Intrinsically Universal Cellular Automata

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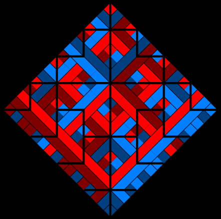
Complexity of Simple Programs, Cork
December 7, 2008



Cellular automata

Definition A **cellular automaton** (**CA**) is a quadruple (d, S, N, f) where S is a **finite set of states**, $N \subseteq_{\text{finite}} \mathbb{Z}^d$ is the **neighborhood** and $f : S^N \longrightarrow S$ is the **local rule**.

A **configuration** $c \in S^{\mathbb{Z}^d}$ is a coloring of $\mathbb{Z}^d$ by S.

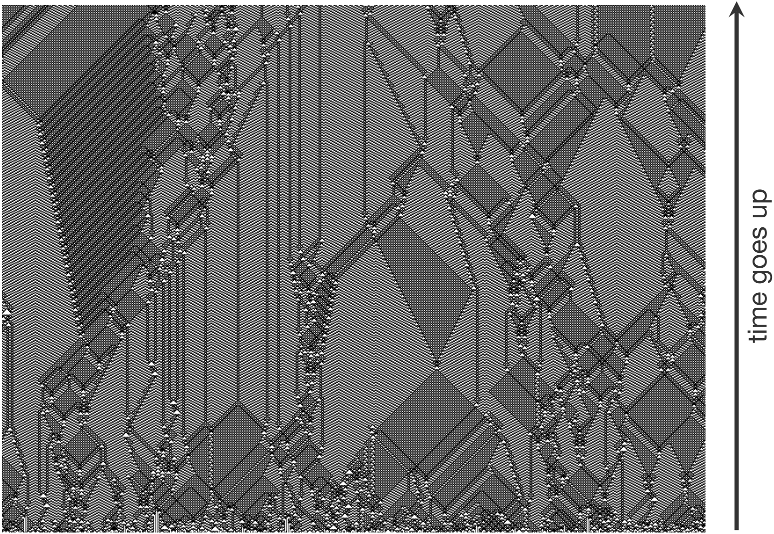
The **global map** $F : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$ applies f uniformly and locally:

$$\forall c \in S^{\mathbb{Z}}, \forall z \in \mathbb{Z}, \qquad F(c)(z) = f(c_{|z+N}).$$

A **space-time diagram** $\Delta \in S^{\mathbb{N} \times \mathbb{Z}^d}$ satisfies, for all $t \in \mathbb{Z}^+$,

$$\Delta(t+1) = F(\Delta(t)).$$

Space-time diagram



$S = \{0, 1, 2\}, r = 1, f(x, y, z) = \lfloor 6430564760289/3^{9x+3y+z} \rfloor \pmod 3$

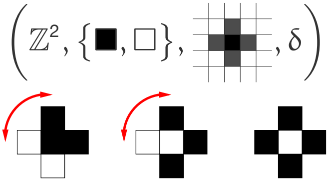
Universality in higher dimensions

Construction of universal CA appeared with CA as a tool to embed computation into the CA world. First, for **2D CA**

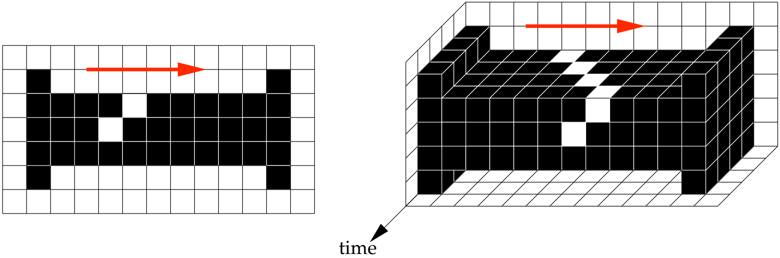
1966	von Neumann	5	25
1968	Codd	5	8
1970	Conway	8	2
1970	Banks	5	2

A natural idea in 2D is to emulate **universal boolean circuits** by embedding ingredients into the CA space: **signals**, **wires**, **turns**, **fan-outs**, **gates**, **delays**, **clocks**, *etc*.

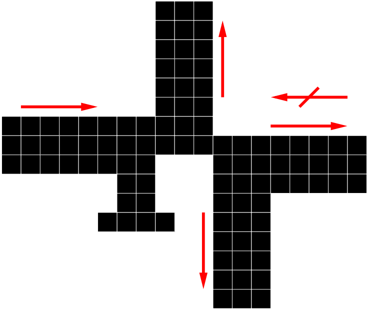
Banks’ 2-state Universal CA

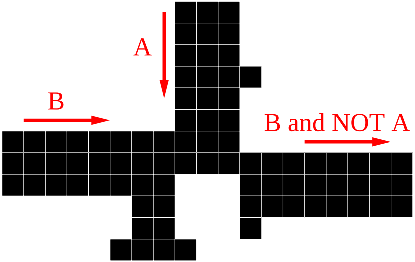


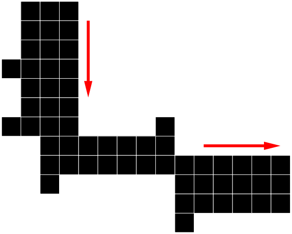
E. R. Banks. Universality in Cellular Automata. 1970

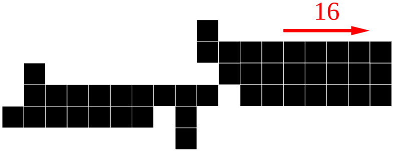


Banks’ CA: gadgets









Universality in 1D

Boolean circuits are **less intuitive** to simulate, but it is easy to simulate **sequential models of coumputation** like Turing machines.

A. R. Smith III. Simple computation-universal cellular spaces. 1971

1971	Smith III	18
1987	Albert & Culik II	14
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A cellular automaton is **Turing-universality** if... What exactly is the **formal definition**? What is a **non universal** CA?

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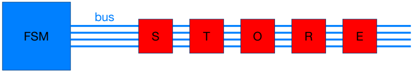
A cellular automaton is **Turing-universality** if... What exactly is the **formal definition**? What is a **non universal** CA?

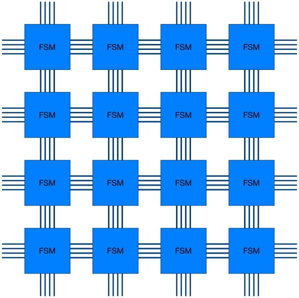
A consensual yet formal definition is unknown and seems difficult to achieve **[Durand & Roka 1999]**.

Another path to universality

Sequential models of computations are basically FSM + storage.



Boolean circuits can also simulate parallel models of computation.



This leads to a notion of **intrinsic universality** that is used implicitly in the literature **[Banks 1970] [Albert & Culik II 1987]**.

1. Intrinsic Universality

Bulking classifications

Idea define a **quasi-order** on cellular automata, **equivalence classes** capturing behaviors.

Grouping quasi-order **[Mazoyer & Rapaport 1999]** was introduced as a classification to capture simple **algebraic properties** of CA.

Bulking quasi-order **[NO PhD 2002]** is an extension of grouping to capture **algorithmic properties** and **intrinsic universality** as a maximal equivalence class.

The study was further developed in **[Theyssier PhD 2005]** where some less strict quasi-order where developed (skipped in this talk).

The sub-automaton relation

A CA $\mathcal{A}$ is **algorithmically simpler** than a CA $\mathcal{B}$ if all the space-time diagrams of $\mathcal{A}$ are space-time diagrams of $\mathcal{B}$.

Formally, $\mathcal{A} \subseteq \mathcal{B}$ if there exists $\varphi : S_{\mathcal{A}} \to S_{\mathcal{B}}$ injective such that

$$\overline{\varphi} \circ G_{\mathcal{A}} = G_{\mathcal{B}} \circ \overline{\varphi}$$

That is, the following diagram commutes:

$$\begin{array}{ccc}
C & \xrightarrow{\ \varphi\ } & \overline{\varphi}(C) \\
{\scriptstyle G_{\mathcal{A}}}\big\downarrow & & \big\downarrow{\scriptstyle G_{\mathcal{B}}} \\
G_{\mathcal{A}}(C) & \xrightarrow[\ \varphi\ ]{} & \overline{\varphi}(G_{\mathcal{A}}(C))
\end{array}$$

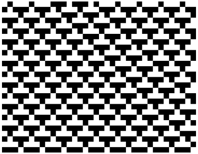
Remark Different elementary relations can be considered.

Bulking

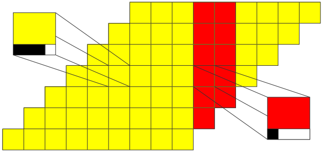
We quotient the set of CA by **discrete affine transformations**, the only geometrical transformations preserving CA.

The $\langle m, n, k\rangle$ transformation of $\mathcal{A}$ satisfies:

$$G_{\mathcal{A}^{\langle m,n,k\rangle}} = \sigma^k \circ o^m \circ G_{\mathcal{A}}^n \circ o^{-m} \ .$$



$\mathcal{A}$

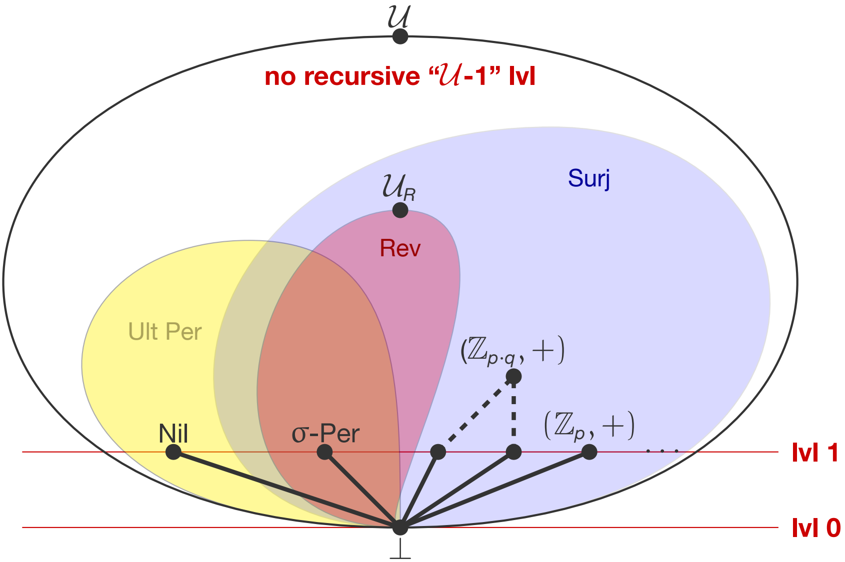


$\mathcal{A}^{\langle 4,4,1\rangle}$

The **bulking quasi-order** is defined by $\mathcal{A} \leqslant \mathcal{B}$ if there exists $\langle m, n, k\rangle$ and $\langle m', n', k'\rangle$ such that

$$\mathcal{A}^{\langle m,n,k\rangle} \subseteq \mathcal{B}^{\langle m',n',k'\rangle} \ .$$

The big picture



Intrinsic universality

A CA $\mathcal{U}$ is **intrinsically universal** if it is maximal for $\leqslant$,
i.e. for all CA $\mathcal{A}$, there exists α such that $\mathcal{A} \sqsubseteq \mathcal{U}^{\alpha}$.

Theorem There exists **Turing universal** CA that are not intrinsically universal.

Turing universality is obtained in a very classical way to ensure compatibility with your own definition.

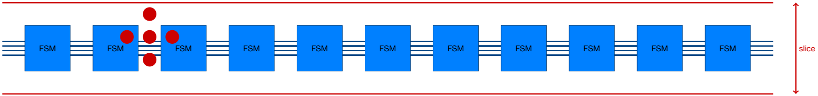
Theorem [NO STACS 2003] It is **undecidable**, given a CA to determine if it is intrinsically universal.

The proof proceeds by reduction of the nilpotency problem on spatially periodic configurations.

2. Constructing small universal CA

Using boolean circuits

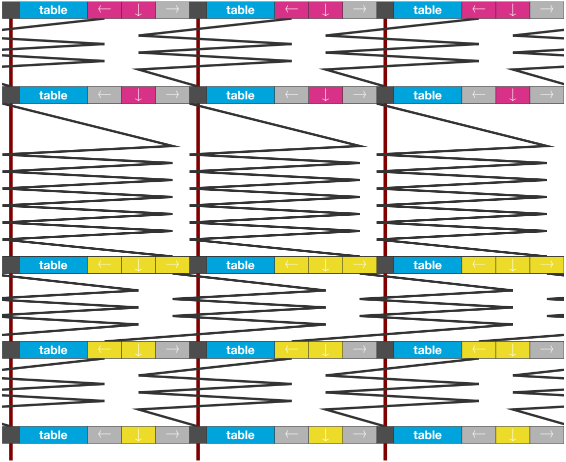
Every 2D intrinsically universal CA can be converted to a 1D intrinsically universal CA **[Banks 1970]**.



Cut **slices** of a periodic configuration, catenate them **horizontally**, use the **adequate neighborhood**.

The neighborhood can be transformed into radius 1 at the cost of **increase of the number of states**.

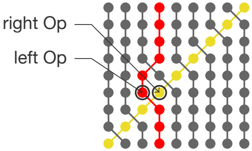
Using highly parallel Turing machines

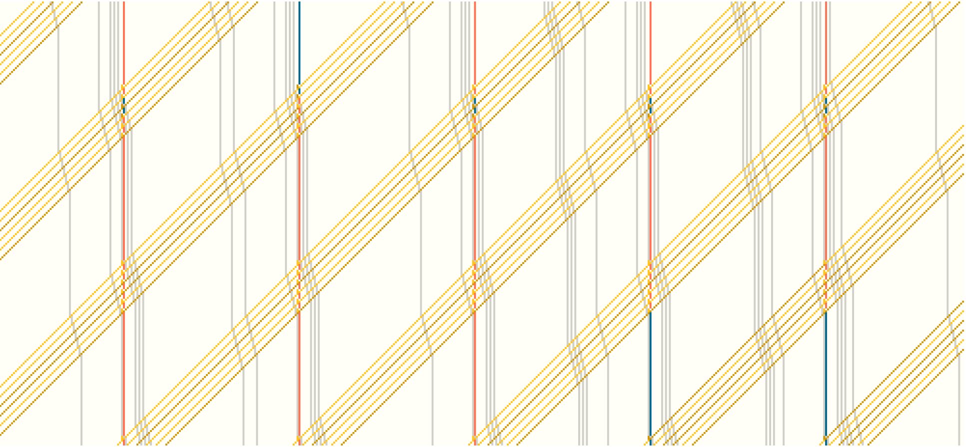


Use one **Turing-like head** per macro-cell, the **moving sequence** being **independent** of the computation.

6 states

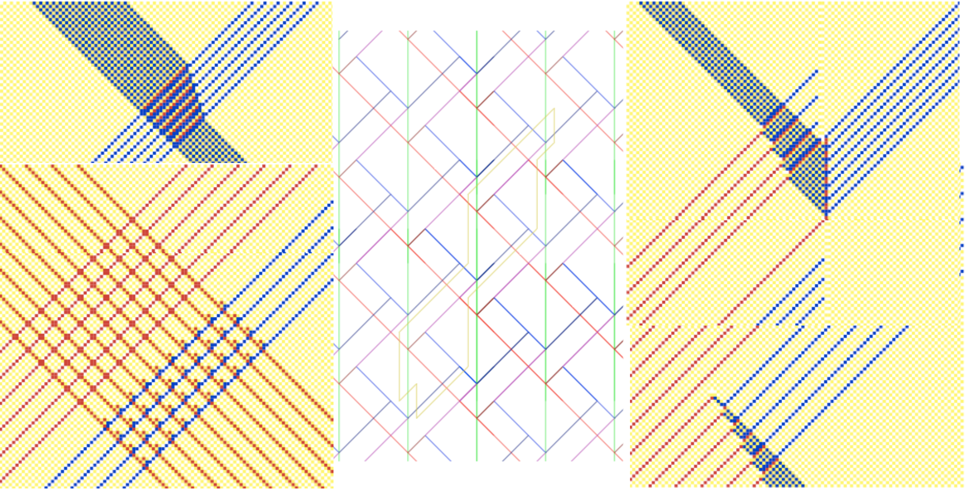
We constructed a **6 states** intrinsically universal CA of radius 1 embedding **boolean circuits** into the line **[NO ICALP 2002]**.





4 states

Using our framework for particles and collisions, this was improved to **4 states** by **arithmetical encoding** **[NO Richard CSP 2008]**.



3. Identifying non universal CA

Proving non universality

We have a **formal** definition of intrinsic universality. How do we prove that a CA is **not universal**?

Easy if the CA has a property that cannot be a property of universal CA: **injectivity**, **surjectivity**, **ultimate periodicity**, **additivity**, etc.

What about non trivial CA?

Maybe **communication complexity** might help?

[Goles, Meunier, Rapaport & Theyssier CSP 2008]

Deciding the pattern problem

Pattern Problem Given an **ultimately periodic configuration** and a **finite pattern**, decide whether the pattern appears in the orbit of the configuration.

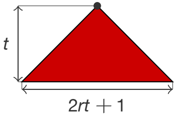
Decidable for simple CA.

$\mathbf{0}'$-complete for intrinsically universal CA.

...for non trivial CA, this requires **intermediate degrees**.

Complexity of the verification problem

Verification Problem Given a **finite ball** of radius rt and a state, decide whether in t steps, the ball reduces to the state.



Constant for trivial CA.

P-complete for intrinsically universal CA.

...for non trivial CA, this requires **separating P from lower classes**.

To go further...

Open Problem Is rule 110 intrinsically universal?

(we know that particles and collisions of Matthew are not enough)

Find better methods and invariants to prove **non universality**.