Combinatorial substitutions and tilings

Thomas Fernique & Nicolas Ollinger

LIF

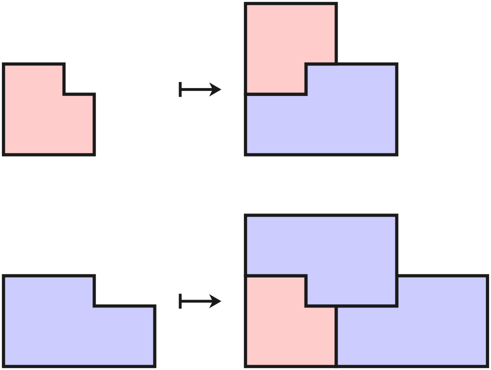
Journées SDA2, Caen — 20 juin 2011

Combinatorial substitutions and tilings

Thomas Fernique & Nicolas Ollinger

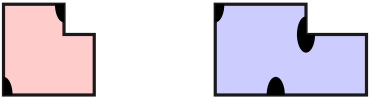
Journées SDA2, Caen — 20 juin 2011

Substitutions

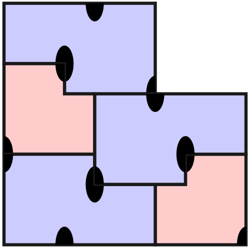


Rewriting rules that can be **iterated** to generate tilings with a **hierarchical** global structure.

Tilings



Local constraints that propagate to enforce some global structure.



(Ammann, Grünbaum, Shephard, 1992)

Goal

Proposition Tilings generated by any **fair substitution** can be enforced by finitely many **local constraints**.

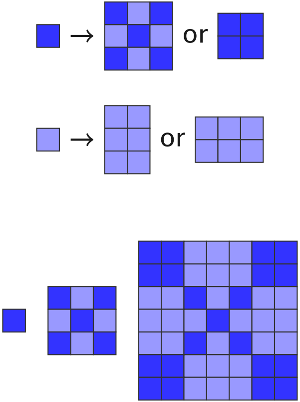
Remark The result is **constructive**: given a substitution, derive the set of local constraints.

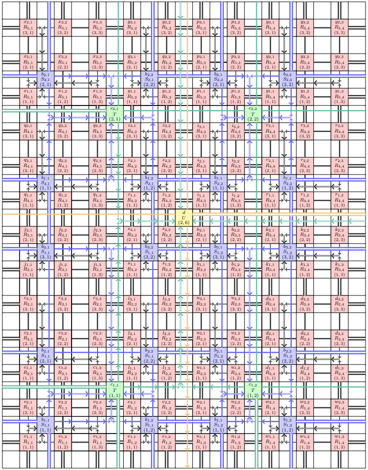
This is well-known old technology!

Remark **substitutions** are at the heart of most classical constructions of **aperiodic** sets of tiles.

We just extend (maybe simplify?) previous similar results.

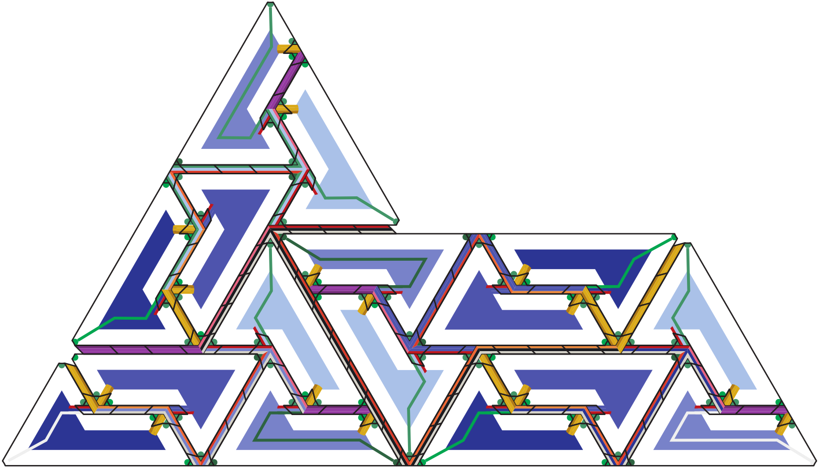
Mozes 1990





Theorem[Mozes 1990] The limit set of a **non-deterministic rectangular substitution** *(+ some hypothesis)* is sofic.

Goodman-Strauss 1998



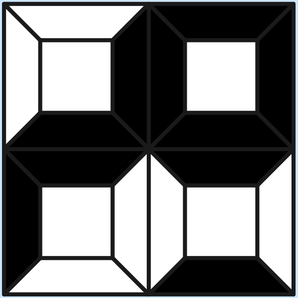
Theorem[Goodman-Strauss 1998] The limit set of **homothetic substitution** *(+ some hypothesis)* is sofic.

1. The classical recipe

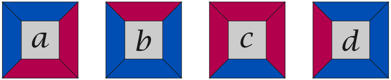
2. Combinatorial substitutions

3. Main result

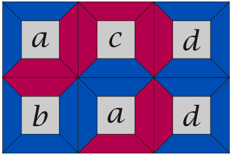
4. Conclusion & Open Pb



Wang tiles

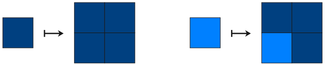


A **tile set** $\tau \subseteq \Sigma^4$ is a finite set of tiles with colored edges.

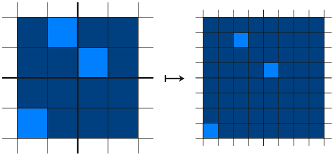


The set of **τ-tilings** $X_\tau \subseteq \tau^{\mathbb{Z}^2}$ is the set of colorings of $\mathbb{Z}^2$ by τ where colors match along edges.

Two-by-two substitutions

$s:$ 

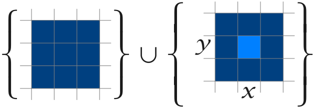
A **2x2 substitution** $s : \Sigma \to \Sigma^{\boxplus}$ maps letters to squares of letters on the same finite alphabet.

$S:$ 

The substitution is extended as a **global map** $S : \Sigma^{\mathbb{Z}^2} \to \Sigma^{\mathbb{Z}^2}$ on colorings of the plane:

$$\forall z \in \mathbb{Z}^2, \forall k \in \boxplus, \quad S(c)(2z+k) = s(c(z))(k)$$

Limit set and history

$\Lambda_S =$ 

$x,y \in \mathbb{Z}^2$

The **limit set** $\Lambda_s \subseteq \Sigma^{\mathbb{Z}^2}$ is the maximal attractor of S:

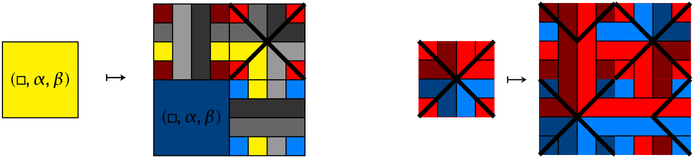
$$\Lambda_s = \bigcap_{t \in \mathbb{N}} \left\langle S^t \left(\mathbb{Z}^2 \right) \right\rangle_\sigma$$

The limit set is the set of colorings admitting an **history** $(c_i)_{i \in \mathbb{N}}$ where $c_i = s(c_{i+1})$.

Idea To encode Λ_s via **local matching rules** decorate s into a **locally checkable** $s^\bullet$ embedding a whole history.

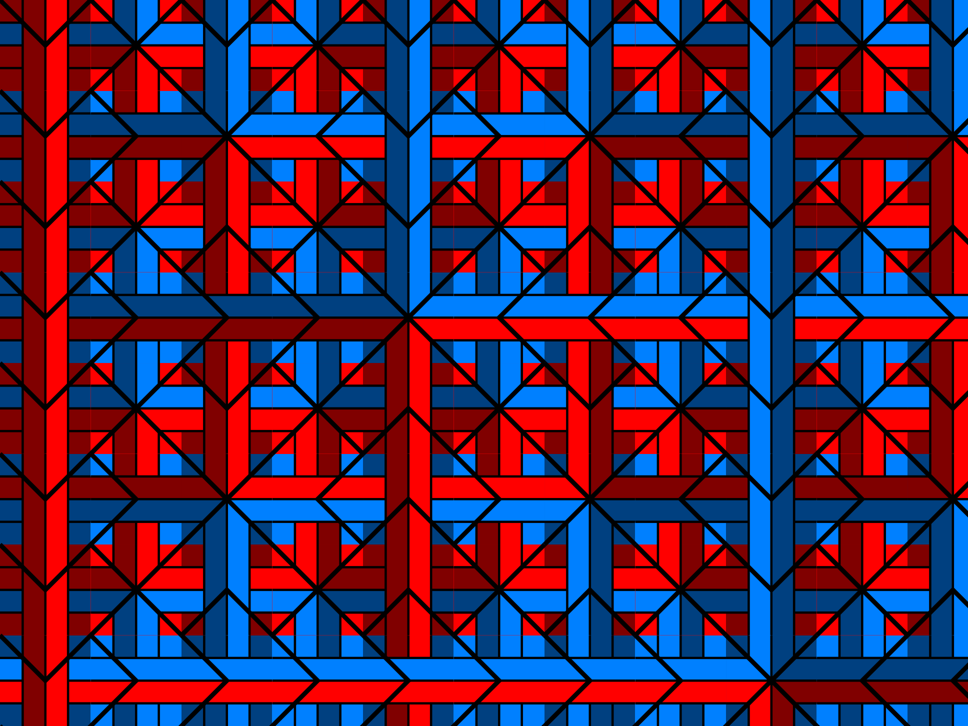
Self-simulation

A tile set τ **simulates** a tile set τ' with an encoding $f : \tau' \to \tau^n$ if tilings by τ decompose via f in tilings by τ'.



Tilings of a **self-simulating** tile set τ with encoding s are (a subset of) the **limit set** of s.

Idea To encode Λ_s via **local matching rules** find **fixpoints** of decorated simulation schemes.

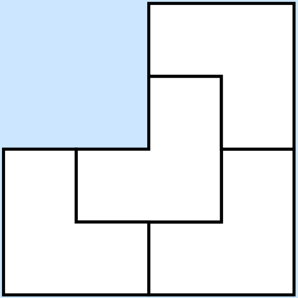


1. The classical recipe

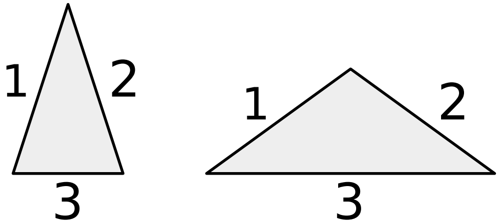
2. Combinatorial substitutions

3. Main result

4. Conclusion & Open Pb

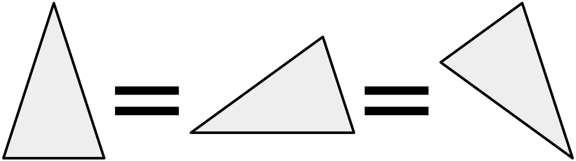


Tiles and tilings



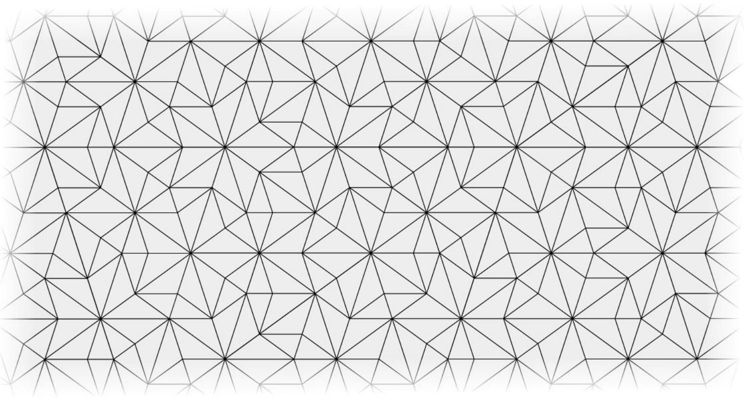
Tile polytope of $\mathbb{R}^d$ with finitely many (numbered) facets.

Tiles and tilings



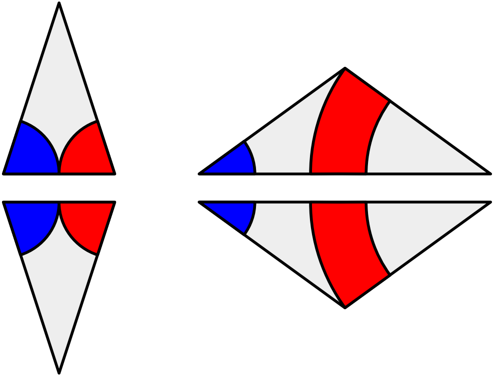
Tiles are here considered up to translations and rotations.

Tiles and tilings



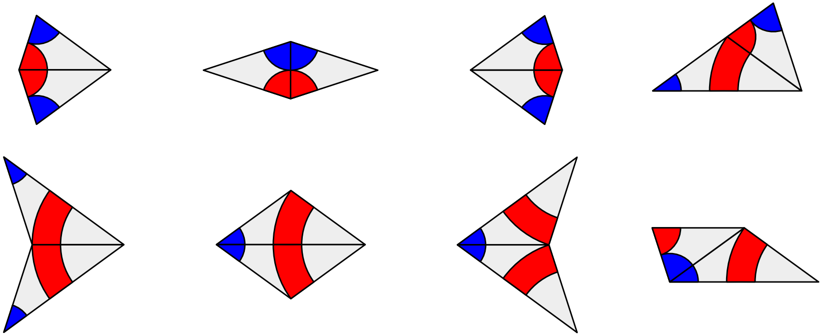
Tiling covering of $\mathbb{R}^d$ by **facet-to-facet** tiles.

Decorations



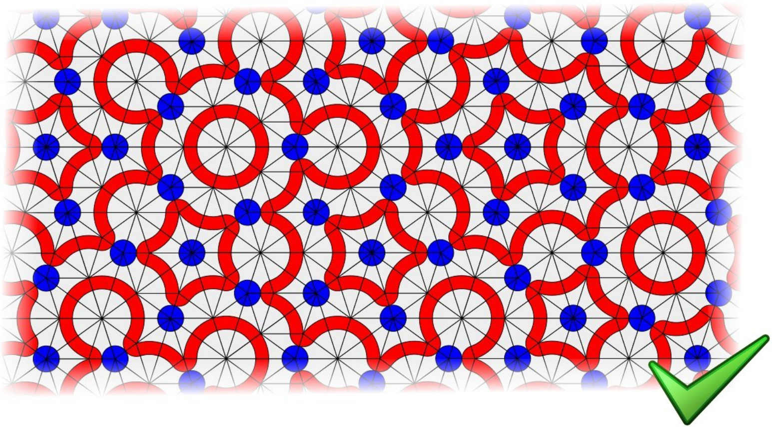
Decoration maps each point of tile boundaries to a color.

Decorations



matching if decorations are equal over common facets.

Decorations



Decorated tiling tiling by matching decorated tiles.

Sofic tilings

Decorated tile set $\tau \rightsquigarrow$ set X_τ of decorated tilings.

Let π be the map which removes tile decorations.

Definition A set of tilings is **sofic** if it can be written as $\pi(X_\tau)$, where τ is a **finite** decorated tile set.

Sofic tilings

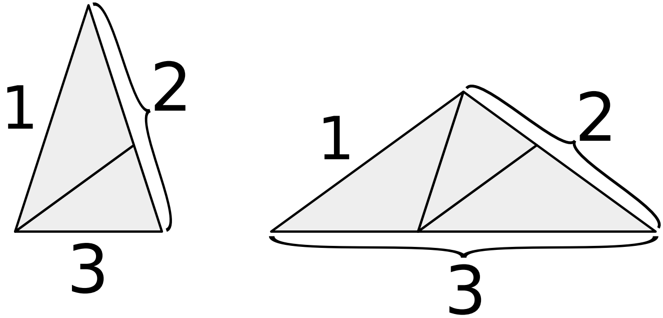
Decorated tile set $\tau \rightsquigarrow$ set X_τ of decorated tilings.

Let π be the map which removes tile decorations.

Definition A set of tilings is **sofic** if it can be written as $\pi(X_\tau)$, where τ is a **finite** decorated tile set.

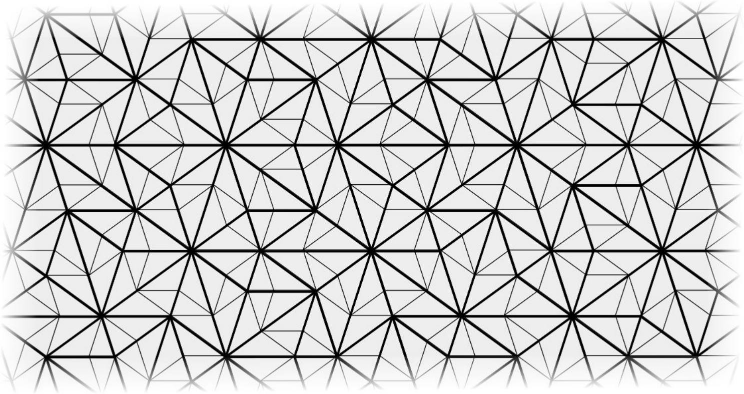
What (interesting) properties on tilings can (or cannot) be enforced by soficity?

Macro-tiles and macro-tilings



Macro-tile finite partial tiling with (numbered) **macro-facets**.

Macro-tiles and macro-tilings



Macro-tiling macro-facet-to-facet tiling by macro-tiles.

Combinatorial substitution

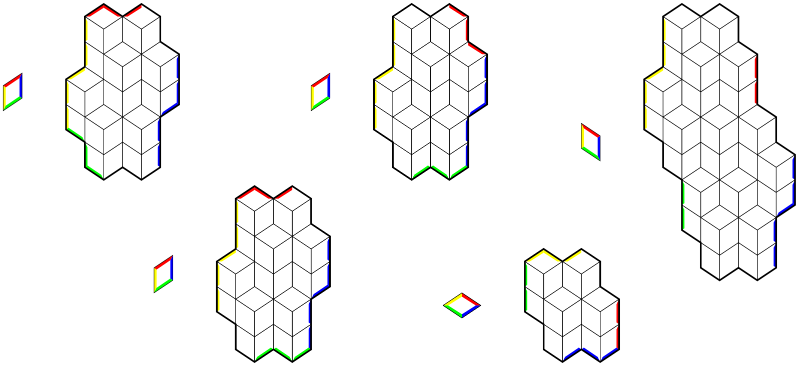
Definition A **combinatorial substitution** is a finite set of pairs (tile, macro-tile).

Let $\sigma = \{(P_i, Q_i)\}_i$ be a combinatorial substitution.

Image under σ of a tiling by the P_i's: macro-tiling by the Q_i's with the same **combinatorial structure**.

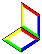
Definition The **limit set** of a combinatorial substitution σ is the set of tilings which admit an infinite sequence of preimages under σ.

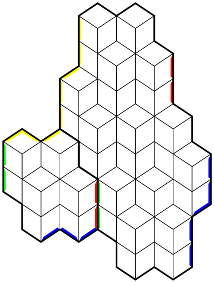
Example



Rauzy combinatorial substitution

Example

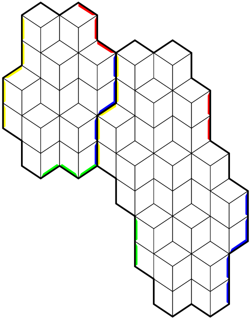




Tiles match in a tiling as macro-tiles in its image...

...and conversely.

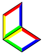
Example

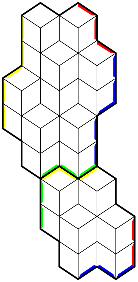


Tiles match in a tiling as macro-tiles in its image. . .

. . . and conversely.

Example

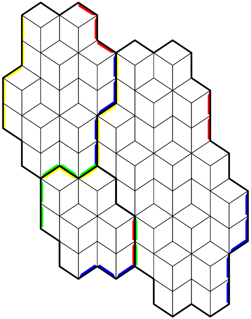




Tiles match in a tiling as macro-tiles in its image...

...and conversely.

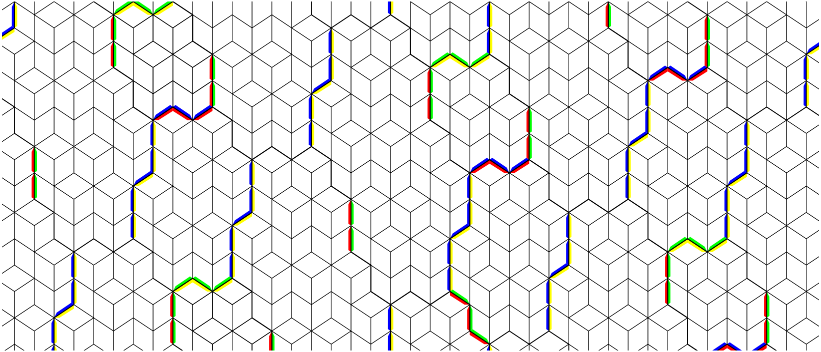
Example



Tiles match in a tiling as macro-tiles in its image...

...and conversely.

Example



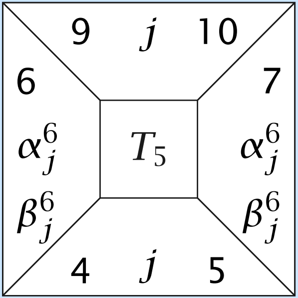
Any tiling decomposes into macro-tiles.

1. The classical recipe

2. Combinatorial substitutions

3. Main result

4. Conclusion & Open Pb



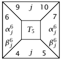
Self-simulation

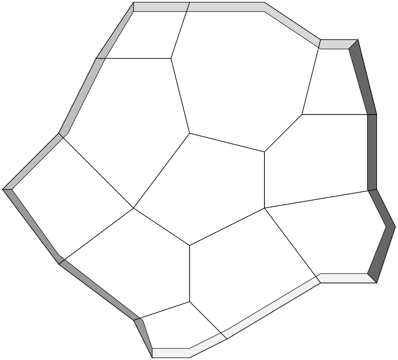
Definition A decorated tile set τ **self-simulates** if it admits tilings and there are τ-macro-tiles s.t.

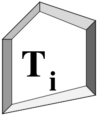
1. any τ-tiling is also a macro-tiling by these τ-macro-tiles;
2. each τ-macro-tile is **combinatorially equivalent** to a τ-tile.

Proposition If τ **self-simulates**, then $\pi(X_\tau)$ is a **subset of the limit set** of the combinatorial substitution with pairs τ-macro-tile/equivalent τ-tile.

A self-simulating decorated tile set τ

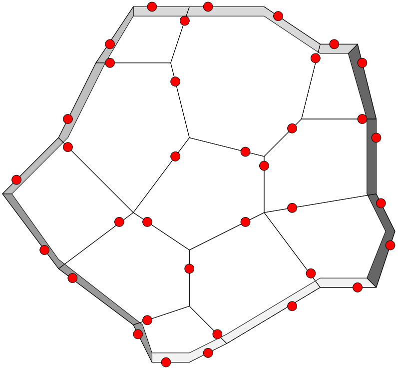


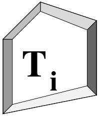




Fix a set of macro-tiles and let $T_1, \ldots, T_n$ be all their tiles.

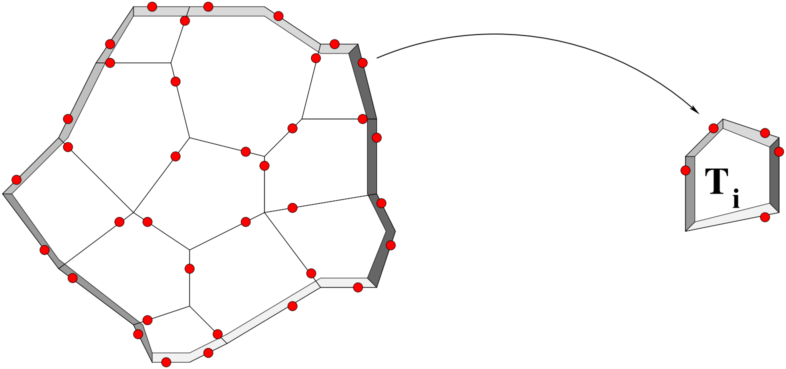
A self-simulating decorated tile set τ





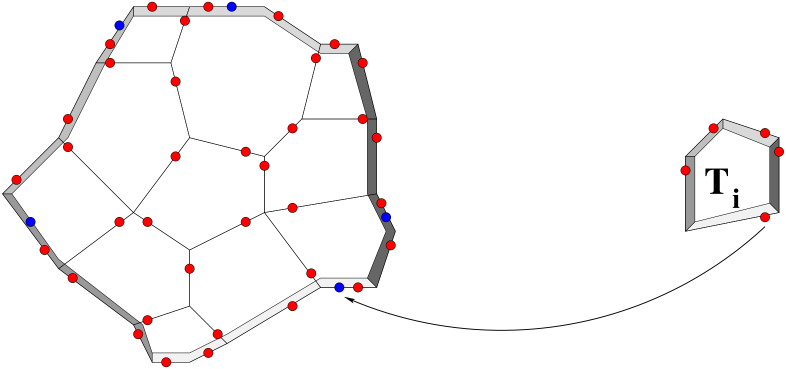
To enforce τ-tilings to be τ-macro-tilings: decorations specify tile neighbors within macro-tiles and mark macro-facets.

A self-simulating decorated tile set τ



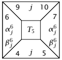
This yields so-called macro-indices on tile facets.

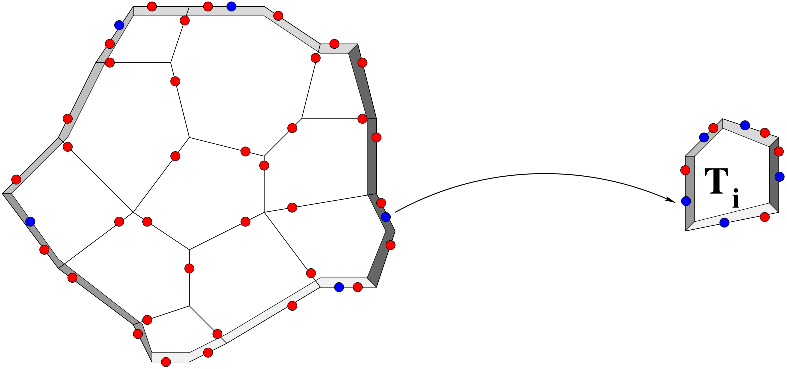
A self-simulating decorated tile set τ



The macro-indices of facets of a τ-tile must then be encoded on the corresponding macro-facets of its simulating τ-macro-tile.

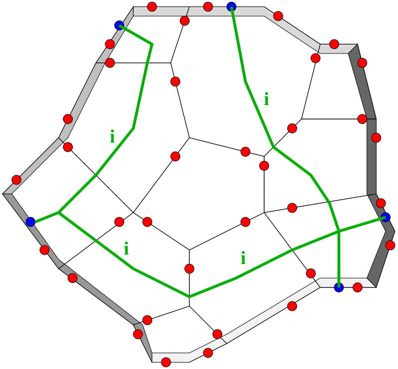
A self-simulating decorated tile set τ

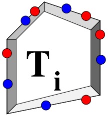




This yields so-called neighbor-indices on tile facets.

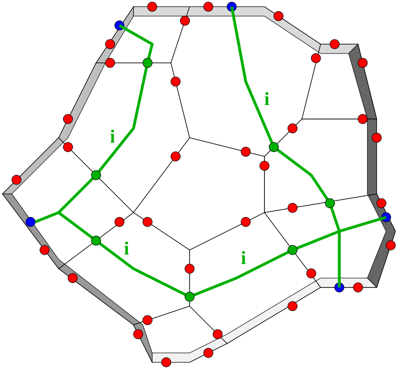
A self-simulating decorated tile set τ

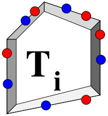




We force these neighbor-indices to come from the same tile T_i, called parent-tile, by carrying its index i between macro-facets, where it is converted into the suitable neighbor-index.

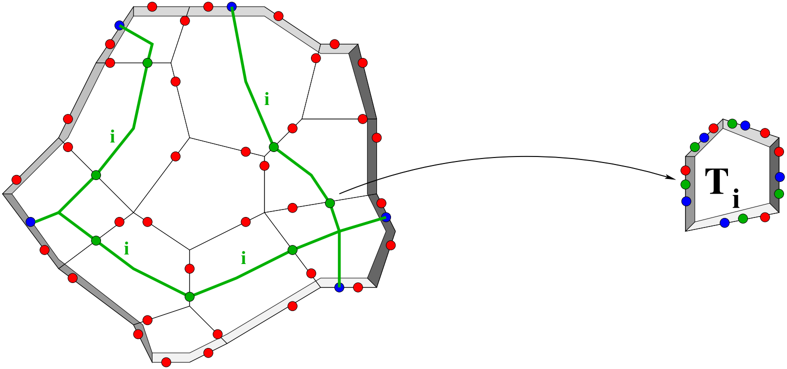
A self-simulating decorated tile set τ





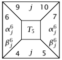
Such tile indices are encoded on facets by so-called parent-index.

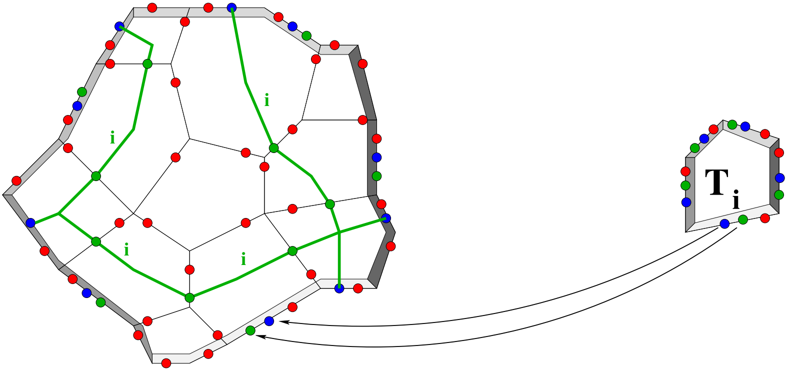
A self-simulating decorated tile set τ



This yields, once again, a new index on each tile facets...

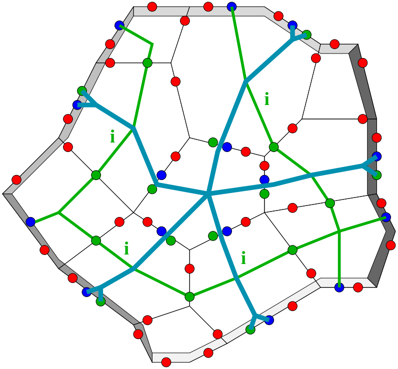
A self-simulating decorated tile set τ

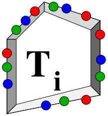




But the trick is that the neighbor-indices and parent-indices of facets of a τ-tile can be encoded on the corresponding big enough macro-facets of the equivalent τ-macro-tile without any new index!

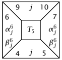
A self-simulating decorated tile set τ

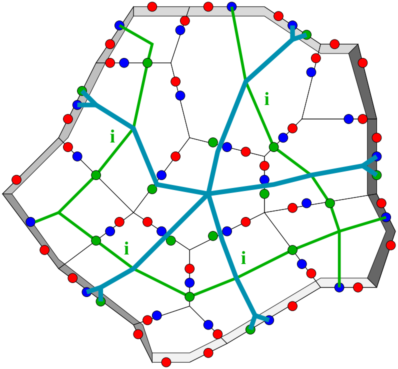


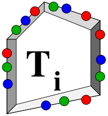


In big enough macro-tiles, we can then carry these pairs of neighbor/parent indices up to a central tile along a star-like network.

A self-simulating decorated tile set τ

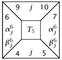


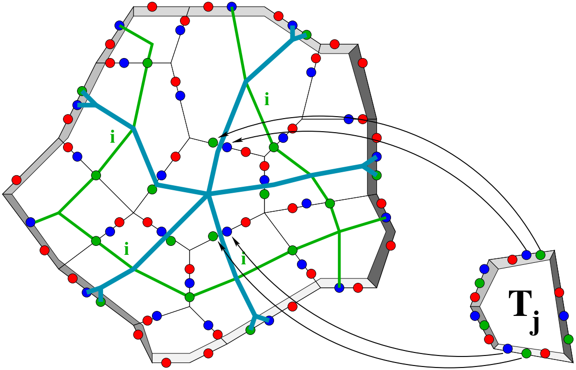




On internal facets not crossed by this network, we copy the macro-index on the neighbor-index (this redundancy is later used).

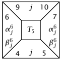
A self-simulating decorated tile set τ

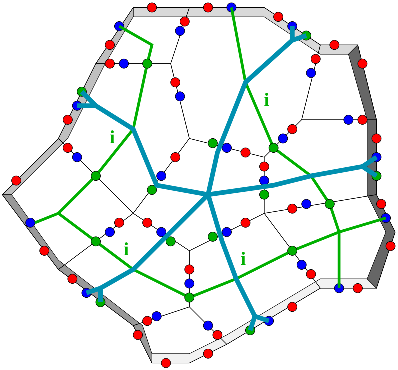




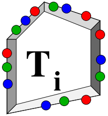
The pairs on a central τ-tile can be those of any non-central τ-tile (from which the central τ-tile is said to derive).

A self-simulating decorated tile set τ



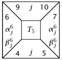


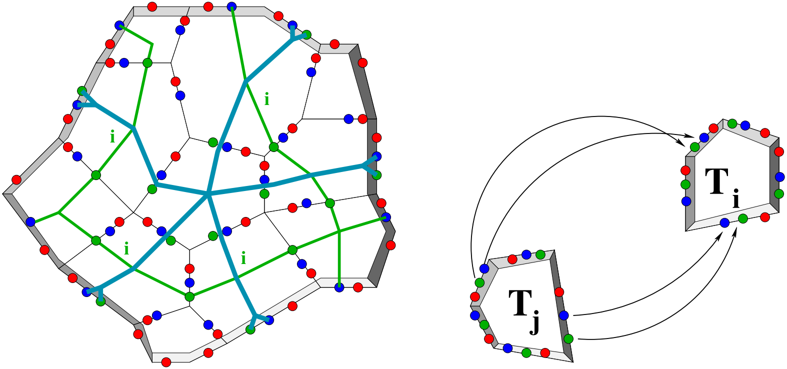
~



The τ-macro-tile with parent-index i is combinatorially equivalent to T_i endowed with the pairs of the central τ-tile.
But is it a τ-tile?

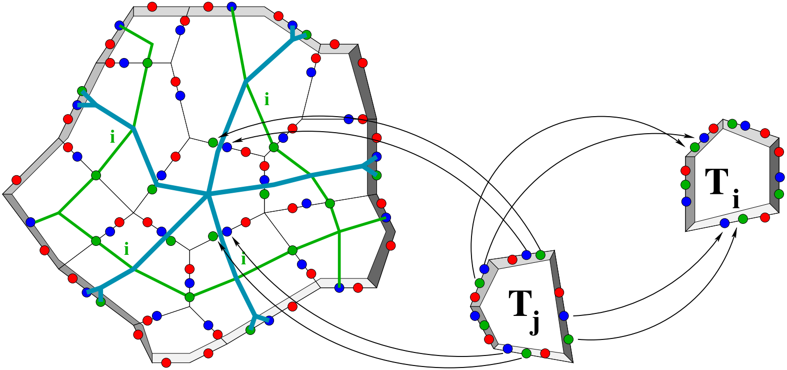
A self-simulating decorated tile set τ





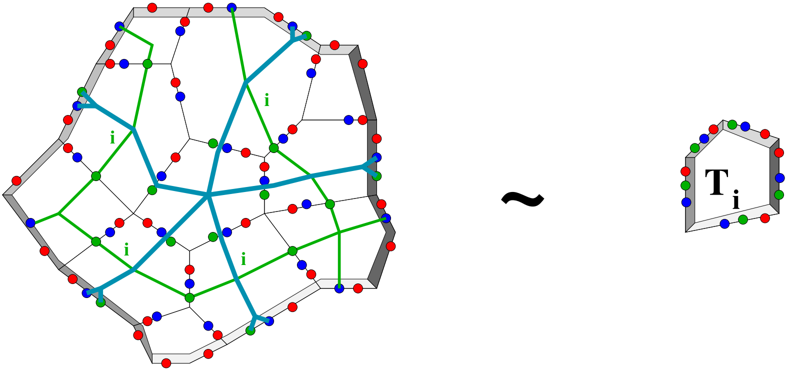
If T_i is a central tile, then its pairs can be derived from any non-central τ-tile (as for any central tile). . .

A self-simulating decorated tile set τ



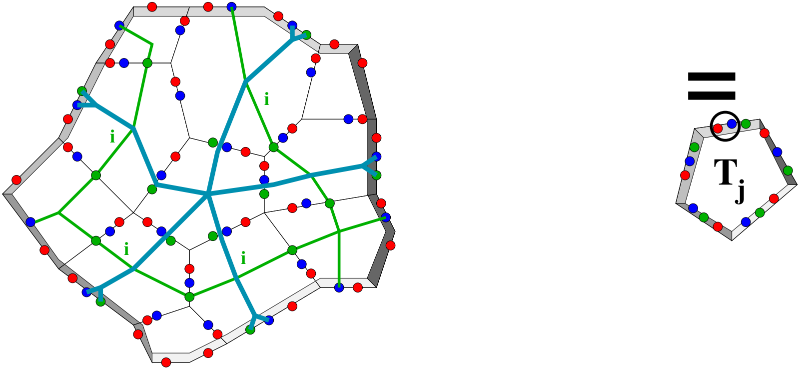
. . . in particular from the non-central τ-tile from which are also derived the pairs of the central τ-tile of our τ-macro-tile.

A self-simulating decorated tile set τ



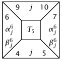
In this case, the equivalent decorated T_i is a derived central τ-tile.

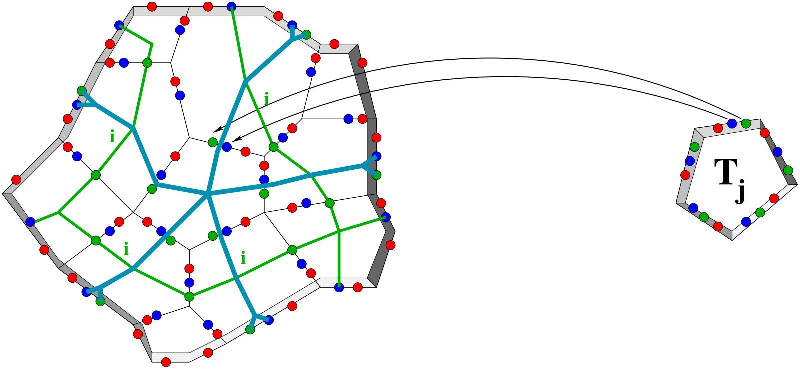
A self-simulating decorated tile set τ



Otherwise, consider the non-central τ-tile from which derives our central τ-tile; at least one facet is internal and not crossed by a network: its neighbor and macro indices are equal (by redundancy).

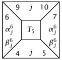
A self-simulating decorated tile set τ

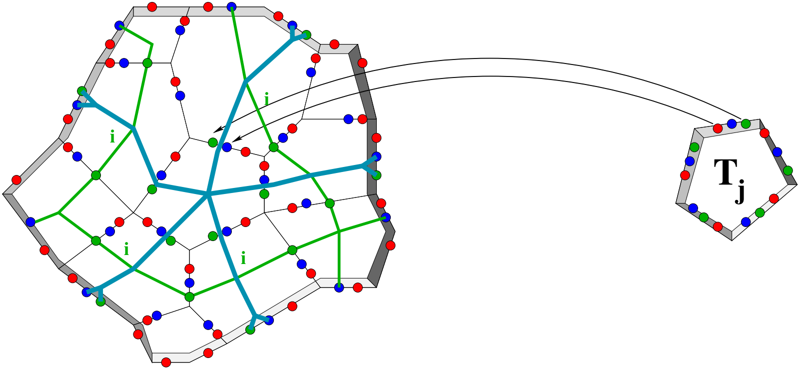




Thus, by copying the neighbor and parent indices (derivation)...

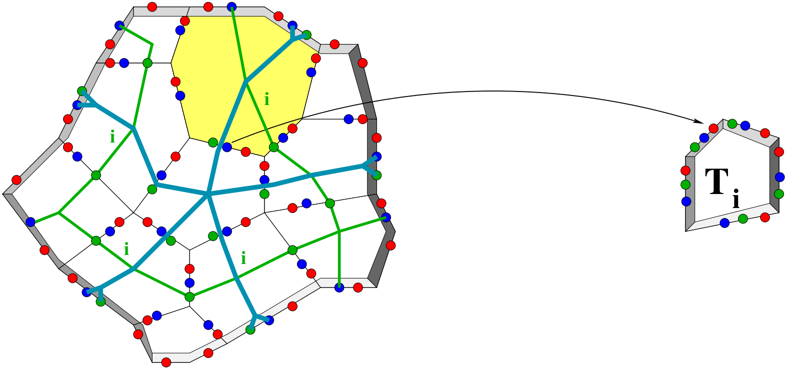
A self-simulating decorated tile set τ





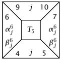
. . . one copies a macro-index on our central τ-tile, and thus on the whole corresponding network branch.

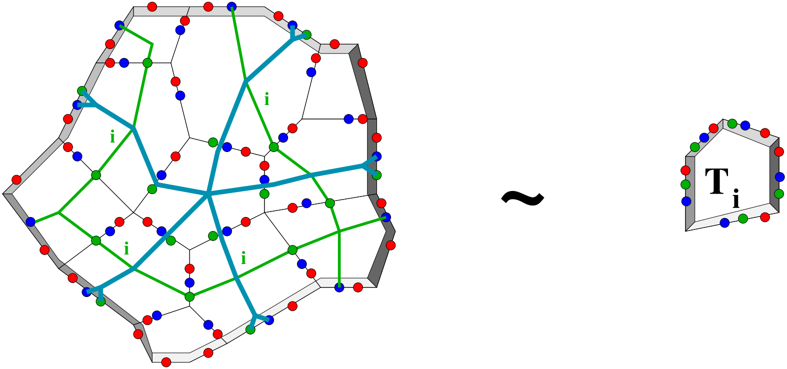
A self-simulating decorated tile set τ



A tile on this k-th branch which also knows the parent-index i can then force this macro-index to be the one on the k-th facet of a decorated T_i (recall that all the decorated T_i have the same one).

A self-simulating decorated tile set τ





In this case, the equivalent decorated T_i is the non-central τ-tile from which derives the central τ-tile of our τ-macro-tile.

1. The classical recipe

2. Combinatorial substitutions

3. Main result

4. Conclusion & Open Pb

? ? ?

Main result

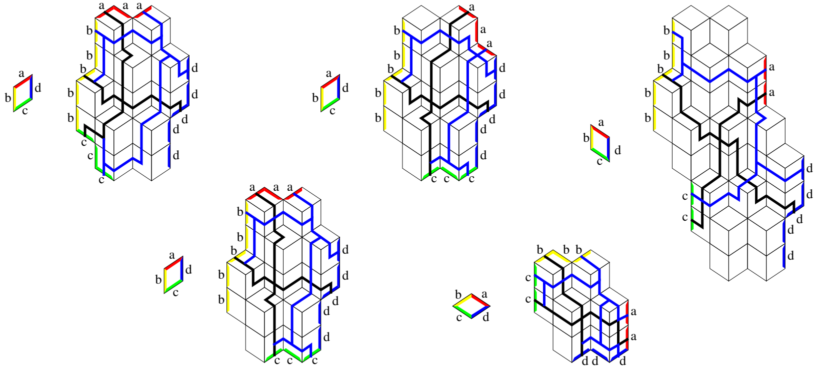
Theorem[FO 2010] The **limit set** of a **good** combinatorial substitution is **sofic**.

Remark No need to care about **geometry**.

But, what is a **good** combinatorial substitution?

Definition A **good** combinatorial substitution is both **connecting** and **consistent**.

Connecting



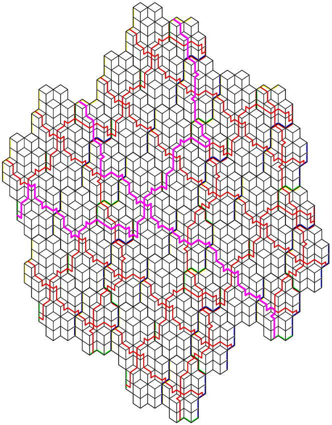
Intuitively A substitution is **connecting** if there is enough space inside macro-tiles to wire the networks.

Consistent

Definition A combinatorial substitution is **consistent** if any tiling by macro-tiles admits a preimage under the substitution.

Remark This is where the **geometrical** consistency hides.

Open Pb **Characterize** consistent combinatorial substitutions.



Conclusion and open problem

???

The **global** hierarchical structure associated to a substitution system can be enforced by **local** matching rules.

Open Pb Is it possible to describe the **geometry** of tiles by finite local **combinatorial** constraints?