L'indécidable dynamique des automates cellulaires

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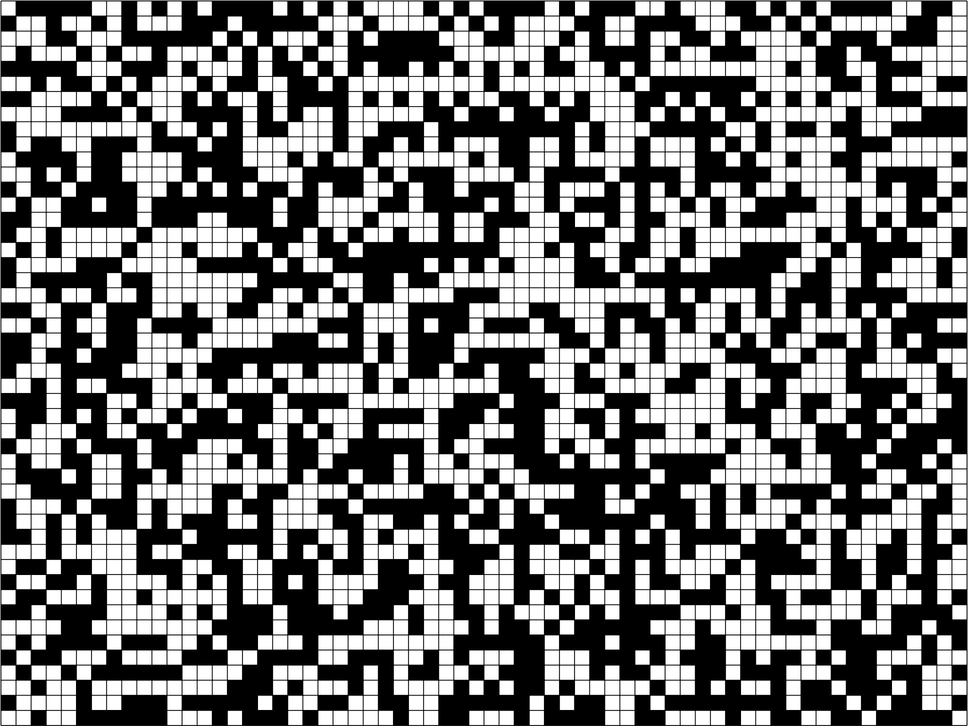
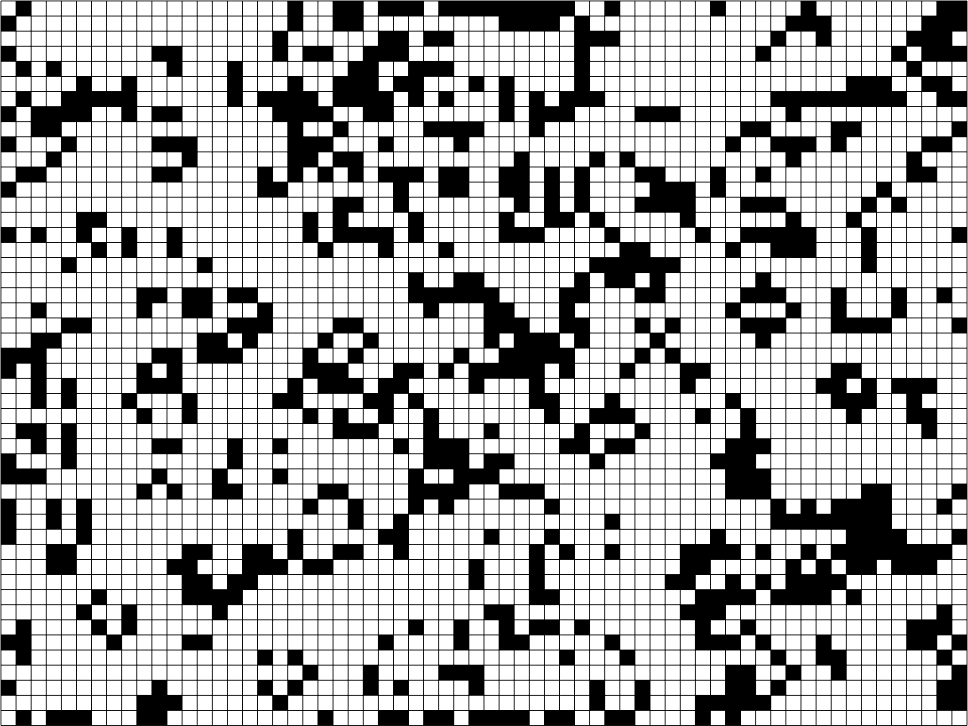
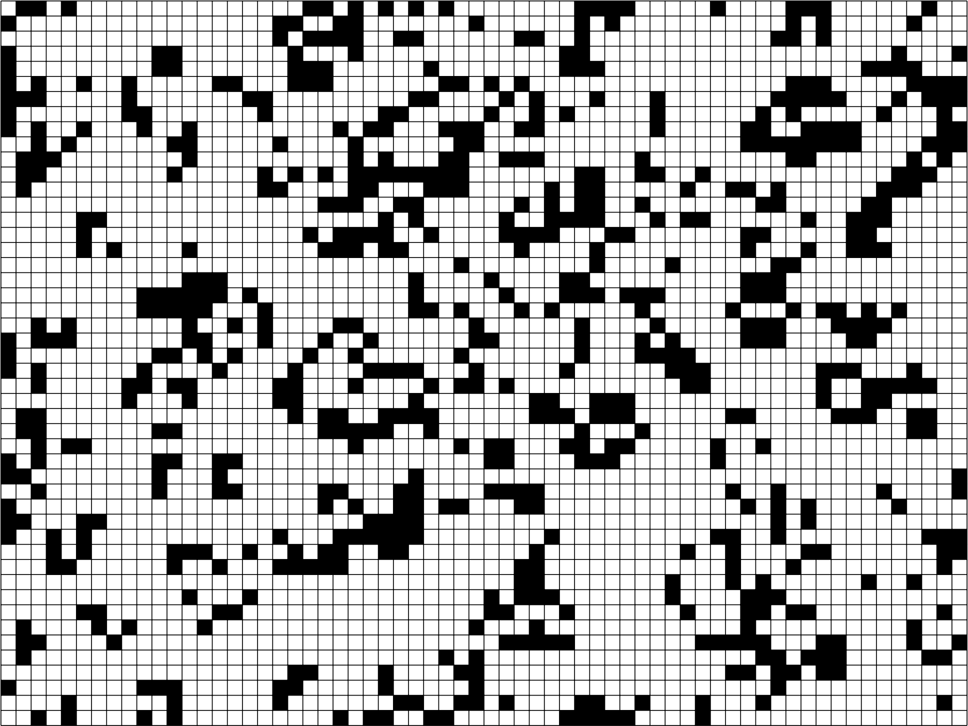
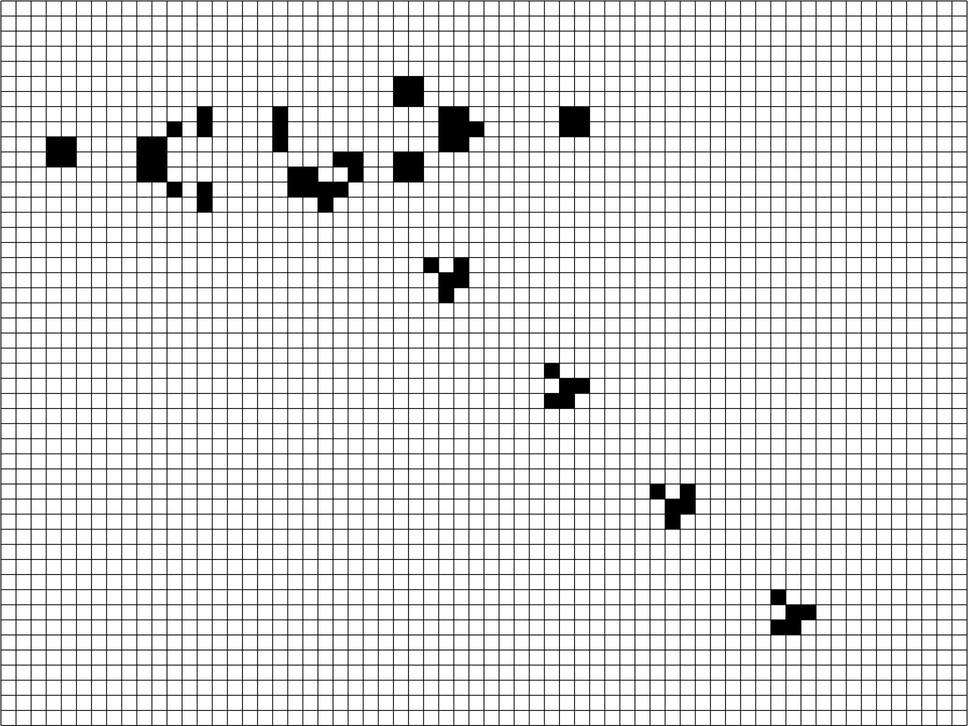
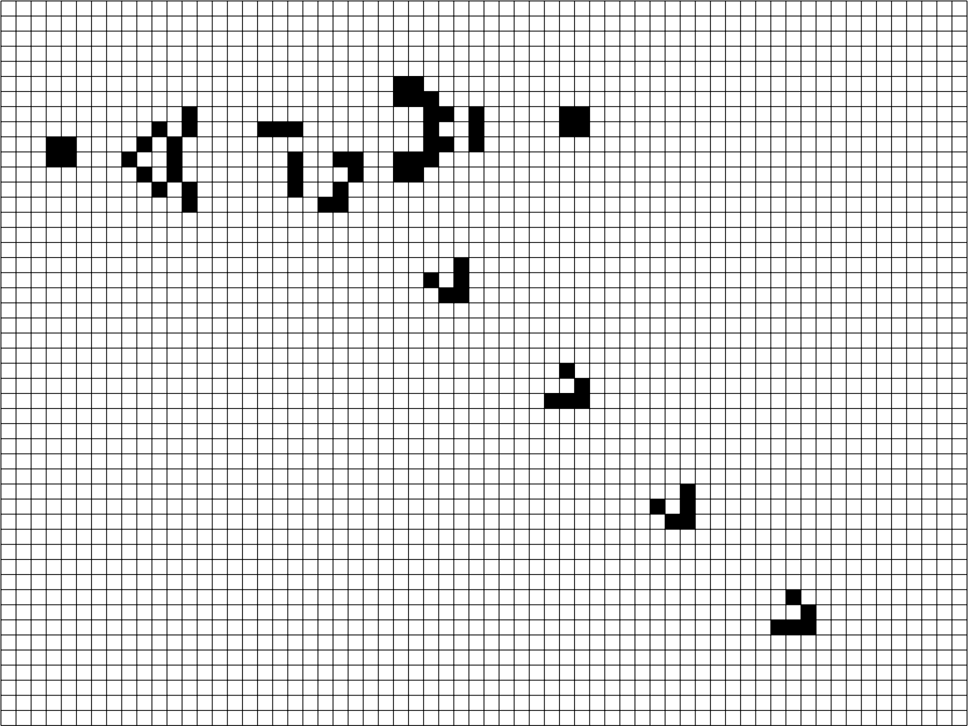
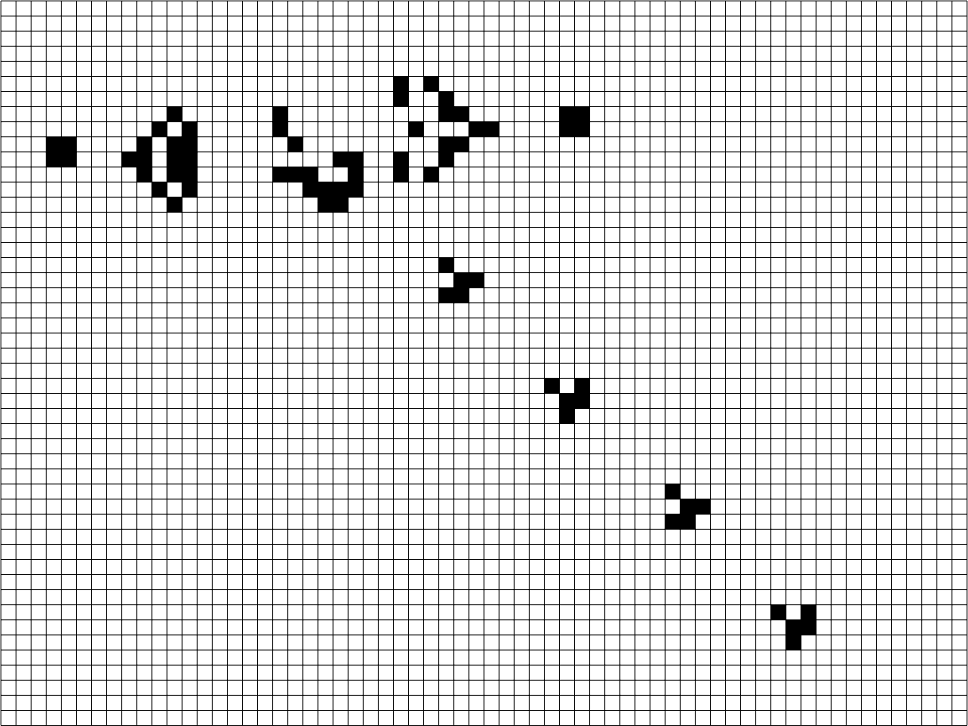


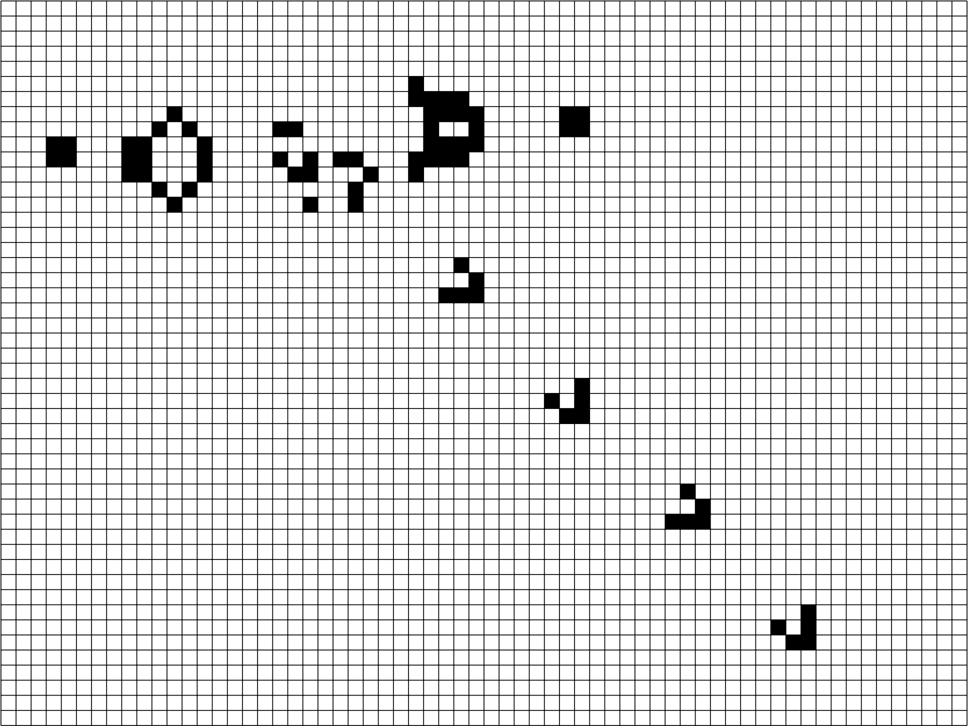


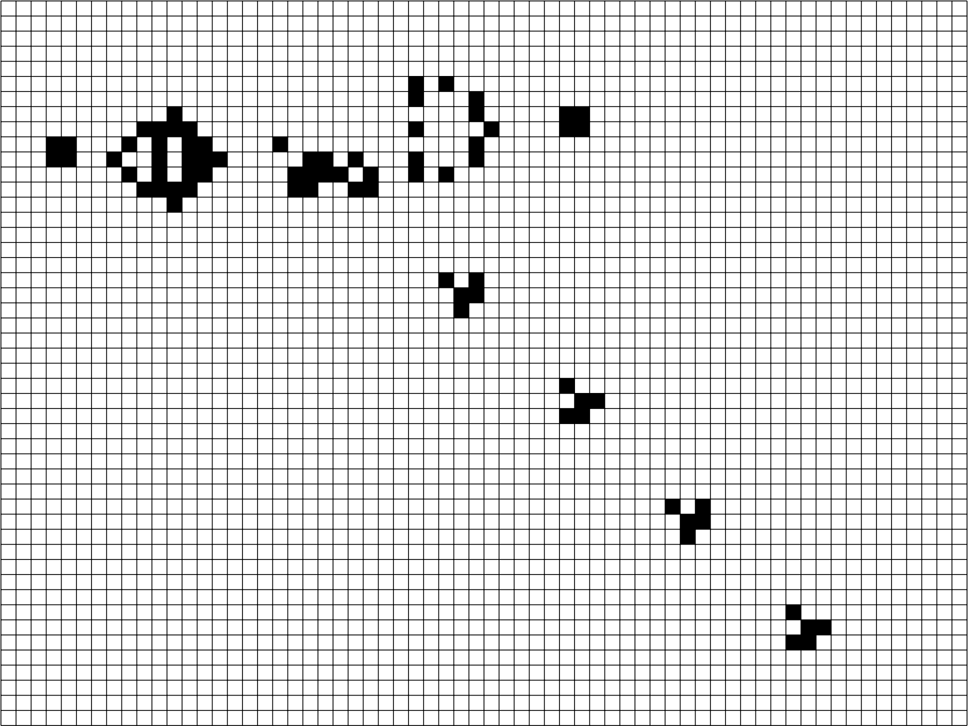


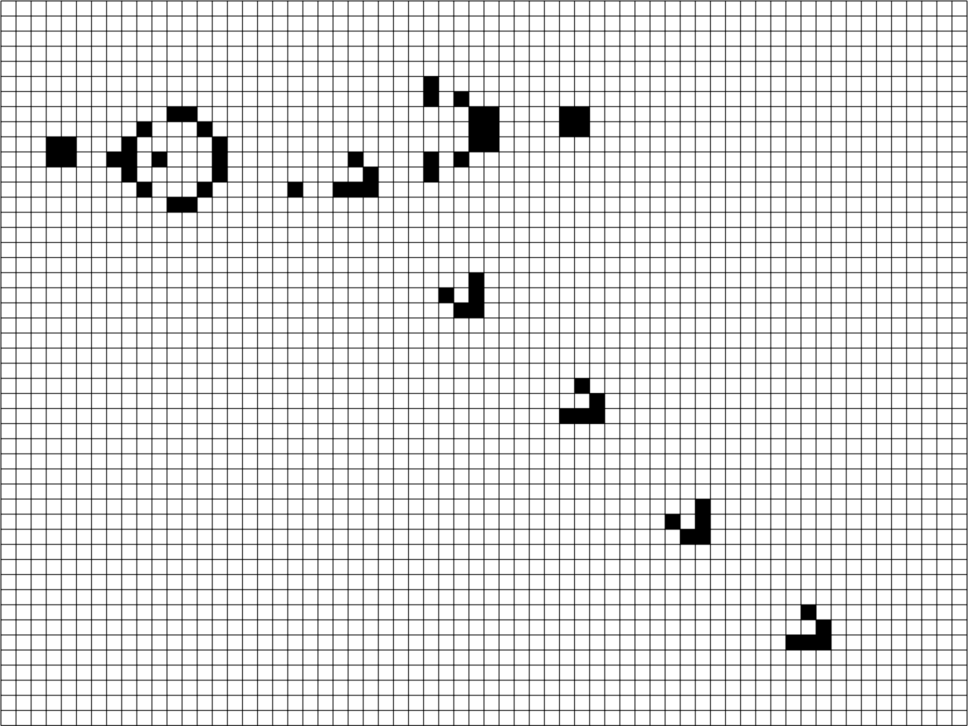


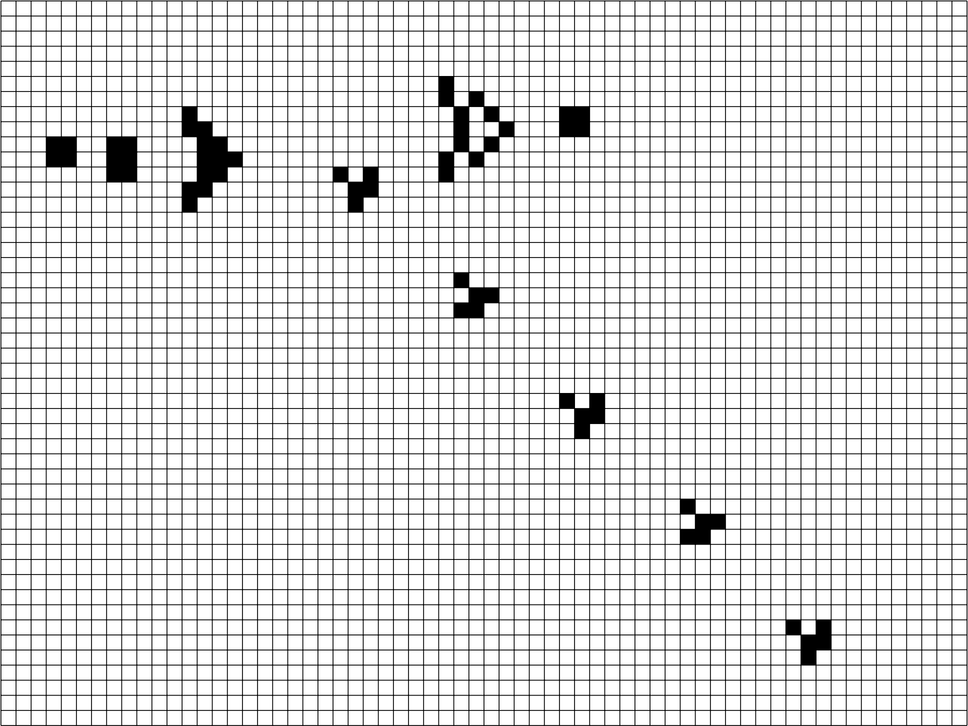


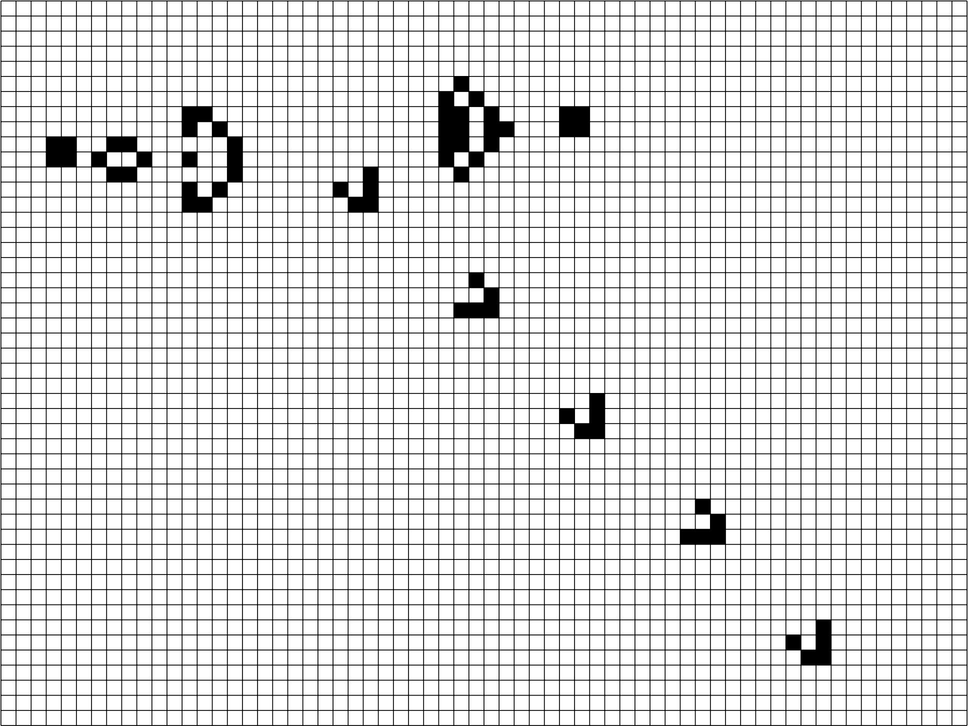


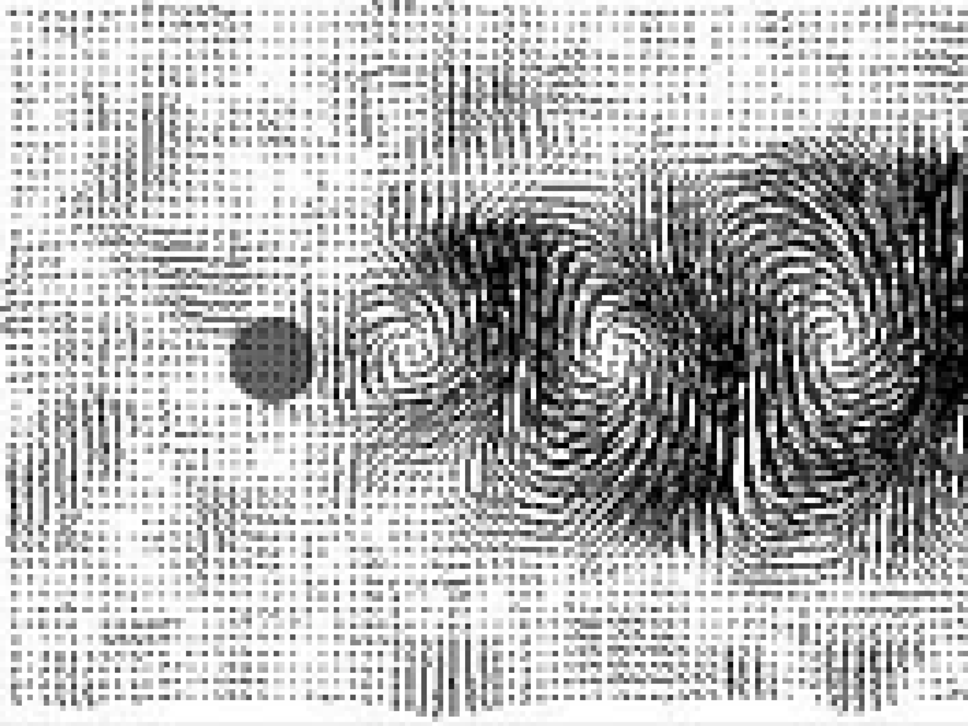






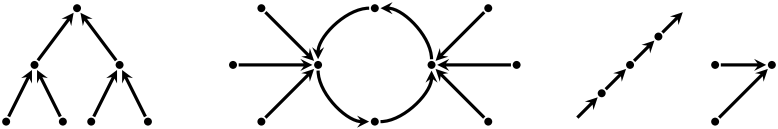






Discrete dynamical systems

Definition A **DDS** is a pair (X, F) where X is a topological space and $F : X \to X$ is a continuous map.



Definition The **orbit** of $x \in X$ is the sequence $(F^n(x))$ obtained by iterating F.

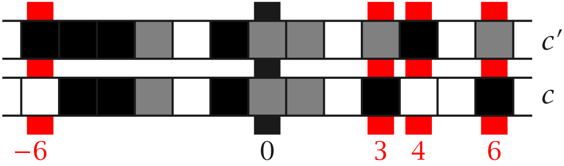
In this talk, $X = S^{\mathbb{Z}}$ is endowed with the **Cantor topology** (product of the discrete topology on S), and F is a continuous map **invariant by translation**.

Cantor topology

Definition The **Cantor topology** on $S^{\mathbb{Z}}$ is the product topology over $\mathbb{Z}$ of the discrete topology on S.

Remark The Cantor topology is **metric** and **compact**.

$$\forall c, c' \in S^{\mathbb{Z}}, d(c, c') = 2^{-\min\left\{|p| \,\middle|\, c_p \neq c'_p\right\}}$$



$d(c, c') = 1/8$

Definition A **subshift** is a non-empty set both topologically closed and closed by translation.

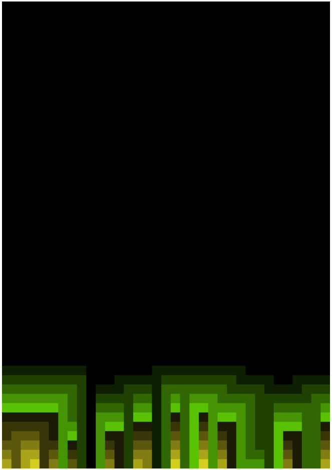
The nilpotency problem (Nil)

Definition A DDS is **nilpotent** if
$\exists z \in X, \forall x \in X, \exists n \in \mathbb{N}, F^n(x) = z.$

Given a recursive encoding of the DDS, can we **decide** nilpotency?

A DDS is **uniformly nilpotent** if
$\exists z \in X, \exists n \in \mathbb{N}, \forall x \in X, F^n(x) = z.$

Given a recursive encoding of the DDS, can we **bound recursively** n?



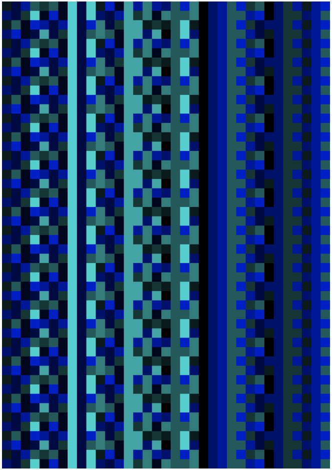
The periodicity problem (Per)

Definition A DDS is **periodic** if
$\forall x \in X, \exists n \in \mathbb{N}, F^n(x) = x$.

Given a recursive encoding of the DDS, can we **decide** periodicity?

A DDS is **uniformly periodic** if
$\exists n \in \mathbb{N}, \forall x \in X, F^n(x) = x$.

Given a recursive encoding of the DDS, can we **bound recursively** n?

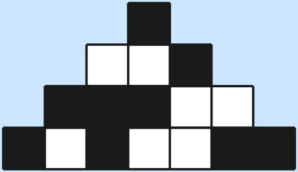


1. **Cellular Automata**

2. Nilpotency and tilings

3. Periodicity and mortality

4. Open problems



Cellular automata

Definition A **CA** is a triple (S, r, f) where S is a **finite set of states**, $r \in \mathbb{N}$ is the **radius** and $f : S^{2r+1} \to S$ is the **local rule** of the cellular automaton.

A **configuration** $c \in S^{\mathbb{Z}}$ is a coloring of $\mathbb{Z}$ by S.

The **global map** $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ applies f uniformly and locally:

$$\forall c \in S^{\mathbb{Z}}, \forall z \in \mathbb{Z}, \qquad F(c)(z) = f(c(z-r), \ldots, c(z+r)).$$

A **space-time diagram** $\Delta \in S^{\mathbb{N} \times \mathbb{Z}}$ satisfies, for all $t \in \mathbb{Z}^+$,

$$\Delta(t+1) = F(\Delta(t)).$$

Space-time diagram

$$S = \{0,1,2\},\ r = 1,\ f(x,y,z) = \left\lfloor 6450288690466/3^{9x+3y+z} \right\rfloor \pmod 3$$

Turing universality

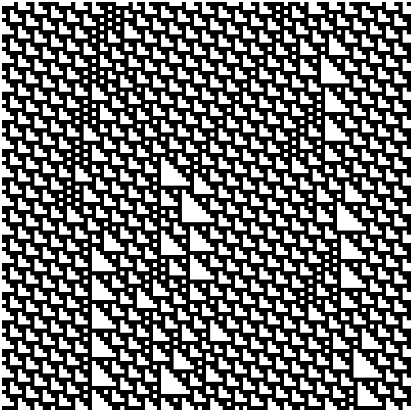
Theorem There exists **Turing-universal** CA.

à la Smith III

...BBBBBabaabBBBBB...
S

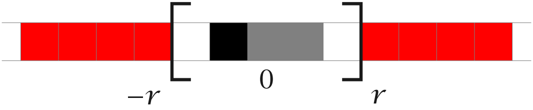
$(S \cup \Sigma, 2, f)$

à la Cook (rule 110)



Curtis-Hedlund-Lyndon's theorem

$$[m] = \left\{ c \in S^{\mathbb{Z}} \,\middle|\, \forall p \in \mathbb{Z}, |p| \leqslant r \Rightarrow c(p) = m(p) \right\}$$



Remark The **clopen sets** are finite unions of cylinders.

Therefore in this topology **continuity** means **locality**.

Theorem [Hedlund69] Cellular automata coincide with continuous maps invariant by translation.

Undecidability results

Theorem Both **Nil** and **Per** are **recursively undecidable**.

The proofs inject **computation** into **dynamics**.

Undecidability is not necessarily a negative result:
it is a **hint of complexity**.

Remark Due to **universe configurations** both nilpotency and periodicity are uniform.

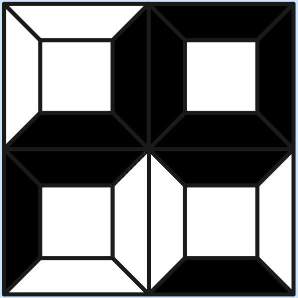
The bounds grow **faster than any recursive function**: there exists simple nilpotent or periodic CA with huge bounds.

1. Cellular Automata

2. Nilpotency and tilings

3. Periodicity and mortality

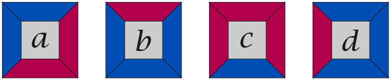
4. Open problems

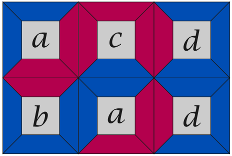


The Domino Problem (DP)

*"Assume we are **given a finite set of square plates** of the same size with **edges colored**, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are **not permitted to rotate or reflect a plate**. The question is to find an effective procedure by which we can **decide**, for each given finite set of plates, **whether we can cover up the whole plane** (or, equivalently, an infinite quadrant thereof) **with copies of the plates subject to the restriction that adjoining edges must have the same color**."*

(Wang, 1961)





Undecidability of DP

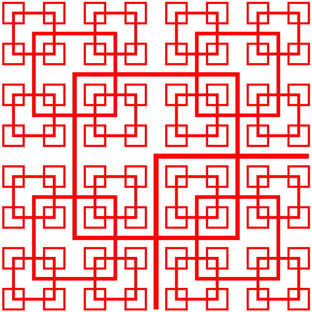
Theorem[Berger64] **DP** is **recursively undecidable**.

Remark To prove it one needs **aperiodic** tile sets.

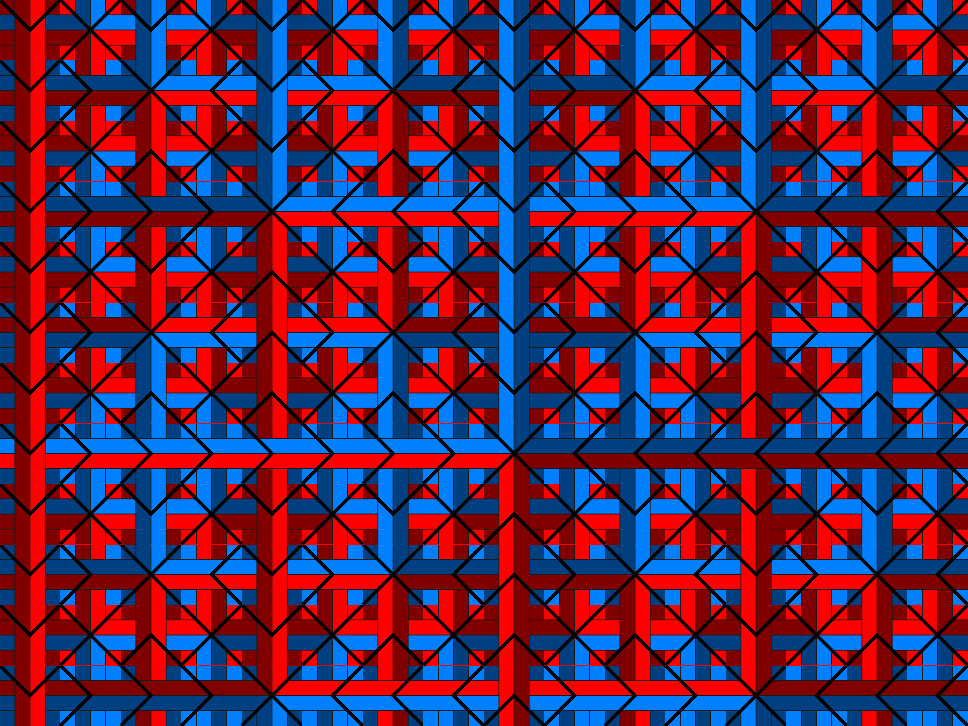
Idea of the proof

Enforce an (aperiodic) **self-similar structure** using local rules.

Insert a **Turing machine** computation **everywhere** using the structure.



Remark Plenty of different proofs!



Nilpotency and limit set

Definition The **limit set** of a CA F is the non-empty subshift

$$\Lambda_F = \bigcap_{n \in \mathbb{N}} F^n\left(S^{\mathbb{Z}}\right)$$

Remark Λ_F is the set of configurations appearing in **biinfinite space-time diagrams** $\Delta \in S^{\mathbb{Z}\times\mathbb{Z}}$ such that $\forall t \in \mathbb{Z}, \Delta(t+1) = F(\Delta(t))$.

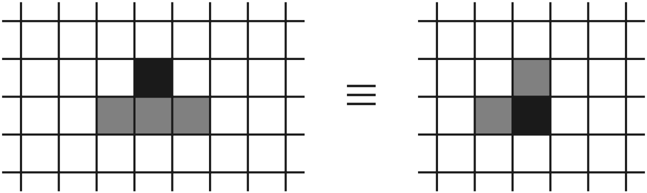
Lemma A CA is nilpotent iff its limit set is a **singleton**.

Reduction

A state $\perp \in S$ is **spreading** if $f(N) = \perp$ when $\perp \in N$.

A CA with a spreading state $\perp$ is not nilpotent iff it admits a biinfinite space-time diagram without $\perp$.

A tiling problem Find a coloring $\Delta \in (S \setminus \{\perp\})^{\mathbb{Z}^2}$ satisfying the tiling constraints given by f.



Theorem[Kari92] NW-DP $\leqslant_m$ Nil

Revisiting DP

Theorem[Kari92] **NW-DP** is **recursively undecidable**.

Remark Reprove of undecidability of **DP** with the additionnal determinism constraint!

Corollary **Nil** is **recursively undecidable**.

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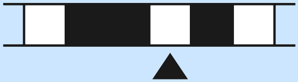

4. Open problems

The Immortality Problem (IP)

"(T_2) To find an effective method, which for every Turing-machine M decides whether or not, for all tapes I (finite and infinite) and all states B, M will eventually halt if started in state B on tape I" *(Büchi, 1962)*

Definition A **TM** is a triple (S, Σ, T) with S the set of states, Σ the alphabet and T a set of instructions of two kinds:

(s, δ, t) : *"in state s move in direction δ and enter state t."*

(s, a, t, b) : *"in state s, reading letter a, write letter b and enter state t."*

A **configuration** $\mathfrak{c} \in S \times \Sigma^{\mathbb{Z}}$ is a pair (s, c) where s is the state and the head points at position 0 of the tape c.

For **deterministic** TM, the **global map** $G : S \times \Sigma^{\mathbb{Z}} \to S \times \Sigma^{\mathbb{Z}}$ which applies instructions is a partial continuous map.

Undecidability of IP

Definition A TM is **mortal** if all configurations are ultimately halting.

Theorem[Hooper66] **IP** is **recursively undecidable**.

Remark To prove it one needs **aperiodic** TM.

Idea of the proof

Simulate 2-counters machines *à la* Minsky $(s, \underline{@}1^m\mathrm{x}2^n\mathrm{y})$

Replace **unbounded searches** by **recursive calls** to initial segments of the simulation.

Periodicity and reversibility

Definition A CA F is **reversible** if there exists a CA G such that $G = F^{-1}$.

Theorem A CA is **reversible** iff it is **bijective**.

Remark **Periodicity** implies **reversibility**.

Definition A TM (S, Σ, T) is **reversible** if (S, Σ, T^{-1}) is deterministic, where

$$\begin{aligned}(s, \delta, t)^{-1} &= (t, \delta, s) \\ (s, a, t, b)^{-1} &= (t, b, s, a)\end{aligned}$$

Reduction

Theorem[KO2008] R-**IP** $\leqslant_m$ **TM-Per** $\leqslant_m$ **Per**

Idea for TM-Per $\leqslant_m$ Per

Let $\mathcal{M} = (S, \Sigma, T)$ be a complete RTM
Let $(S', 2, f)$ be the RCA with set of states
$\Sigma \times (S \times \{+, -\} \cup \{\leftarrow, \rightarrow\})$ simulating $\mathcal{M}$ on $+$ and $\mathcal{M}^{-1}$ on $-$.
In case of local inconsistency, invert polarity.
The RCA is periodic iff $\mathcal{M}$ is periodic.

Revisiting IP

Theorem[KO2008] **R-IP** is **recursively undecidable**.

Remark Reprove of undecidability of **IP** with the additionnal reversibility constraint!

Corollary **TM-Per** and **Per** are **recursively undecidable**.

Program it!

```
def [s|search_1|t_0, t_1, t_2⟩ :
  s. @_α ⊢ @_α, l
  l. →, u
  u. x ⊢ x, t_0
   | 1x ⊢ 1x, t_1
   | 11x ⊢ 11x, t_2
   | 111 ⊢ 111, c
  call [c|check_1|p⟩ from 1
  p. 111 ⊢ 111, l

def [s|search_2|t_0, t_1, t_2⟩ :
  s. x ⊢ x, l
  l. →, u
  u. y ⊢ y, t_0
   | 2y ⊢ 2y, t_1
   | 22y ⊢ 22y, t_2
   | 222 ⊢ 222, c
  call [c|check_2|p⟩ from 2
  p. 222 ⊢ 222, l

def [s|test1|z, p⟩ :
  s. @_α x ⊢ @_α x, z
   | @_α 1 ⊢ @_α 1, p

def [s|endtest2|z, p⟩ :
  s. xy ⊢ xy, z
   | x2 ⊢ x2, p

def [s|test2|z, p⟩ :
  [s|search_1|t_0, t_1, t_2⟩
  [t_0|endtest2|z_0, p_0⟩
  [t_1|endtest2|z_1, p_1⟩
  [t_2|endtest2|z_2, p_2⟩
  ⟨z_0, z_1, z_2|search_1|z]
  ⟨p_0, p_1, p_2|search_1|p]

def [s|mark_1|t, co⟩ :
  s. y1 ⊢ 2y, t
   | yx ⊢ yx, co

def [s|endinc_1|t, co⟩ :
  [s|search_2|r_0, r_1, r_2⟩
  [r_0|mark_1|t_0, co_0⟩
  [r_1|mark_1|t_1, co_1⟩
  [r_2|mark_1|t_2, co_2⟩
  ⟨t_2, t_0, t_1|search_2|t]
  ⟨co_0, co_1, co_2|search_2|co]

def [s|inc2_1|t, co⟩ :
  [s|search_1|r_0, r_1, r_2⟩
  [r_0|endinc_1|t_0, co_0⟩
  [r_1|endinc_1|t_1, co_1⟩
  [r_2|endinc_1|t_2, co_2⟩
  ⟨t_0, t_1, t_2|search_1|t]
  ⟨co_0, co_1, co_2|search_1|co]

def [s|dec2_1|t⟩ :
  ⟨s, co|inc2_1|t]

def [s|mark_2|t, co⟩ :
  s. y2 ⊢ 2y, t
   | yx ⊢ yx, co

def [s|endinc_2|t, co⟩ :
  [s|search_2|r_0, r_1, r_2⟩
  [r_0|mark_2|t_0, co_0⟩
  [r_1|mark_2|t_1, co_1⟩
  [r_2|mark_2|t_2, co_2⟩
  ⟨t_2, t_0, t_1|search_2|t]
  ⟨co_0, co_1, co_2|search_2|co]

def [s|inc2_2|t, co⟩ :
  [s|search_1|r_0, r_1, r_2⟩
  [r_0|endinc_2|t_0, co_0⟩
  [r_1|endinc_2|t_1, co_1⟩
  [r_2|endinc_2|t_2, co_2⟩
  ⟨t_0, t_1, t_2|search_1|t]
  ⟨co_0, co_1, co_2|search_1|co]

def [s|dec2_2|t⟩ :
  ⟨s, co|inc2_2|t]

def [s|pushinc_1|t, co⟩ :
  s. x2 ⊢ 1x, c
   | xy1 ⊢ 1xy, pt
   | xyx ⊢ 1yx, pco
  [c|endinc_1|pt0, pco0⟩
  pt0. →, t0
  t0. 2 ⊢ 2, pt
  pt. ←, t
  pco0. x ⊢ 2, pco
  pco. ←, zco
  zco. 1 ⊢ x, co

def [s|inc1_1|t, co⟩ :
  [s|search_1|r_0, r_1, r_2⟩
  [r_0|pushinc_1|t_0, co_0⟩
  [r_1|pushinc_1|t_1, co_1⟩
  [r_2|pushinc_1|t_2, co_2⟩
  ⟨t_2, t_0, t_1|search_1|t]
  ⟨co_0, co_1, co_2|search_1|co]

def [s|dec1_1|t⟩ :
  ⟨s, co|inc1_1|t]

def [s|pushinc_2|t, co⟩ :
  s. x2 ⊢ 1x, c
   | xy2 ⊢ 1xy, pt
   | xyy ⊢ 1yy, pco
  [c|endinc_2|pt0, pco0⟩
  pt0. →, t0
  t0. 2 ⊢ 2, pt
  pt. ←, t
  pco0. x ⊢ 2, pco
  pco. ←, zco
  zco. 1 ⊢ x, co

def [s|inc1_2|t, co⟩ :
  [s|search_1|r_0, r_1, r_2⟩
  [r_0|pushinc_2|t_0, co_0⟩
  [r_1|pushinc_2|t_1, co_1⟩
  [r_2|pushinc_2|t_2, co_2⟩
  ⟨t_2, t_0, t_1|search_1|t]
  ⟨co_0, co_1, co_2|search_1|co]

def [s|dec1_2|t⟩ :
  ⟨s, co|inc1_2|t]

def [s|init_1|r⟩ :
  s. →, u
  u. 11 ⊢ xy, e
  e. ←, r

def [s|RCM_1|co_1, co_2⟩ :
  [s|init_1|s_0⟩
  [s_0|test1|s_1z, n⟩
  [s_1|inc1_1|s_2, co_1⟩
  [s_2|inc2_1|s_3, co_2⟩
  [s_3|test1|n', s_1p⟩
  ⟨s_1z, s_1p|test1|s_1]

def [s|init_2|r⟩ :
  s. →, u
  u. 22 ⊢ xy, e
  e. ←, r

def [s|RCM_2|co_1, co_2⟩ :
  [s|init_2|s_0⟩
  [s_0|test1|s_1z, n⟩
  [s_1|inc1_2|s_2, co_1⟩
  [s_2|inc2_2|s_3, co_2⟩
  [s_3|test1|n', s_1p⟩
  ⟨s_1z, s_1p|test1|s_1]

fun [s|check_1|t⟩ :
  [s|RCM_1|co_1, co_2, ...⟩
  ⟨co_1, co_2, ...|RCM_1|t]

fun [s|check_2|t⟩ :
  [s|RCM_2|co_1, co_2, ...⟩
  ⟨co_1, co_2, ...|RCM_2|t]
```

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? ? ?

Conclusion

We have proven the **undecidability** of **dynamical properties**.

The results extend to larger families of dynamical properties.

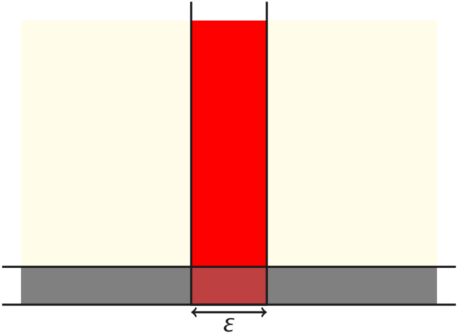
We consider the behaviour of the model starting from **arbitrary initial configurations**.

We use variations of two problems (**DP** and **IP**) introduced by Büchi and Wang to solve the $\forall\exists\forall$ class of the **classical decision problem** and later proven undecidable by Berger and Hooper, two PhD students of Wang.

Open Problem

Definition A CA F is **positively expansive** if

$$\exists \varepsilon > 0,\ \forall x \neq y,\ \exists n \geqslant 0,\ d\left(F^n(x), F^n(y)\right) \geqslant \varepsilon$$



Question Is positive expansivity **decidable**?