Tiling the Plane with a Fixed Number of Polyominoes

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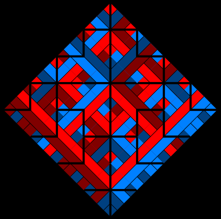
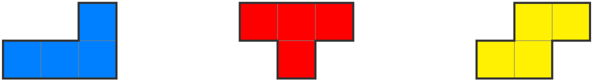
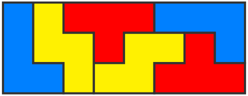


Polyominoes

A **polyomino** is a simply connected tile obtained by gluing together rookwise connected unit squares.



A **tiling** of a region by a set of polyominoes is a partition of the region into images of the tiles by isometries.

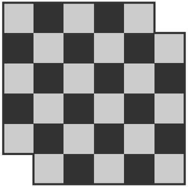


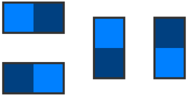
A **tiling by translation** is a tiling where isometries are restricted to translations.

Tiling finite regions

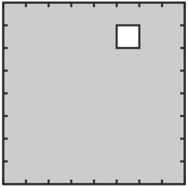
The combinatorics of tilings of finite regions is challenging, polyominoes make great puzzles.

Can you tile with dominoes a $2m \times 2n$ rectangle with two opposite corners cut? **[Golomb 1965]**





Can you tile with L-tiles a $2^n \times 2^n$ square with one cut unit square? **[Golomb 1965]**



Tiling the plane

In this talk, we consider tilings of **the whole Euclidian plane** by finite sets of polyominoes.

A tiling is **discrete** if all the unit squares composing images of the polyominoes are aligned on the grid $\mathbb{Z}^2$.

Lemma A set of tiles admits a tiling iff it admits a discrete tiling.

Sketch of the proof Non-discrete tilings have countably many infinite parallel fracture lines. By shifting along fracture lines, one constructs a discrete tiling from any non-discrete tiling.

The k-Polyomino Problem

Polyomino Problem

Given a finite set of polyominoes, decide if it can tile the plane.

***k*-Polyomino Problem**

Given a set of k polyominoes, decide if it can tile the plane.

Lemma Finite sets of polyominoes tiling the plane are **co-re**.

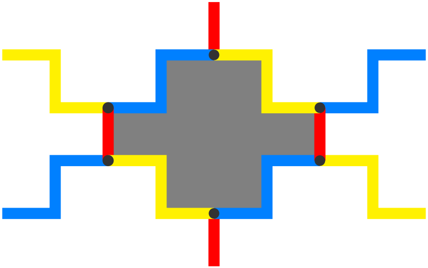
Sketch of the proof Consider tilings of finite regions covering larger and larger squares. If the set does not tile the plane, by compacity, there exists a size of square it cannot cover with tiles.

1. well known facts

One polyomino by translation

[Wijshoff and van Leeuwen 1984] A **single polyomino** that tiles the plane **by translation** tiles it biperiodically. The problem is decidable.

[Beauquier and Nivat 1991] A single polyomino tiles the plane by translation iff it is a *pseudo-hexagon* (contour word $uvw\tilde{u}\tilde{v}\tilde{w}$).

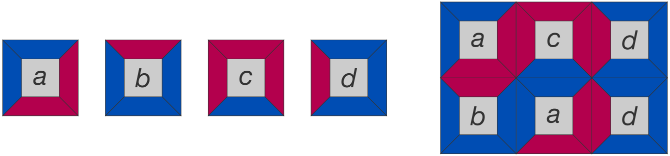


[Gambini and Vuillon 2007] This can be tested in $O(n^2)$.

The Domino Problem

"Assume we are **given a finite set of square plates** of the same size with **edges colored**, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are **not permitted to rotate or reflect a plate**. The question is to find an effective procedure by which we can **decide**, for each given finite set of plates, **whether we can cover up the whole plane** (or, equivalently, an infinite quadrant thereof) **with copies of the plates subject to the restriction that adjoining edges must have the same color**."

(Wang, 1961)



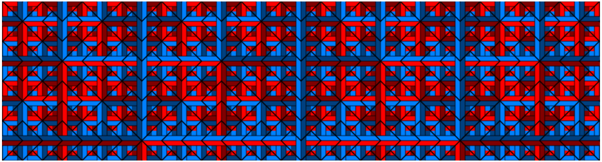
The Domino Problem is undecidable

Tile sets without tilings are recursively enumerable.

A set of Wang tiles with a periodic tiling admits a biperiodic tiling.

Tile sets with a biperiodic tiling are recursively enumerable.

Undecidability is to be found in **aperiodic tile sets**, tile sets that only admit aperiodic tilings.



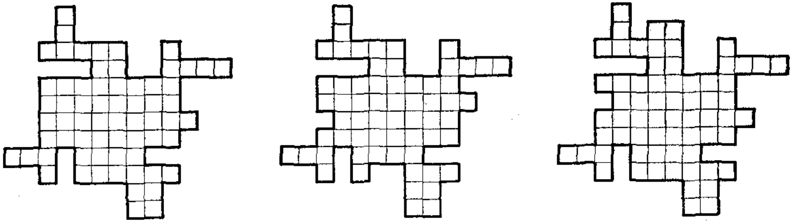
Theorem [Berger 1964] **DP** is undecidable.

The Polyomino Problem is undecidable

Wang tiles are oriented unit squares with colors.

Colors can be encoded by **bumps and dents**.

A Wang tile can be **encoded** as a big pseudo-square polyomino with bumps and dents in place of colors.



[Golomb 1970] The Polyomino Problem is undecidable.

Fixed number of polyominoes

The reduction of Golomb encodes N Wang tiles into N polyominoes.

What about the k-Polyomino Problem?

(1) either it is decidable for all k and the family of algorithms is not itself recursive *(eg. set of Wang tiles with k colors)*;

(2) either there exists a frontier between decidable and undecidable cases *(eg. Post Correspondence Problem)*.

We will show that (2) holds.

2. the 5-Polyomino Problem is undecidable

Dented polyominoes

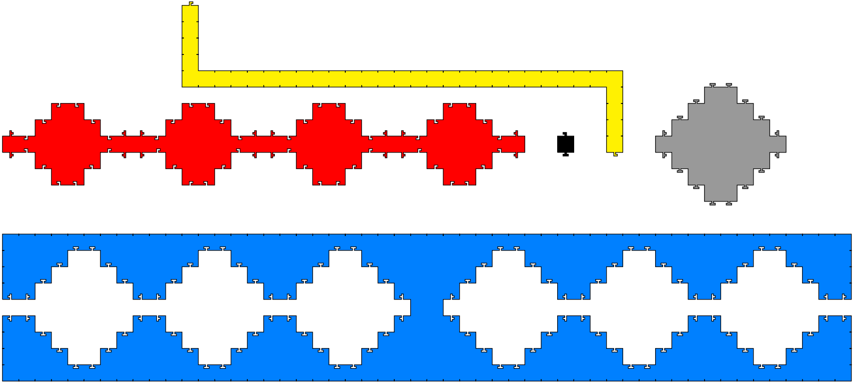
Computing with polyominoes relies on several levels of encoding. To lever the complexity of the tiles, we use dented polyominoes.

A **dented polyomino** is a polyomino with edges labeled by a **dent shape** and an **orientation**. When considering tilings, dents and bumps have to match.

Lemma Every set of k dented polyominoes can be encoded as a set of k polyominoes, preserving the set of tilings.

Sketch of the proof Scale each polyomino by a factor far larger than bumps, then add bumps and dents along edges.

5 tiles

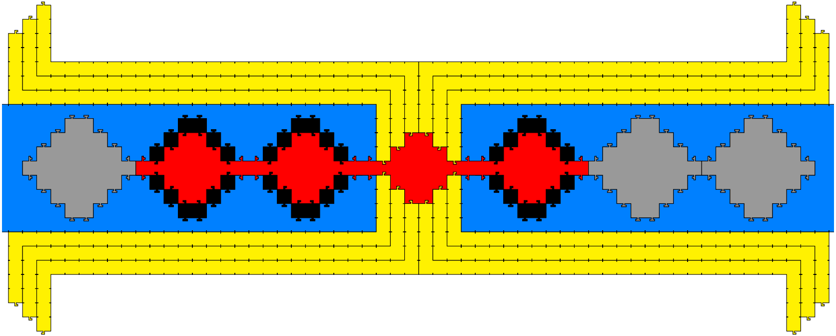


	blank	*bit*	*marker*	*inside*
shape				
bump		**wire**, **tooth**	**meat**, **filler**	**tooth**, **filler**
dent		**meat**	**jaw**	**jaw**

Encoding Wang tiles

A **meat** is placed in between two **jaws** to select a tile. The gaps inside the **jaws** are filled by **fillers** and **teeth**.

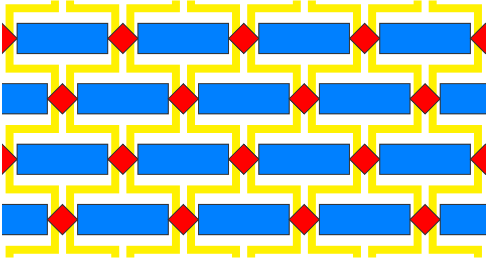
Wires connect Wang tiles.



Encoding a tiling by Wang tiles

Wang tiles are encoded and placed on a **regular grid**.

Tiles of a same diagonal are placed on a horizontal line sharing jaws.



Every tiling is coding

It remains to show to **difficult part of the proof**.

Why does every tiling codes a tiling by Wang tiles?

(1) The polyominoes locally enforce Wang tiles coding;

(2) Details on the encoding of colors enforce a same orientation for all Wang tiles in the plane.

Theorem The 5-Polyomino Problem is undecidable.

3. consequences and related open problems

Tiling by translation

Previous encoding uses **1** meat, **1** jaw, **1** filler, **4** wires, **4** teeth.

Theorem The 11-Polyomino Translation Problem is undecidable.

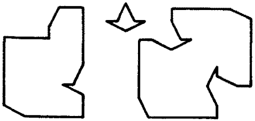
The problem is decidable for a single polyomino and undecidable for 11 polyominoes. What about $2 \leqslant k \leqslant 10$?

Even for $k = 2$, it seems that it is not trivial...

Aperiodic set of polyominoes

A weaker property is the existence of **aperiodic** sets of polyominoes.

If all sets of polyominoes are biperiodic for a given *k*, the *k*-Polyomino Problem is decidable.



[Ammann et al 1992] There exists an aperiodic set of 3 polyominoes.

[Ammann et al 1992] There exists an aperiodic set of 8 polyominoes for tiling by translation.

Open problem

Tiling Study $1 \leqslant k \leqslant 4$, aperiodicity for $1 \leqslant k \leqslant 2$.

Tiling by translation Study $2 \leqslant k \leqslant 10$, aperiodicity for $2 \leqslant k \leqslant 7$.

The following (old) problem is still open...

Open Problem Does there exist an aperiodic polyomino?