Indécidabilité, pavages et polyominos

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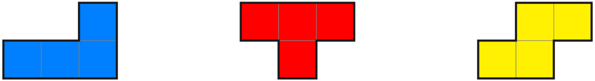
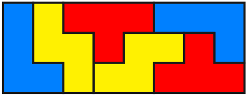
Polyominoes

Definition A **polyomino** is a simply connected tile obtained by gluing together rookwise connected unit squares.



Definition A **tiling** of a region by a set of polyominoes is a partition of the region into images of the tiles by isometries.



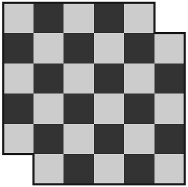
Definition A **tiling by translation** is a tiling where isometries are restricted to translations.

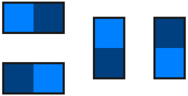
Tiling finite regions

Remark The combinatorics of tilings of finite regions is challenging, polyominoes make great puzzles.

Can you tile with dominoes a $2m \times 2n$ rectangle with two opposite corners cut?

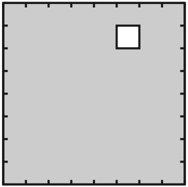
[Golomb 1965]





Can you tile with L-tiles a $2^n \times 2^n$ square with one cut unit square?

[Golomb 1965]



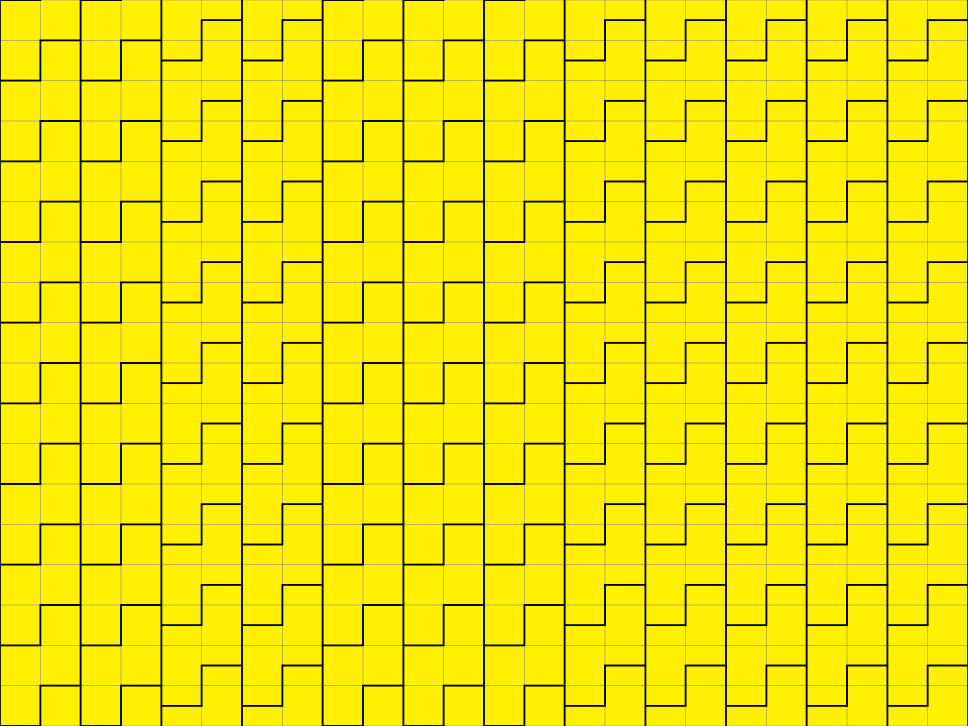
Tiling the plane

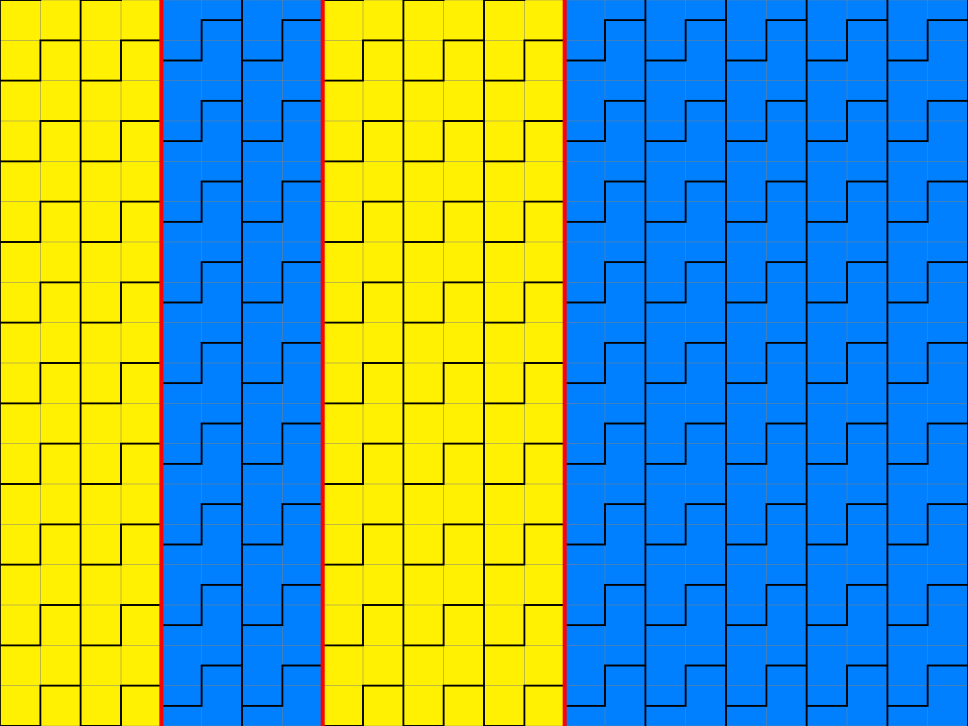
We consider tilings of **the whole Euclidian plane** by finite sets of polyominoes.

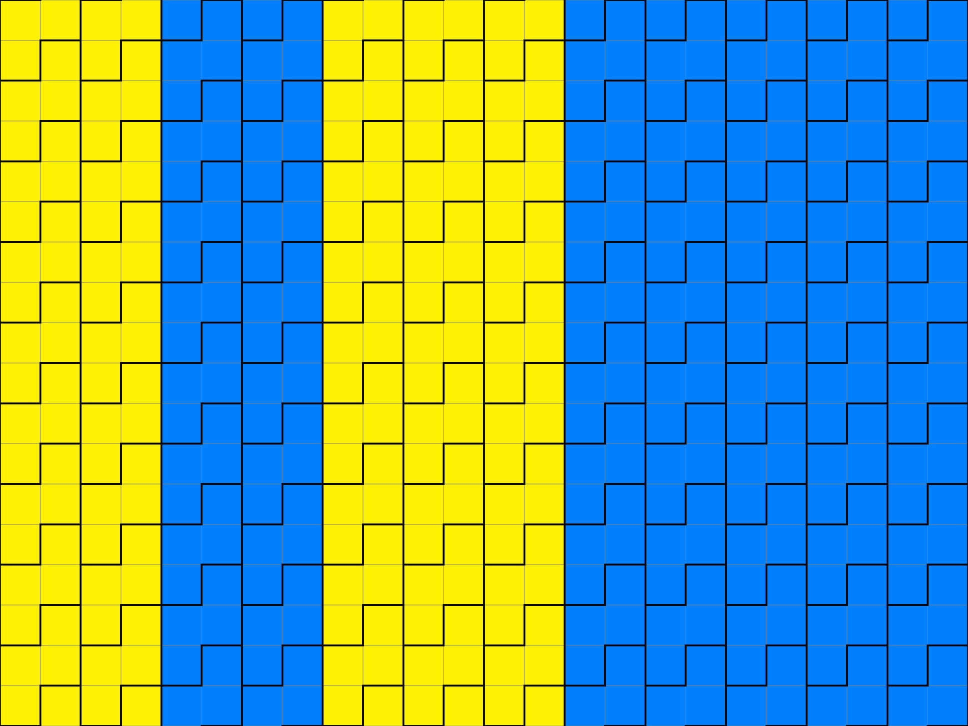
Definition A tiling is **discrete** if all the unit squares composing images of the polyominoes are aligned on the grid $\mathbb{Z}^2$.

Lemma A tile set admits a tiling iff it admits a discrete tiling.

Sketch of the proof Non-discrete tilings have countably many infinite parallel fracture lines. By shifting along fracture lines, one constructs a discrete tiling from any non-discrete tiling.







The k-Polyomino Problem

We consider the **complexity** of the following two problems:

Polyomino Problem Given a finite set of polyominoes, decide if it can tile the plane.

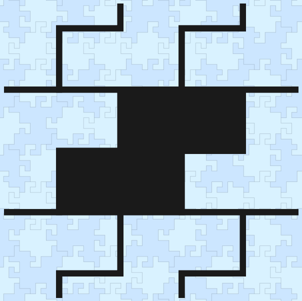
k-Polyomino Problem Given a set of k polyominoes, decide if it can tile the plane.

1. Complexity of tiling

2. The Polyomino Problem

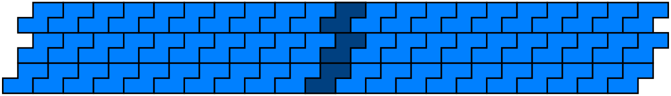
3. The k-Polyomino Problem

4. Conclusion

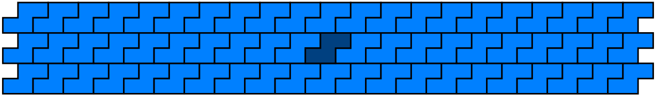


Periodic Tilings

Definition A tiling is **periodic** with period p if it is invariant by a **translation** of vector p.



Lemma If a finite set of polyominoes admits a **periodic** tiling then it admits a **biperiodic** tiling.

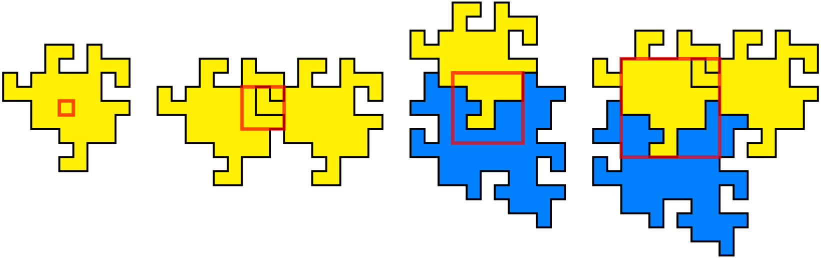


Lemma Finite sets of polyominoes tiling the plane biperiodically are **re** (**recursively enumerable**).

co-Tiling

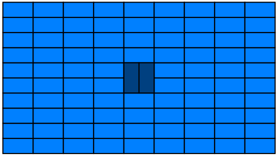
Lemma Finite sets of polyominoes tiling the plane are **co-re**.

Sketch of the proof Consider tilings of finite regions covering larger and larger squares. If the set does not tile the plane, by compacity, there exists a size of square it cannot cover with tiles.



Aperiodicity

Definition A tiling is **aperiodic** if it admits no non-trivial period.



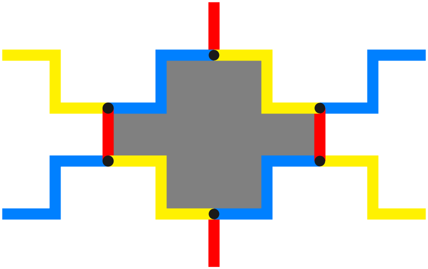
Definition A set of polyominoes is **aperiodic** if it admits a tiling and all its tilings are aperiodic.

Remark When there is **no aperiodic** set of tiles, the (k-)Polyomino Problem is **decidable**.

One polyomino by translation

Theorem[Wijshoff and van Leeuwen 1984] A **single polyomino** that tiles the plane **by translation** tiles it biperiodically. The problem is decidable.

Theorem[Beauquier and Nivat 1991] A single polyomino tiles the plane by translation iff it is a **pseudo-hexagon**.



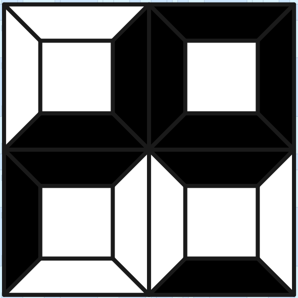
[Gambini and Vuillon 2007] This can be tested in $O(n^2)$.

1. Complexity of tiling

2. The Polyomino Problem

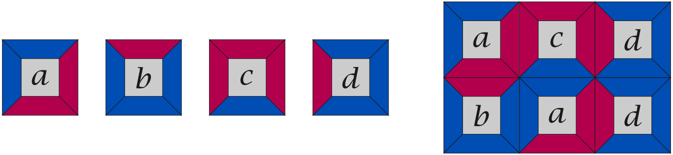
3. The k-Polyomino Problem

4. Conclusion

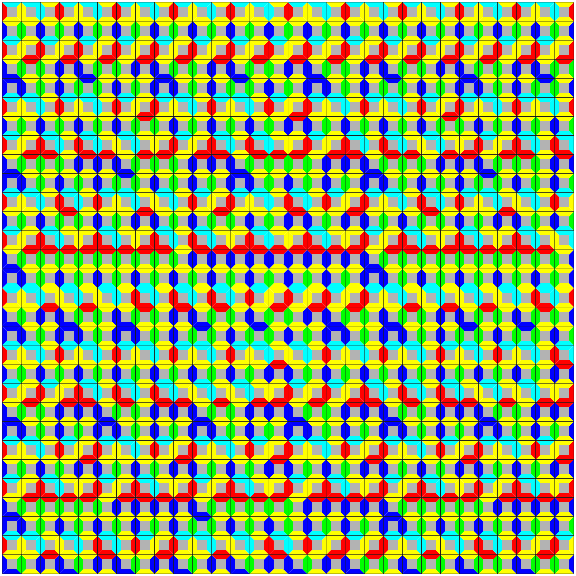


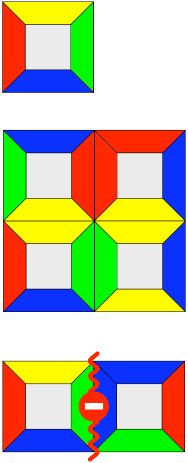
The Domino Problem (DP)

"Assume we are ***given a finite set of square plates*** *of the same size with* ***edges colored****, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are* ***not permitted to rotate or reflect a plate****. The question is to find an effective procedure by which we can* ***decide****, for each given finite set of plates,* ***whether we can cover up the whole plane*** *(or, equivalently, an infinite quadrant thereof)* ***with copies of the plates subject to the restriction that adjoining edges must have the same color****."* (Wang, 1961)



Wang tiles





Undecidability of DP

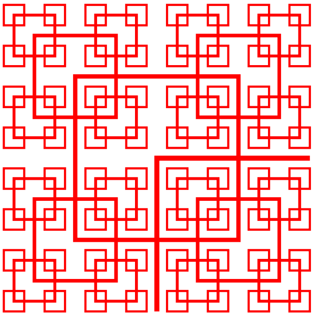
Theorem[Berger 1964] **DP** is **recursively undecidable**.

Remark To prove it one needs aperiodic tile sets.

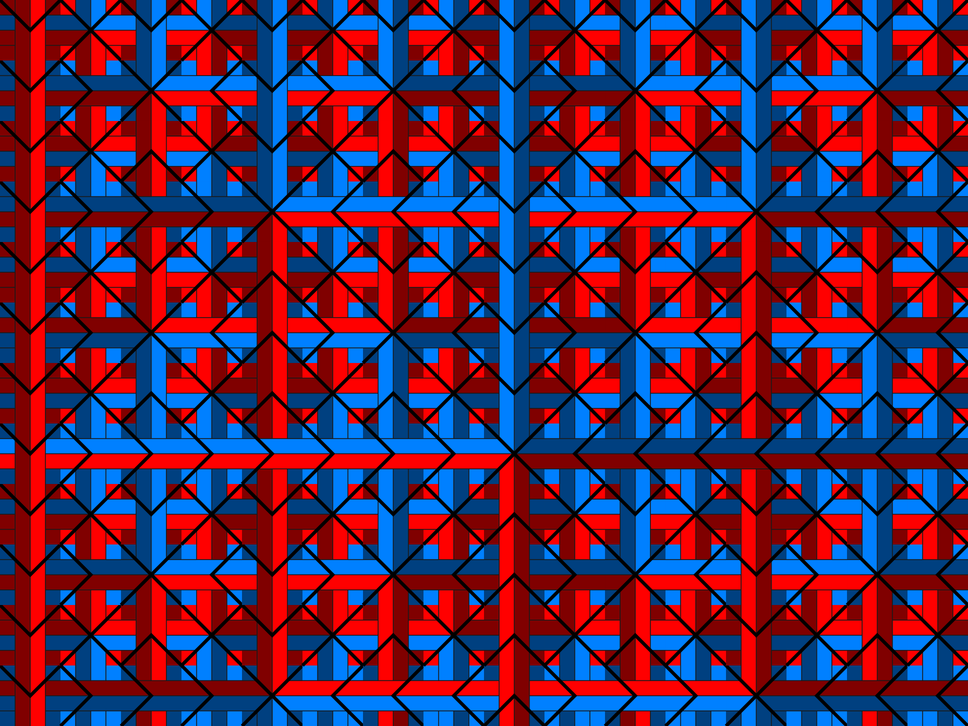
Idea of the proof

Enforce an (aperiodic) **self-similar structure** using local rules.

Insert a **Turing machine** computation **everywhere** using the structure.



Remark Plenty of different proofs!

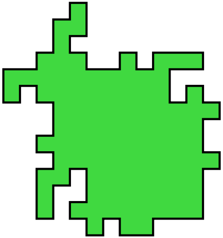


The Polyomino Problem is undecidable

Wang tiles are oriented unit squares with colors.

Colors can be encoded by **bumps and dents**.

A Wang tile can be **encoded** as a big pseudo-square polyomino with bumps and dents in place of colors.



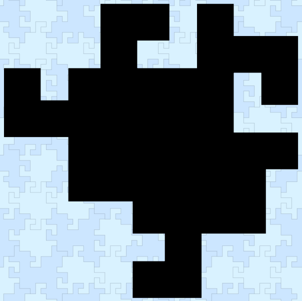
Theorem[Golomb 1970] The Polyomino Problem is **recursively undecidable**.

1. Complexity of tiling

2. The Polyomino Problem

3. The k-Polyomino Problem

4. Conclusion



Fixed number of polyominoes

Remark The reduction of Golomb encodes N Wang tiles into N polyominoes.

What about the **k-Polyomino Problem**?

(1) either it is decidable for all k and the family of algorithms is not itself recursive *(eg. set of Wang tiles with k colors)*;

(2) either there exists a frontier between decidable and undecidable cases *(eg. Post Correspondence Problem)*.

We will show that (2) holds.

Dented polyominoes

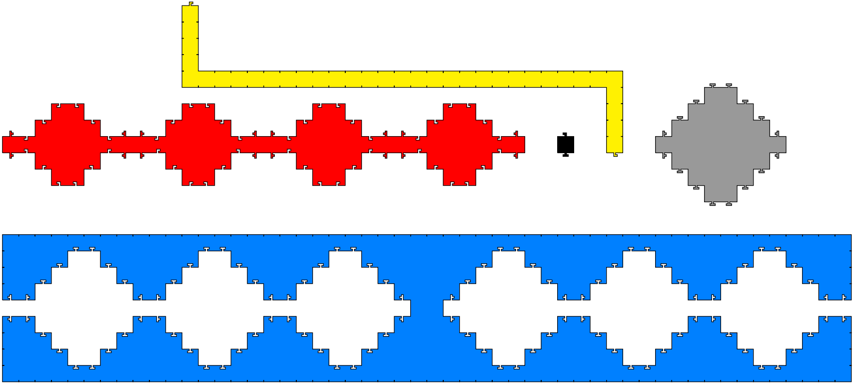
Computing with polyominoes relies on several levels of encoding. To lever the complexity of the tiles, we use dented polyominoes.

Definition A **dented polyomino** is a polyomino with edges labeled by a **dent shape** and an **orientation**. When considering tilings, dents and bumps have to match.

Lemma Every set of k dented polyominoes can be encoded as a set of k polyominoes, preserving the set of tilings.

Sketch of the proof Scale each polyomino by a factor far larger than bumps, then add bumps and dents along edges.

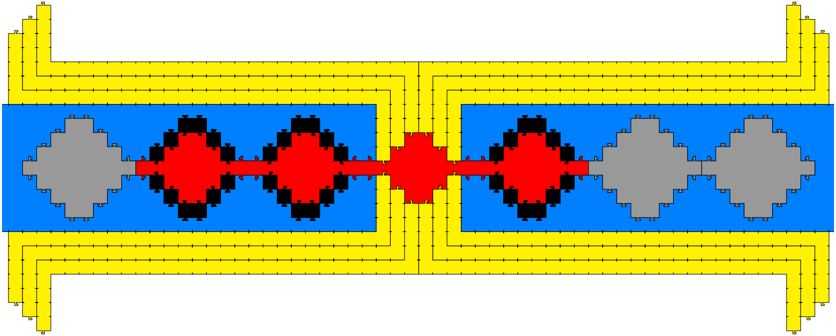
5 tiles



	blank	*bit*	*marker*	*inside*
shape				
bump		wire, **tooth**	**meat**, **filler**	**tooth**, **filler**
dent		**meat**	**jaw**	**jaw**

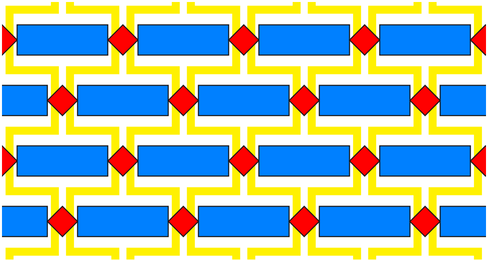
Encoding Wang tiles

A **meat** is placed in between two **jaws** to select a tile. The gaps inside the **jaws** are filled by **fillers** and **teeth**.
Wires connect Wang tiles.



Encoding a tiling by Wang tiles

Wang tiles are encoded and placed on a **regular grid**.
Tiles of a same diagonal are placed on a horizontal line sharing jaws.



Every tiling is coding

It remains to show to **difficult part of the proof**.

Why does every tiling codes a tiling by Wang tiles?

(1) The polyominoes locally enforce Wang tiles coding;

(2) Details on the encoding of colors enforce a same orientation for all Wang tiles in the plane.

Theorem[O 2009] The 5-Polyomino Problem is **undecidable**.

Tiling by translation

Previous encoding uses:

1 meat, **1** jaw, **1** filler, **4** wires, **4** teeth.

Theorem[O 2009] The 11-Polyomino Translation Problem is **undecidable**.

The problem is decidable for a single polyomino and undecidable for 11 polyominoes. What about $2 \leqslant k \leqslant 10$?

Even for $k = 2$, it seems that it is not trivial...

1. Complexity of tiling
2. The Polyomino Problem
3. The k-Polyomino Problem
4. **Conclusion**

???

Nonrecursive tilings

???

Patch Extension Problem Given a finite patch of tiles decide if it can be extended to a tiling.

Theorem[Myers 1974] There exists sets of Wang tiles for which the Patch Extension Problem is **undecidable**.

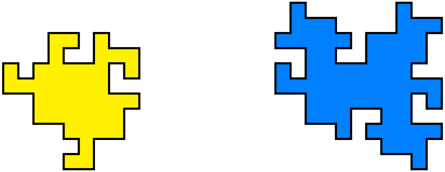
Remark The proof is **constructive** and gives an explicit aperiodic set of tiles.

Corollary There exists sets of 5 polyominoes for which the Patch Extension Problem is **undecidable**.

Aperiodic sets of polyominoes

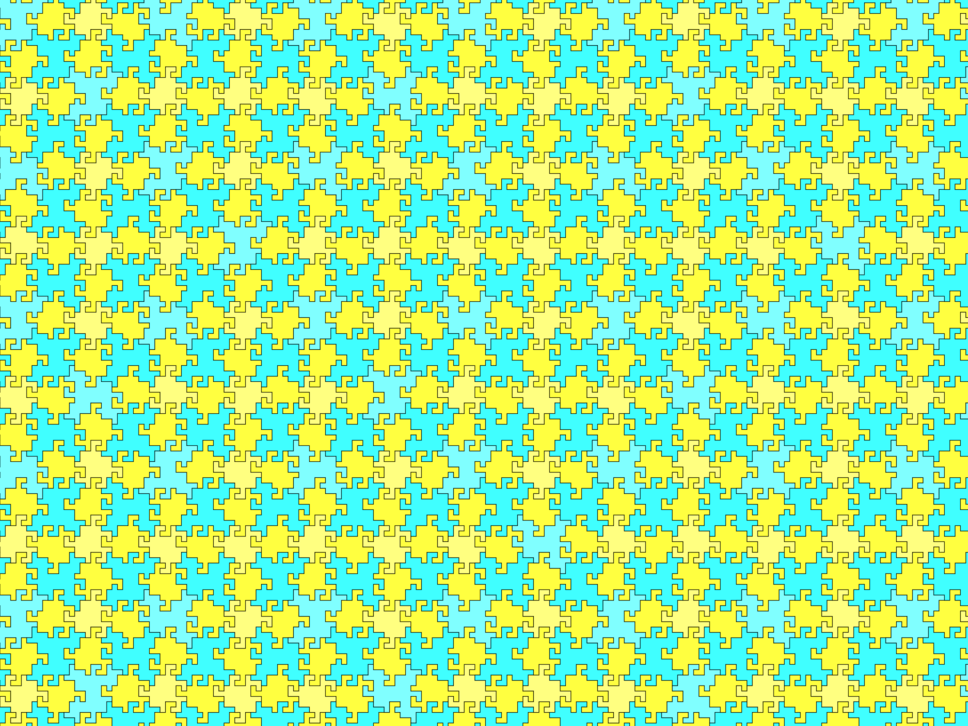
???

Remark If all tiling sets of polyominoes admit a biperiodic tiling for a given k, the k-Polyomino Problem is decidable.



Theorem[Goodman-Strauss 1999] There exists an **aperiodic set of 2 polyominoes**. **(as interpreted by Jolivet)**

Theorem[Ammann et al 1992] There exists an aperiodic set of **8 polyominoes for tiling by translation**.



Open problem

	Tiling	1	$2-4$	$5+$
aperiodic set	No	?	Yes	Yes
k-**PP**	$O(n^2)$	?	?	undecidable
Tiling by translation	1	$2-7$	$8-10$	$11+$

The following (old) problem is still open...

Open Problem Does there exists an aperiodic polyomino?