Periodicity and Immortality in Reversible Computing

Jarkko Kari (Dpt. of Mathematics, University of Turku, Finland)
Nicolas Ollinger (LIF, Aix-Marseille Université, CNRS, France)

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E. Ochmański and J. Tyszkiewicz (Eds.): **MFCS 2008**, LNCS 5162, pp. 419–430, 2008.

In this talk

We investigate the **(un)decidability** of **dynamical properties** of three models of **reversible** computation.

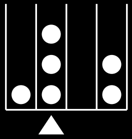
We consider the behavior of the models starting from **arbitrary initial configurations**.

Immortality is the property of having at least one non-halting orbit.

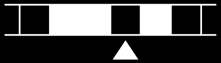
Periodicity is the property of always eventually returning back to the starting configuration.

Models of reversible computation

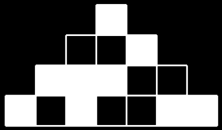
Counter Machines (**CM**)



Turing Machines (**TM**)



Cellular Automata (**CA**)



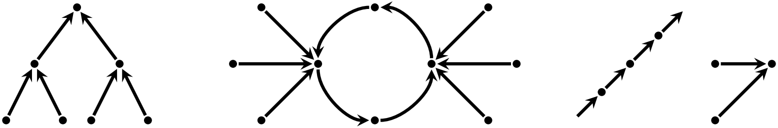
A machine is **deterministic** if there exists at most one transition from each configuration.

A machine is **reversible** if there exists **another machine** that can inverse **each step** of computation.

Discrete Dynamical Systems

A **DDS** $\mathcal{S}$ is a pair (X, F) where X is a **topological space** and $F : X \longrightarrow X$ is a **partial** and **continuous** map.

In the case of **TM** and **CA**, X is the compact and metrizable product of the discrete topology.



The **orbit** of $x \in X$ is the sequence $(F^n(x))$ obtained by iterating F.

A model $\mathcal{M}$ is a recursive family of **DDS**.

The immortality problem (**IP**)

A configuration on which *F* is undefined is **halting**.

A configuration is **mortal** if its orbit is eventually halting.

Halting Problem Given $\mathcal{S} \in \mathcal{M}$, is $x_0 \in X$ mortal for $\mathcal{S}$?

$\mathcal{S}$ is **mortal** if all its configurations are mortal.

$\mathcal{S}$ is **uniformly mortal** if a uniform bound *n* exists such that F^n is halting for all configuration.

Immortality Problem Given $\mathcal{S} \in \mathcal{M}$, is $\mathcal{S}$ immortal?

When *X* is compact and the set of halting configurations is open, uniform mortality is the same as mortality.

The periodicity problem (**PP**)

$\mathcal{S}$ is **complete** if F is total.

A configuration x is ***n*-periodic** if $F^n(x) = x$.

$\mathcal{S}$ is **periodic** if all its configurations are periodic.

$\mathcal{S}$ is **uniformly periodic** if a uniform bound n exists such that F^n is the identity map.

Periodicity Problem Given $\mathcal{S} \in \mathcal{M}$, is $\mathcal{S}$ periodic?

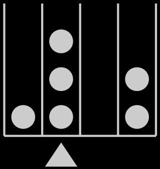
When X is compact and the set of n-periodic configurations is open, uniform periodicity is the same as periodicity.

Results

	HP	IP	PP
RCA		[KL08]	Thm 12
RTM	[Lecerf63]	Thm 7	Thm 8
RCM	[Morita96]	Thm 1	Thm 3

$\longrightarrow$ denotes many-one reductions.

1. Reversible Counter Machines



A ***k*-CM** is a triple (S, k, T) where S is a finite set of states and $T \subseteq S \times \{0,+\}^k \times \mathbb{Z}_k \times \{-,0,+\} \times S$ is a set of instructions.

$(s, u, i, \phi, t) \in T$: *"in state s with counter values u,*
apply ϕ to counter i and enter to state t."

DDS $(S \times \mathbb{N}^k, G)$ where $G(\mathfrak{c})$ is the unique $\mathfrak{c}'$ such that $\mathfrak{c} \vdash \mathfrak{c}'$.

Immortality

[Minsky67] Every recursive function is computed by a 2-DCM and thus **HP** is undecidable for 2-DCM.

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Idea for new proof *Enforce infinite orbits to go through unbounded initial segments of an orbit from x_0 to reduce **HP**.* $\Diamond$

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[Morita96] Every *k*-DCM is **simulated** by a 2-RCM.

Idea *Encode a stack with two counters to keep an history of simulated instructions.* $\Diamond$

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Theorem 1 **IP** is undecidable for 2-RCM.

Idea *Morita's simulation preserves immortality.* $\diamond$

Periodicity

Theorem 3 **PP** is undecidable for 2-RCM.

Idea Reduce **IP** to **PP**:

(1) **IP** is still undecidable for 2-RCM with **mortal reverse**
(add a constantly incremented counter to the *k*-DCM)

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(1) **IP** is still undecidable for 2-RCM with **mortal reverse**
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(2) Let $\mathcal{M} = (S, 2, T)$ be a 2-RCM with mortal reverse.
$\mathcal{M}$ admits no periodic orbit.
Let $\mathcal{M}'$ be the 2-RCM with set of states $S \times \{+, -\}$ simulating $\mathcal{M}$ on $+$ and $\mathcal{M}^{-1}$ on $-$ and inversing polarity on halting states.

Periodicity

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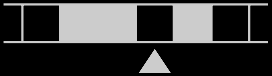
(3) $\mathcal{M}'$ is periodic iff $\mathcal{M}$ is mortal. $\Diamond$



2. Reversible Turing Machines

A **TM** is a triple (S, Σ, T) where S is a finite set of states, Σ a finite alphabet and $T \subseteq (S \times \{\leftarrow, \rightarrow\} \times S) \cup (S \times \Sigma \times S \times \Sigma)$ is a set of instructions.

(s, δ, t) : *"in state s move according to δ and enter state t."*
(s, a, t, b) : *"in state s, reading letter a, write letter b and enter state t."*

DDS $(S \times \Sigma^{\mathbb{Z}}, G)$ where $G(\mathfrak{c})$ is the unique $\mathfrak{c}'$ such that $\mathfrak{c} \vdash \mathfrak{c}'$.

Immortality: a first attempt

"(T_2) To find an effective method, which for every Turing-machine *M* decides whether or not, for all tapes *I* (finite and infinite) and all states *B*, *M* will eventually halt if started in state *B* on tape *I*" *(Büchi, 1962)*

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Problem The simulation **does not** preserve immortality due to **unbounded searches**. We need to rewrite Hooper's proof for reversible machines.

Immortality: simulating RCM

Theorem 7 **IP** is undecidable for RTM.

Reduction reduce **HP** for 2-RCM $(s, \underline{@}1^m x 2^n y)$

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$\underline{@}$111111111111x2222y *search* x $\longrightarrow$
s

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$@\underset{s'_1}{\underline{1}}11111111111x2222y$ *bounded search 1*

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$$\mathtt{@1}\underset{s'_2}{\underline{\mathtt{1}}}\mathtt{11111111111x2222y}$$ *bounded search 2*

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Hooper's trick use bounded searches with **recursive calls** to initial segments of the simulation of increasing sizes:

$@11\underline{1}111111111x2222y$ *bounded search 3*

s'_3

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$@\underline{@}_s xy 1111111111x2222y$ *recursive call*

s_0

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$@@_s11111x22222\underset{\overline{s}_c}{\underline{y}}x2222y$ *ultimately in case of collision...*

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$@\underline{@}_s xy1111111111x2222y$ *...revert to clean*

s_b

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pop and continue bounded search 1

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$\mathtt{@111}\underline{\mathtt{@}}_{\mathtt{s}}\mathtt{xy1111111x2222y}$ *recursive call*

$\mathtt{s}_{\mathtt{0}}$

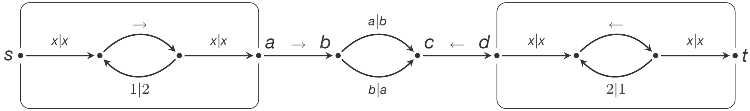
The RTM is immortal iff the 2-RCM is mortal on $(s_0, (0, 0))$.

Programming tips and tricks (1/2)

We designed a TM programming language called Gnirut:

`http://www.lif.univ-mrs.fr/~nollinge/rec/gnirut/`

First ingredient use **macros** to avoid repetitions:



$[s|\text{search}|a\rangle$ $\langle d|\text{search}|t]$

def $[s|\text{search}|t\rangle$:

$\quad s.\ \mathrm{x} \vdash \mathrm{x}, u$

$\quad u.\ \ \longrightarrow, r$

$\quad r.\ 1 \vdash 2, u \mid \mathrm{x} \vdash \mathrm{x}, t$

$[s|\text{search}|a\rangle$

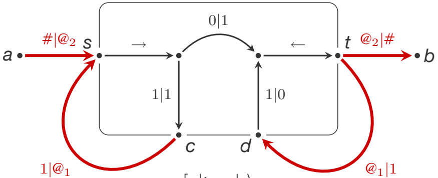
$a.\ \ \longrightarrow, b$

$b.\ a \vdash \mathrm{b}, c \mid \mathrm{b} \vdash a, c$

$c.\ \ \longleftarrow, d$

$\langle d|\text{search}|t]$

Second ingredient use **recursive calls**:



$[s|\mathrm{incr}|t\rangle$

fun $[s|\mathrm{incr}|t\rangle$:
  $s.\ \longrightarrow, r$
  $r.\ 0 \vdash 1, b \mid 1 \vdash 1, c$
  call $[c|\mathrm{incr}|d\rangle$ from 1 $\Leftarrow$ **call 1**
  $d.\ 1 \vdash 0, b$
  $b.\ \longleftarrow, t$

call $[a|\mathrm{incr}|b\rangle$ from # $\Leftarrow$ **call 2**

Immortality: skeleton

$$[s|\mathrm{check}_1|t\rangle \quad \text{satisfies} \quad s.\,\underline{@_\alpha}1^m\mathsf{x} \vdash \underline{@_\alpha}1^m\mathsf{x}, t \quad \text{or} \quad s.\,\underline{@_\alpha}1^\omega \uparrow \quad \text{or halt.}$$

Immortality: skeleton

$[s|\mathrm{check}_1|t\rangle$ satisfies $s.\,\underline{@_\alpha}1^m\mathtt{x} \vdash \underline{@_\alpha}1^m\mathtt{x}, t$ or $s.\,\underline{@_\alpha}1^\omega \uparrow$ or halt.

$[s|\mathrm{search}_1|t_0, t_1, t_2\rangle$ satisfies $s.\,\underline{@_\alpha}1^m\mathtt{x} \vdash @_\alpha 1^m\underline{\mathtt{x}}, t_{m[3]}$ or $\ldots$

Immortality: skeleton

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RCM ingredients:

- testing counters $[s|\mathrm{test1}|z, p\rangle$ and $[s|\mathrm{test2}|z, p\rangle$
- increment counter $[s|\mathrm{inc1}|t, co\rangle$ and $[s|\mathrm{inc2}|t, co\rangle$
- decrement counter $[s|\mathrm{dec1}|t, co\rangle$ and $[s|\mathrm{dec2}|t, co\rangle$

Immortality: skeleton

$[s|\mathrm{check}_1|t\rangle$ satisfies $s.\,\underline{@_\alpha}1^m x \vdash \underline{@_\alpha}1^m x, t$ or $s.\,\underline{@_\alpha}1^\omega \uparrow$ or halt.

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Simulator $[s|\mathrm{RCM}_\alpha|co_1, co_2, \ldots\rangle$ initialize then compute

Immortality: skeleton

$[s|\mathrm{check}_1|t\rangle$ satisfies $s.\ \underline{@_\alpha}1^m\mathtt{x} \vdash \underline{@_\alpha}1^m\mathtt{x}, t$ or $s.\ \underline{@_\alpha}1^\omega \uparrow$ or halt.

$[s|\mathrm{search}_1|t_0, t_1, t_2\rangle$ satisfies $s.\ \underline{@_\alpha}1^m\mathtt{x} \vdash @_\alpha 1^m\underline{\mathtt{x}}, t_{m[3]}$ or $\dots$

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Simulator $[s|\mathrm{RCM}_\alpha|co_1, co_2, \dots\rangle$ initialize then compute

$$\boxed{[s|\mathrm{check}_\alpha|t\rangle = [s|\mathrm{RCM}_\alpha|co_1, co_2, \dots\rangle + \langle co_1, co_2, \dots|\mathrm{RCM}_\alpha|s]}$$

Program it!

def $[s|\mathrm{search}_1|t_0,t_1,t_2\rangle$:
$s.\ \underline{@_\alpha} \vdash \underline{@_\alpha}, l$
$l.\ \rightarrow, u$
$u.\ \underline{x} \vdash \underline{x}, t_0$
$|\ \underline{1}x \vdash 1\underline{x}, t_1$
$|\ \underline{1}1x \vdash 11\underline{x}, t_2$
$|\ \underline{1}11 \vdash \underline{1}11, c$
call $[c|\mathrm{check}_1|p\rangle$ from 1
$p.\ \underline{1}11 \vdash 11\underline{1}, l$

def $[s|\mathrm{search}_2|t_0,t_1,t_2\rangle$:
$s.\ \underline{x} \vdash \underline{x}, l$
$l.\ \rightarrow, u$
$u.\ \underline{y} \vdash \underline{y}, t_0$
$|\ \underline{2}y \vdash 2\underline{y}, t_1$
$|\ \underline{2}2y \vdash 22\underline{y}, t_2$
$|\ \underline{2}22 \vdash \underline{2}22, c$
call $[c|\mathrm{check}_2|p\rangle$ from 2
$p.\ \underline{2}22 \vdash 22\underline{2}, l$

def $[s|\mathrm{test1}|z,p\rangle$:
$s.\ \underline{@_\alpha}x \vdash \underline{@_\alpha}x, z$
$|\ \underline{@_\alpha}1 \vdash \underline{@_\alpha}1, p$

def $[s|\mathrm{endtest2}|z,p\rangle$:
$s.\ \underline{x}y \vdash \underline{x}y, z$
$|\ \underline{x}2 \vdash \underline{x}2, p$

def $[s|\mathrm{test2}|z,p\rangle$:
$[s|\mathrm{search}_1|t_0,t_1,t_2\rangle$
$[t_0|\mathrm{endtest2}|z_0,p_0\rangle$
$[t_1|\mathrm{endtest2}|z_1,p_1\rangle$
$[t_2|\mathrm{endtest2}|z_2,p_2\rangle$
$\langle z_0,z_1,z_2|\mathrm{search}_1|z]$
$\langle p_0,p_1,p_2|\mathrm{search}_1|p]$

def $[s|\mathrm{mark}_1|t,co\rangle$:
$s.\ \underline{y}1 \vdash 2\underline{y}, t$
$|\ \underline{y}x \vdash \underline{y}x, co$

def $[s|\mathrm{endinc}_1|t,co\rangle$:
$[s|\mathrm{search}_2|r_0,r_1,r_2\rangle$
$[r_0|\mathrm{mark}_1|t_0,co_0\rangle$
$[r_1|\mathrm{mark}_1|t_1,co_1\rangle$
$[r_2|\mathrm{mark}_1|t_2,co_2\rangle$
$\langle t_2,t_0,t_1|\mathrm{search}_2|t]$
$\langle co_0,co_1,co_2|\mathrm{search}_2|co]$

def $[s|\mathrm{inc2}_1|t,co\rangle$:
$[s|\mathrm{search}_1|r_0,r_1,r_2\rangle$
$[r_0|\mathrm{endinc}_1|t_0,co_0\rangle$
$[r_1|\mathrm{endinc}_1|t_1,co_1\rangle$
$[r_2|\mathrm{endinc}_1|t_2,co_2\rangle$
$\langle t_0,t_1,t_2|\mathrm{search}_1|t]$
$\langle co_0,co_1,co_2|\mathrm{search}_1|co]$

def $[s|\mathrm{dec2}_1|t\rangle$:
$\langle s,co|\mathrm{inc2}_1|t]$

def $[s|\mathrm{mark}_2|t,co\rangle$:
$s.\ \underline{y}2 \vdash 2\underline{y}, t$
$|\ \underline{y}x \vdash \underline{y}x, co$

def $[s|\mathrm{endinc}_2|t,co\rangle$:
$[s|\mathrm{search}_2|r_0,r_1,r_2\rangle$
$[r_0|\mathrm{mark}_2|t_0,co_0\rangle$
$[r_1|\mathrm{mark}_2|t_1,co_1\rangle$
$[r_2|\mathrm{mark}_2|t_2,co_2\rangle$
$\langle t_2,t_0,t_1|\mathrm{search}_2|t]$
$\langle co_0,co_1,co_2|\mathrm{search}_2|co]$

def $[s|\mathrm{inc2}_2|t,co\rangle$:
$[s|\mathrm{search}_1|r_0,r_1,r_2\rangle$
$[r_0|\mathrm{endinc}_2|t_0,co_0\rangle$
$[r_1|\mathrm{endinc}_2|t_1,co_1\rangle$
$[r_2|\mathrm{endinc}_2|t_2,co_2\rangle$
$\langle t_0,t_1,t_2|\mathrm{search}_1|t]$
$\langle co_0,co_1,co_2|\mathrm{search}_1|co]$

def $[s|\mathrm{dec2}_2|t\rangle$:
$\langle s,co|\mathrm{inc2}_2|t]$

def $[s|\mathrm{pushinc}_1|t,co\rangle$:
$s.\ \underline{x}2 \vdash 1\underline{x}, c$
$|\ \underline{x}y1 \vdash 1x\underline{y}, pt$
$|\ \underline{x}yx \vdash 1\underline{y}x, pco$
$[c|\mathrm{endinc}_1|pt0,pco0\rangle$
$pt0.\ \rightarrow, t0$
$t0.\ 2 \vdash 2, pt$
$pt.\ \leftarrow, t$
$pco0.\ x \vdash 2, pco$
$pco.\ \leftarrow, zco$
$zco.\ 1 \vdash x, co$

def $[s|\mathrm{inc1}_1|t,co\rangle$:
$[s|\mathrm{search}_1|r_0,r_1,r_2\rangle$
$[r_0|\mathrm{pushinc}_1|t_0,co_0\rangle$
$[r_1|\mathrm{pushinc}_1|t_1,co_1\rangle$
$[r_2|\mathrm{pushinc}_1|t_2,co_2\rangle$
$\langle t_2,t_0,t_1|\mathrm{search}_1|t]$
$\langle co_0,co_1,co_2|\mathrm{search}_1|co]$

def $[s|\mathrm{dec1}_1|t\rangle$:
$\langle s,co|\mathrm{inc1}_1|t]$

def $[s|\mathrm{pushinc}_2|t,co\rangle$:
$s.\ \underline{x}2 \vdash 1\underline{x}, c$
$|\ \underline{x}y2 \vdash 1x\underline{y}, pt$
$|\ \underline{x}yy \vdash 1\underline{y}y, pco$
$[c|\mathrm{endinc}_2|pt0,pco0\rangle$
$pt0.\ \rightarrow, t0$
$t0.\ 2 \vdash 2, pt$
$pt.\ \leftarrow, t$
$pco0.\ x \vdash 2, pco$
$pco.\ \leftarrow, zco$
$zco.\ 1 \vdash x, co$

def $[s|\mathrm{inc1}_2|t,co\rangle$:
$[s|\mathrm{search}_1|r_0,r_1,r_2\rangle$
$[r_0|\mathrm{pushinc}_2|t_0,co_0\rangle$
$[r_1|\mathrm{pushinc}_2|t_1,co_1\rangle$
$[r_2|\mathrm{pushinc}_2|t_2,co_2\rangle$
$\langle t_2,t_0,t_1|\mathrm{search}_1|t]$
$\langle co_0,co_1,co_2|\mathrm{search}_1|co]$

def $[s|\mathrm{dec1}_2|t\rangle$:
$\langle s,co|\mathrm{inc1}_2|t]$

def $[s|\mathrm{init}_1|r\rangle$:
$s.\ \rightarrow, u$
$u.\ \underline{1}1 \vdash \underline{x}y, e$
$e.\ \leftarrow, r$

def $[s|\mathrm{RCM}_1|co_1,co_2\rangle$:
$[s|\mathrm{init}_1|s_0\rangle$
$[s_0|\mathrm{test1}|s_{1z},n\rangle$
$[s_1|\mathrm{inc1}_1|s_2,co_1\rangle$
$[s_2|\mathrm{inc2}_1|s_3,co_2\rangle$
$[s_3|\mathrm{test1}|n',s_{1p}\rangle$
$\langle s_{1z},s_{1p}|\mathrm{test1}|s_1]$

def $[s|\mathrm{init}_2|r\rangle$:
$s.\ \rightarrow, u$
$u.\ \underline{2}2 \vdash \underline{x}y, e$
$e.\ \leftarrow, r$

def $[s|\mathrm{RCM}_2|co_1,co_2\rangle$:
$[s|\mathrm{init}_2|s_0\rangle$
$[s_0|\mathrm{test1}|s_{1z},n\rangle$
$[s_1|\mathrm{inc1}_2|s_2,co_1\rangle$
$[s_2|\mathrm{inc2}_2|s_3,co_2\rangle$
$[s_3|\mathrm{test1}|n',s_{1p}\rangle$
$\langle s_{1z},s_{1p}|\mathrm{test1}|s_1]$

fun $[s|\mathrm{check}_1|t\rangle$:
$[s|\mathrm{RCM}_1|co_1,co_2,\ldots\rangle$
$\langle co_1,co_2,\ldots|\mathrm{RCM}_1|t]$

fun $[s|\mathrm{check}_2|t\rangle$:
$[s|\mathrm{RCM}_2|co_1,co_2,\ldots\rangle$
$\langle co_1,co_2,\ldots|\mathrm{RCM}_2|t]$

Periodicity

Theorem 8 **PP** is undecidable for RTM.

Idea Reduce **IP** to **PP**:

(1) **IP** is still undecidable for RTM **without periodic orbit**.

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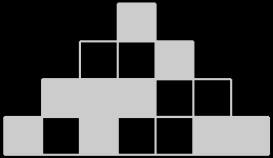
(3) $\mathcal{M}'$ is periodic iff $\mathcal{M}$ is mortal. $\Diamond$

3. Reversible Cellular Automata



A **CA** is a triple (S, r, f) where S is a finite set of states, r the radius and $f : S^{2r+1} \longrightarrow S$ the local rule.

DDS $(S^{\mathbb{Z}}, G)$ where $\forall z \in \mathbb{Z},\; G(c)(z) = f(c(z-r), \ldots, c(z+r))$

Periodicity

Theorem 12 **PP** is undecidable for RCA.

Idea Reduce **PP** for RTM to **PP** for RCA:

(1) **PP** is still undecidable for **complete** RTM.

Periodicity

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$\Sigma \times (S \times \{+, -\} \cup \{\leftarrow, \rightarrow\})$ simulating $\mathcal{M}$ on $+$ and $\mathcal{M}^{-1}$ on $-$ on two levels.

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(3) In case of local inconsistency, invert polarity.

(4) The RCA is periodic iff $\mathcal{M}$ is periodic. $\Diamond$

Open Problems with conjectures

Conjecture 1 It is undecidable whether a given complete 2-**RCM** admits a periodic configuration. *(proven if you remove complete or replace 2 by 3)*

Conjecture 2 There exists a complete **RTM** without a periodic configuration. *(known for DTM [BCN02])*

Conjecture 3 It is undecidable whether a given complete **RTM** admits a periodic configuration. *(known for DTM [BCN02])*