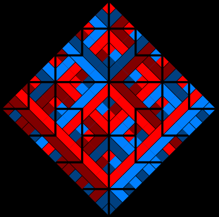
Two-by-two Substitution Systems and the Undecidability of the Domino Problem

Nicolas Ollinger (LIF, Aix-Marseille Université, CNRS, France)

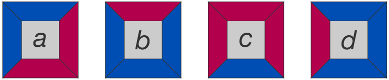
CiE 2008, Αθήνα

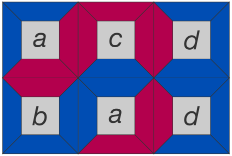


The Domino Problem (**DP**)

"Assume we are **given a finite set of square plates** of the same size with **edges colored**, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are **not permitted to rotate or reflect a plate**. The question is to find an effective procedure by which we can **decide**, for each given finite set of plates, **whether we can cover up the whole plane** (or, equivalently, an infinite quadrant thereof) **with copies of the plates subject to the restriction that adjoining edges must have the same color**."

(Wang, 1961)





The need for aperiodic tilings

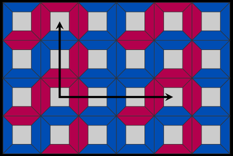
A tiling is *periodic* if it admits two non-collinear periodicity vectors.

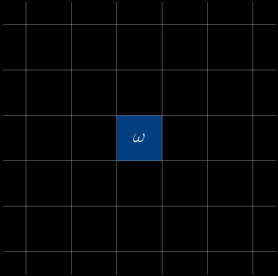
A tile set is *periodic* if it admits a periodic tiling.

Lemma. Periodic tile sets are recursively enumerable.

Lemma. Non-tiling tile sets are recursively enumerable.

Corollary. If **DP** is not recursive, there exists *aperiodic* tile sets.





König's lemma

The need for aperiodic tilings

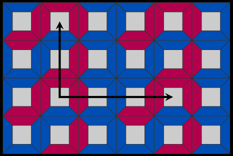
A tiling is *periodic* if it admits two non-collinear periodicity vectors.

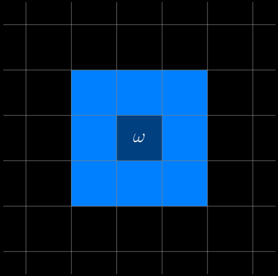
A tile set is *periodic* if it admits a periodic tiling.

Lemma. Periodic tile sets are recursively enumerable.

Lemma. Non-tiling tile sets are recursively enumerable.

Corollary. If **DP** is not recursive, there exists *aperiodic* tile sets.





König's lemma

The need for aperiodic tilings

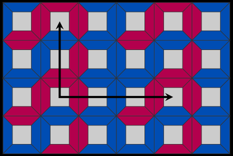
A tiling is *periodic* if it admits two non-collinear periodicity vectors.

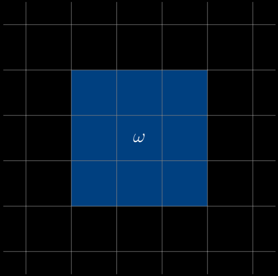
A tile set is *periodic* if it admits a periodic tiling.

Lemma. Periodic tile sets are recursively enumerable.

Lemma. Non-tiling tile sets are recursively enumerable.

Corollary. If **DP** is not recursive, there exists *aperiodic* tile sets.





König's lemma

The need for aperiodic tilings

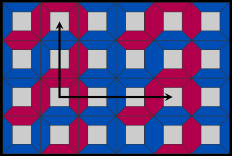
A tiling is *periodic* if it admits two non-collinear periodicity vectors.

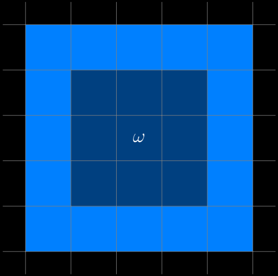
A tile set is *periodic* if it admits a periodic tiling.

Lemma. Periodic tile sets are recursively enumerable.

Lemma. Non-tiling tile sets are recursively enumerable.

Corollary. If **DP** is not recursive, there exists *aperiodic* tile sets.





König's lemma

The need for aperiodic tilings

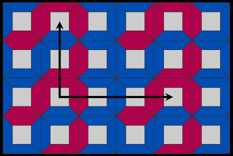
A tiling is *periodic* if it admits two non-collinear periodicity vectors.

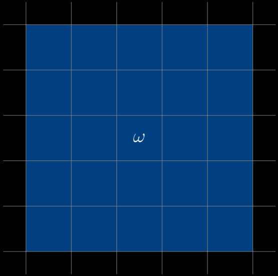
A tile set is *periodic* if it admits a periodic tiling.

Lemma. Periodic tile sets are recursively enumerable.

Lemma. Non-tiling tile sets are recursively enumerable.

Corollary. If **DP** is not recursive, there exists *aperiodic* tile sets.





König's lemma

Undecidability of **DP**: a short history

1964 Berger proves the undecidability of **DP**.

Two main type of related activities in the literature:

(1) construct aperiodic tile sets (small ones);

(2) give a full proof of the undecidability of **DP** (implies (1)).

From 104 tiles (Berger, 1964) to 13 tiles (Culik, 1996) aperiodic sets.

Seminal self-similarity based proofs *(reduction from **HP**)*:

- Berger, 1964 *(20426 tiles, a full PhD thesis)*
- Robinson, 1971 *(56 tiles, 17 pages, long case analysis)*
- Durand et al, 2007 *(Kleene's fixpoint existence argument)*

Tiling rows seen as transducer trace based proof:
Kari, 2007 *(affine maps, short concise proof, reduction from **IP**)*

In this talk

A new self-similarity based construction building on classical proof schemes with concise arguments and few tiles:

1. two-by-two substitution systems and aperiodicity
2. an aperiodic tile set of 104 tiles
3. enforcing any substitution and reduction from **HP** *(sketch)*

This work combines tools and ideas from:

[Berger 64] *The Undecidability of the Domino Problem*

[Robinson 71] *Undecidability and nonperiodicity for tilings of the plane*

[Grünbaum Shephard 89] *Tilings and Patterns, an introduction*

[Durand Levin Shen 05] *Local rules and global order, or aperiodic tilings*

1. two-by-two substitution systems

Two-by-two substitution systems

A 2×2 substitution system maps a finite alphabet to 2×2 squares of letters on that alphabet.

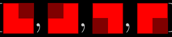
$$s : \Sigma \to \Sigma^{\boxplus}$$

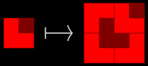
The substitution is iterated to generate bigger squares.

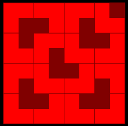
$$S : \Sigma^{\mathcal{P}} \to \Sigma^{\square(\mathcal{P})}$$

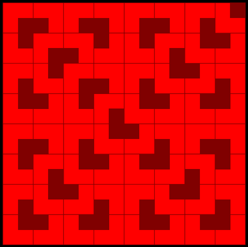
$$\forall z \in \mathcal{P}, \forall c \in \boxplus,$$
$$S(\mathcal{C})(2z + c) = s(\mathcal{C}(z))(c)$$

$$S(u \cdot \mathcal{C}) = 2u \cdot S(\mathcal{C})$$

$\Sigma = \{$  $\}$

$s :$  *+ rotations*





Coloring the whole plane *via* limit sets

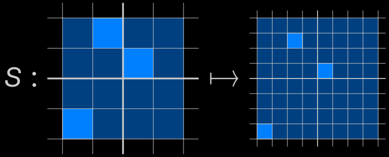
What is a coloring of the plane generated by a substitution?

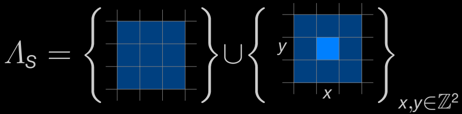
With tilings in mind the set of colorings should be closed by translation and compact.

We take the limit set of iterations of the (continuous) global map closed up to translations.

$\Lambda_S = \bigcap_n \Lambda_S^n$ where $\Lambda_S^0 = \Sigma^{\mathbb{Z}^2}$

$\Lambda_S^{n+1} = \{u \cdot S(\mathcal{C}) | \mathcal{C} \in \Lambda_S^n, u \in \boxplus\}$





Unambiguous substitutions are aperiodic

A substitution is *aperiodic* if its limit set Λ_S is aperiodic.

A substitution is *unambiguous* if, for every coloring $\mathcal{C}$ from its limit set Λ_S, there exists a unique coloring $\mathcal{C}'$ and a unique translation $u \in \boxplus$ satisfying $\mathcal{C} = u \cdot S(\mathcal{C}')$.

Proposition 3. Unambiguity implies aperiodicity.

Sketch of the proof. Consider a periodic coloring with minimal period p, its preimage has period $p/2$. $\diamond$

Idea. Construct a tile set whose tilings are in the limit set of an unambiguous substitution system.

2. an aperiodic tile set of 104 tiles

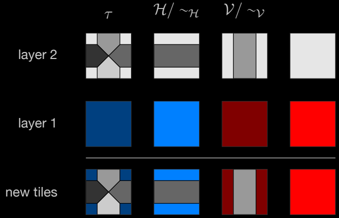
Coding tile sets into tile sets

A tile set τ is a triple $(T, \mathcal{H}, \mathcal{V})$ where $\mathcal{H}$ and $\mathcal{V}$ define horizontal and vertical matching constraints.

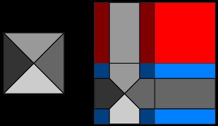
The set of tilings of τ is X_τ.

A tile set $(T', \mathcal{H}', \mathcal{V}')$ *codes* a tile set $(T, \mathcal{H}, \mathcal{V})$, according to a *coding rule* $t : T \longrightarrow T'^{\boxplus}$ if t is injective and

$$X_{\tau'} = \{u \cdot t(\mathcal{C}) | \mathcal{C} \in X_\tau, u \in \boxplus\}.$$



coding tile set



coding rule

Aperiodicity *via* unambiguous self-coding

A tile set $(T, \mathcal{H}, \mathcal{V})$ *codes* a substitution $s : T \longrightarrow T^{\boxplus}$ if it codes itself according to the coding rule s.

Proposition 4. A tile set both admitting a tiling and coding an unambiguous substitution is aperiodic.

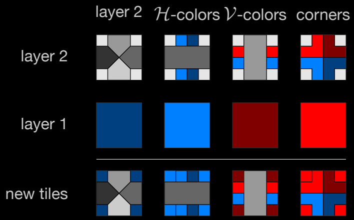
Sketch of the proof. $X_\tau \subseteq \Lambda_S$ and $X_\tau \neq \emptyset$. $\diamond$

A coding scheme with fixpoint?

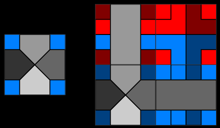
Better scheme: not strictly increasing the number of tiles.

Problem. it cannot encode any layered tile set, constraints between layer 1 and layer 2 are checked edge by edge.

Solution. add a third layer with one bit of information per edge.



coding tile set



coding rule

Canonical substitution

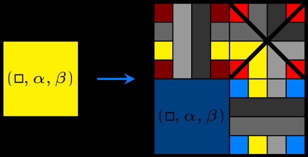
Copy the tile in the SW corner but for layer 1.

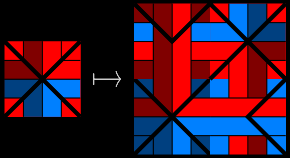
Put the only possible X in NE that carry layer 1 of the original tile on SW wire.

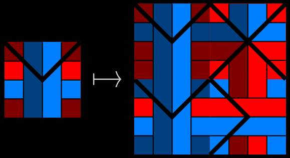
Propagate wires colors.

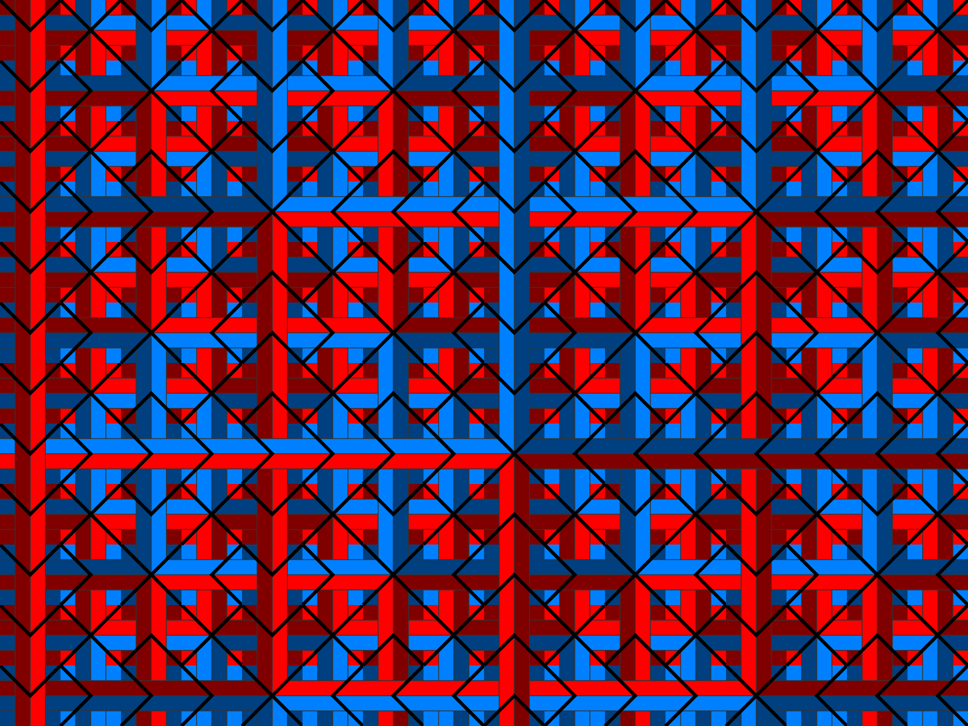
Let H et V tile propagate layer 3 arrows.

The substitution is injective.









Aperiodicity: sketch of the proof

1. The tile set admits a tiling:

Generate a valid tiling by iterating the substitution rule: $X_\tau \cap \Lambda_S \neq \emptyset$.

2. The substitution is unambiguous:

It is injective and the projectors have disjoined images.

3. The tile set codes the substitution:

(a) each tiling is an image of the canonical substitution
Consider any tiling, level by level, short case analysis.

(b) the preimage of a tiling is a tiling
Straightforward by construction (preimage remove constraints).

3. enforcing any substitution

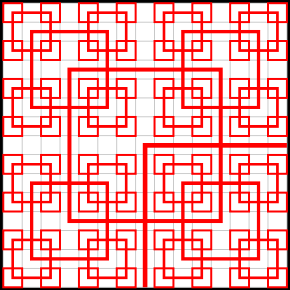
Enforcing substitutions *via* tilings

Let π map every tile of $\tau(s')$ to $s'(a)(u)$ where a and u are the letter and the value of ⊞ on layer 1.

Theorem 2. Let s' be any substitution system. The tile set $\tau(s')$ enforces s':

$$\pi\left(X_{\tau(s')}\right) = \Lambda_{S'}.$$

Idea. Every tiling of $\tau(s')$ codes an history of S' and every history of S' can be encoded into a tiling of $\tau(s')$.



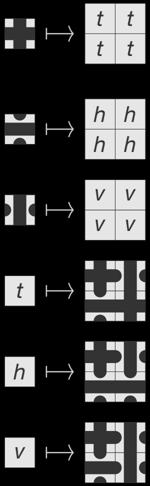
a

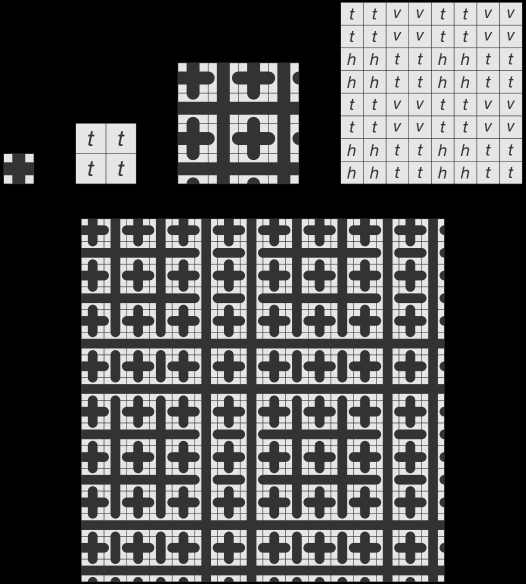
$b = s(a)\binom{1}{1}$

a

$b = s(a)\binom{0}{0}$

Infinitely many squares of unbounded size





Reducing **HP** to **DP**

Any tiling by previous tile set contains infinitely many finite squares of unbounded size.

In each square, simulate the computation of the given Turing machine from an empty tape.

Initial computation is enforced in the SW corner.

Remove the halting state.

The tile set tiles the plan iff the Turing machine does not halt.