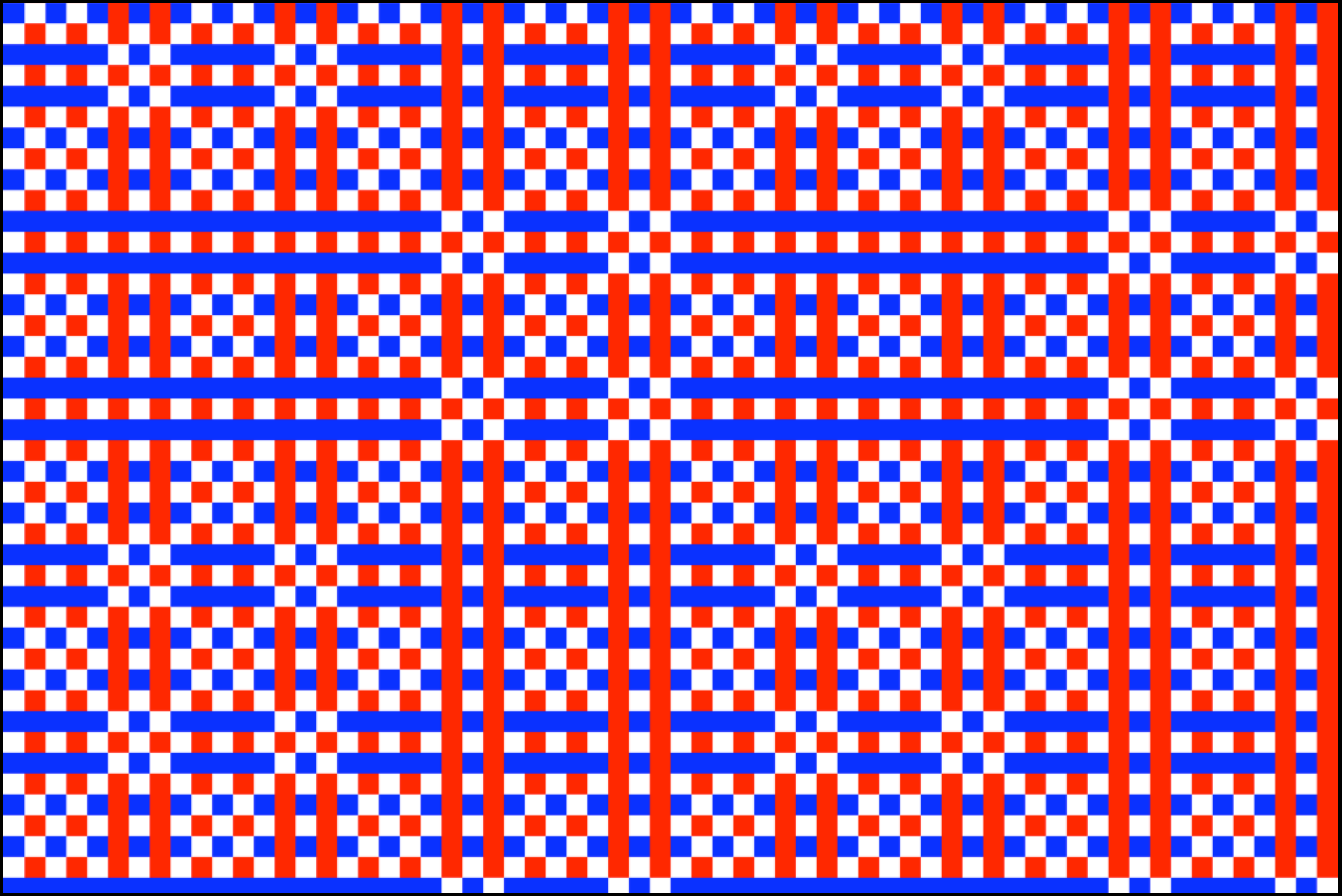


a

$$b = s(a)\binom{1}{1}$$



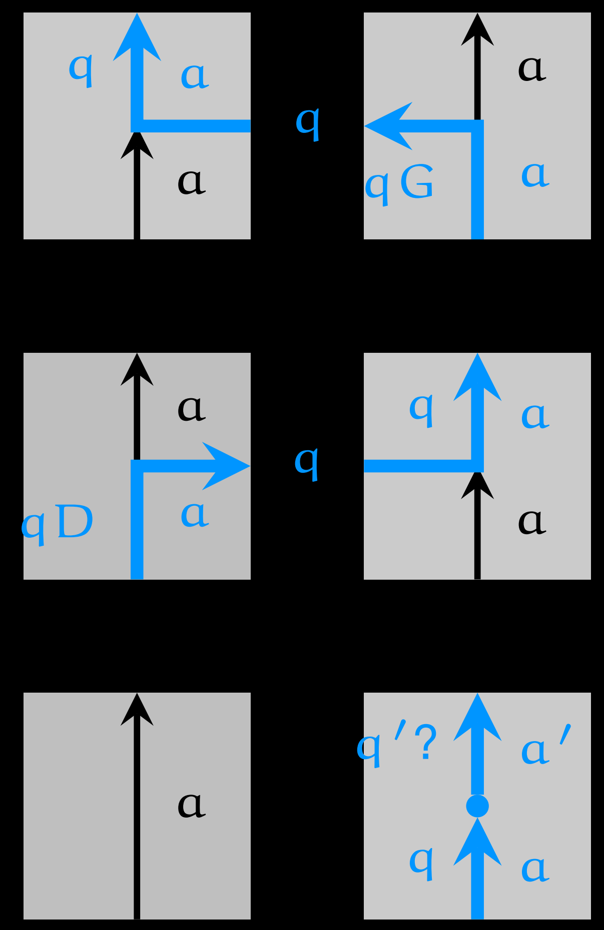
The O2 substitution



O2 iteration

Turing computation

- Turing machines can be simulated with tilesets.
- Put an initial computation at the SW corner of each O2 computation square.
- Remove the halting state.
- The tileset tiles the plane iff the Turing machine does not halt.



That’s all folks!