Substitutions et pavages I : indécidabilité et pavabilité

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GdT GAMoC — 3 novembre 2011

Avant-propos

Indécidabilité, pavages et polyominos

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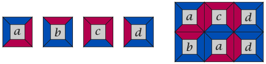
Séminaire du LIFO — 21 février 2011

The Domino Problem (DP)

"Assume we are ***given a finite set of square plates*** *of the same size with* ***edges colored****, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are* ***not permitted to rotate or reflect a plate****. The question is to find an effective procedure by which we can* ***decide****, for each given finite set of plates,* ***whether we can cover up the whole plane*** *(or, equivalently, an infinite quadrant thereof)* ***with copies of the plates subject to the restriction that adjoining edges must have the same color****."* (Wang, 1961)



Undecidability of DP

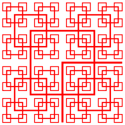
Theorem[Berger 1964] DP is **recursively undecidable**.

Remark To prove it one needs **aperiodic** tile sets.

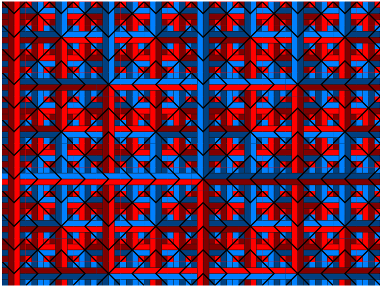
Idea of the proof

Enforce an (aperiodic) **self-similar structure** using local rules.

Insert a **Turing machine** computation **everywhere** using the structure.



Remark Plenty of different proofs!



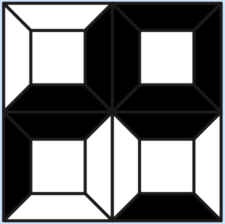
1. Tilings

2. Soficity

3. Substitutions

4. 104

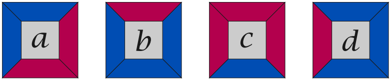
5. Conclusion

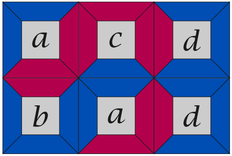


The Domino Problem (DP)

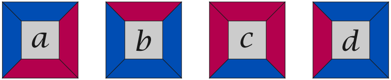
*"Assume we are **given a finite set of square plates** of the same size with **edges colored**, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are **not permitted to rotate or reflect a plate**. The question is to find an effective procedure by which we can **decide**, for each given finite set of plates, **whether we can cover up the whole plane** (or, equivalently, an infinite quadrant thereof) **with copies of the plates subject to the restriction that adjoining edges must have the same color**."*

(Wang, 1961)

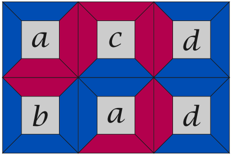




Wang tiles



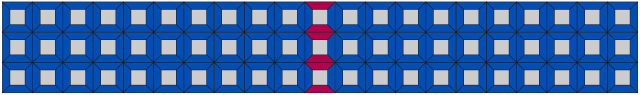
A **tile set** $\tau \subseteq \Sigma^4$ is a finite set of tiles with colored edges.



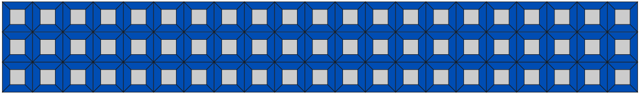
The set of **τ-tilings** $X_\tau \subseteq \tau^{\mathbb{Z}^2}$ is the set of colorings of $\mathbb{Z}^2$ by τ where colors match along edges.

Periodic Tilings

Definition A tiling is **periodic** with period p if it is invariant by a **translation** of vector p.



Lemma If a finite set of tiles admits a **periodic** tiling then it admits a **biperiodic** tiling.

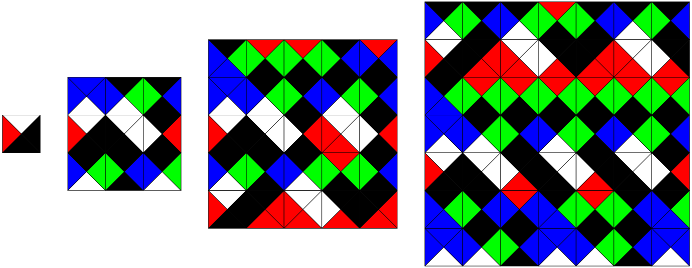


Lemma Finite sets of tiles tiling the plane biperiodically are **re** (**recursively enumerable**).

co-Tiling

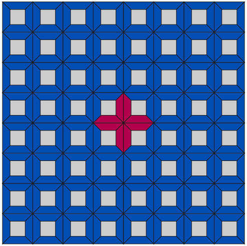
Lemma Finite sets of tiles tiling the plane are **co-re**.

Sketch of the proof Consider tilings of larger and larger square regions. If the set does not tile the plane, by compacity, there exists a size of square it cannot cover with tiles.



Aperiodicity

Definition A tiling is **aperiodic** if it admits no non-trivial period.



Definition A set of tiles is **aperiodic** if it admits a tiling and all its tilings are aperiodic.

Remark If there were **no aperiodic** finite set of tiles, the Domino Problem would be **decidable**.

Undecidability of DP

Theorem[Berger 1964] **DP** is **undecidable**.

Remark To prove it one needs **aperiodic** tile sets.

Seminal self-similarity based proofs *(reduction from **HP**)*:

- Berger, 1964 *(20426 tiles, a full PhD thesis)*
- Robinson, 1971 *(56 tiles, 17 pages, long case analysis)*
- Durand et al, 2007 *(Kleene's fixpoint existence argument)*

Tiling rows seen as transducer trace based proof:
Kari, 2007 *(affine maps, short & concise, reduction from **IP**)*

In this talk [CiE 2008]

A new self-similarity based construction building on classical proof schemes with concise arguments and few tiles:

1. two-by-two substitution systems and aperiodicity
2. an aperiodic tile set of 104 tiles
3. enforcing any substitution and reduction from **HP**

This work combines tools and ideas from:

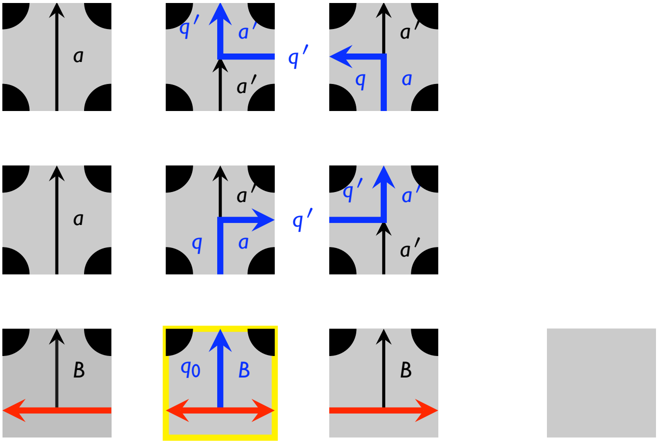
[Berger 64] *The Undecidability of the Domino Problem*

[Robinson 71] *Undecidability and nonperiodicity for tilings of the plane*

[Grünbaum Shephard 89] *Tilings and Patterns, an introduction*

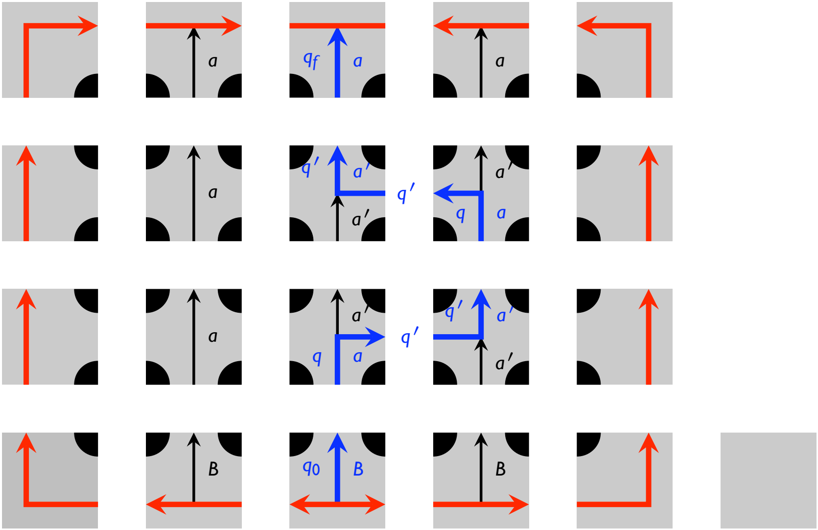
[Durand Levin Shen 05] *Local rules and global order, or aperiodic tilings*

Tiling with a fixed tile

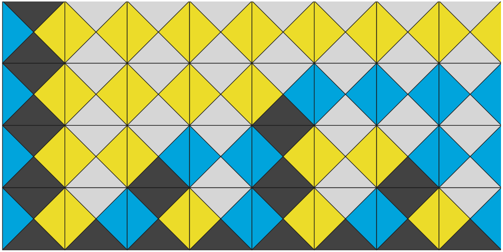


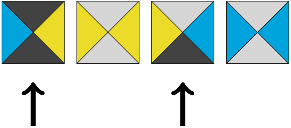
No **halting** tile.

Finite Tiling



Tiling with diagonal constraints





1. Tilings

2. Soficity

3. Substitutions

4. 104

5. Conclusion

a	b	b
a	a	b
a	a	b

Topology

Definition A **topological space** is a pair $(E, \mathcal{O})$ where $\mathcal{O} \subseteq \mathcal{P}(E)$ is the set of **open** subsets satisfying:

- $\mathcal{O}$ contains both $\emptyset$ and E;
- $\mathcal{O}$ is closed under union;
- $\mathcal{O}$ is closed under finite intersection.

S is endowed with the **discrete topology**: $\mathcal{O} = \mathcal{P}(S)$.

$S^{\mathbb{Z}^d}$ is endowed with the **Cantor topology**: the product topology of the discrete topology.

$$\mathcal{O} = \left\{ \prod X_i \,\middle|\, X_i \subseteq S \wedge \mathrm{Card}(\{i | X_i \neq S\}) < \omega \right\}$$

Cantor topology is **metric** and **compact**.

Cylinders

Definition The **cylinder** $[m] \subseteq S^{\mathbb{Z}^d}$ with radius $r \geqslant -1$ generated by the pattern $m \in S^{[-r,r]^d}$ is

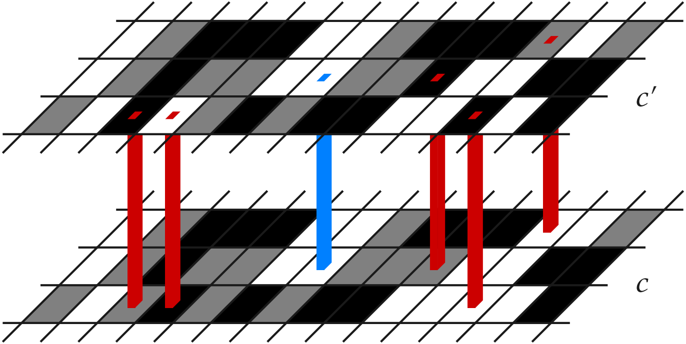
$$[m] = \left\{ c \in S^{\mathbb{Z}^d} \,\middle|\, \forall p \in \mathbb{Z}^d, ||p||_\infty \leqslant r \Rightarrow c(p) = m(p) \right\}$$

Proposition Cylinders are a countable **clopen generating set**.

Notation $[m] \prec [m']$ means $[m]$ is a **sub-cylinder** of $[m']$, *i.e.* $[m'] \subset [m]$.

Metric

Proposition Cantor topology is **metric**



$$d(c, c') = 1/8$$

$$\forall c, c' \in S^{\mathbb{Z}^d}, \quad d(c, c') = 2^{-\min\left\{\|p\|_\infty \,\middle|\, c_p \neq c'_p\right\}}$$

Compact

$$\begin{matrix} a & b & b \\ a & a & b \\ a & a & b \end{matrix}$$

Proposition Every sequence of configurations $(c_i) \in S^{\mathbb{Z}^d\mathbb{N}}$ admits a converging subsequence.

Proof by **extraction**:

By recurrence, let $\left(c_i^0\right) = (c_i)$.

It is alway possible to find:

- a cylinder $[m_n]$ of radius n and
- an infinite subsequence $\left(c_i^{n+1}\right)$ de $\left(c_{i+1}^n\right)$

such that for all $i \in \mathbb{N}$, $c_i^{n+1} \in [m_n]$.

By construction $[m_{n+1}] \subset [m_n]$ and $\left(c_0^{i+1}\right)$ is a converging subsequence of (c_i) (to $\bigcap[m_i]$): $\delta(c_0^{n+1}, c_0^{n+2}) \leqslant 2^{-n}$.

König trees

$\begin{matrix} a & b & b \\ a & a & b \\ a & a & b \end{matrix}$

Remark Cantor topology is essentially **combinatorial**.

Remark Main properties can be obtained using **extraction**.

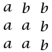
König's Lemma Every infinite tree with finite branching admis an infinite branch.

Definition The **König tree** $\mathcal{A}_C$ of a set of configurations $C \subseteq S^{\mathbb{Z}^d}$ is the tree (V_C, E_C) where

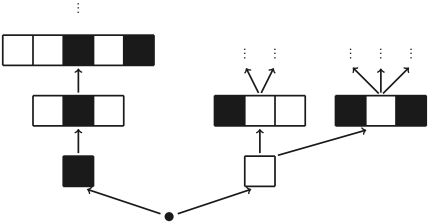
$$\begin{aligned} V_C &= \{[m] | C \cap [m] \neq \emptyset\} \\ E_C &= \left\{([m],[m']) \middle| [m] \prec [m'] \wedge \mathrm{r}([m']) = \mathrm{r}([m]) + 1\right\} \end{aligned}$$

The root of the tree is the cylinder $[] = S^{\mathbb{Z}^d}$ of radius -1.

Toppings



The König tree of a **non empty** set of configurations is an **infinite tree** with finite branching.



To each infinite branch $([m_i])$ is associated a **unique configuration** $\bigcap[m_i]$.

Definition The **topping** $\overline{\mathcal{A}_C}$ of a König tree is the set of configurations associated to its infinite branches.

König topology

The **König topology** is defined by its closed sets: toppings of König trees.

The **complementary of a closed set** is the **union of cylinders** that are not nodes of the tree.

Cantor and **König** topologies are the **same**.

Most **topological concepts** can be explained using **trees**:

- dense sets;
- closed sets with non empty interior;
- compacity;
- Baire's theorem.

Continuity

Proposition **clopen** sets are finite unions of cylinders.

Definition A mapping $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is **local** in $p \in \mathbb{Z}^d$ if there exists a radius r such that:

$$\forall c, c' \in S^{\mathbb{Z}^d}, \quad \left[c_{|r}\right] = \left[c'_{|r}\right] \Rightarrow G(c)_p = G(c')_p \quad .$$

Proposition A mapping $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is **continuous** if and only if it is **local in every point**.

Cellular automata

Definition A **CA** is a tuple (d, S, r, f) where S is a **finite set of states**, $r \in \mathbb{N}$ is the **neighborhood radius** and $f : S^{(2r+1)^d} \to S$ is the **local rule** of the cellular automaton.

A **configuration** $c \in S^{\mathbb{Z}^d}$ is a coloring of $\mathbb{Z}^d$ by S.

The **global map** $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ applies f uniformly and locally:

$$\forall c \in S^{\mathbb{Z}}, \forall z \in \mathbb{Z}^d, \qquad F(c)(z) = f(c(z-r), \ldots, c(z+r)).$$

Curtis-Hedlund-Lyndon Theorem

$\begin{matrix} a & b & b \\ a & a & b \\ a & a & b \end{matrix}$

Definition The **translation** $\sigma_k : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ with vector $k \in \mathbb{Z}^d$ satisfies:

$$\forall c \in S^{\mathbb{Z}^d}, \forall p \in \mathbb{Z}^d, \quad \sigma_k(c)_p = c_{p-k} \quad .$$

Theorem[Hedlund 1969] Continuous mapping commuting with translations are exactly global maps of CA.

Symbolic dynamics

A central object in **symbolic dynamics** is the **subshift**.

Definition A **subshift** of $S^{\mathbb{Z}^d}$ is a set of configurations both **closed** and **invariant by translation**.

Ex $\ldots abaababaaa \ldots$

$$X = \left\{ c \in \{a, b\}^{\mathbb{Z}} \middle| \forall p \in \mathbb{Z}, c_p = b \Rightarrow c_{p+1} = a \right\}$$

Langage of a subshift

$\begin{smallmatrix} a & b & b \\ a & a & b \\ a & a & b \end{smallmatrix}$

Definition The **language** $L(X)$ of a **subshift** X is the set of finite patterns appearing in X.

Proposition A **subshift** is characterized by its **language**.

$$\overline{L} = \left\{ c \in S^{\mathbb{Z}^d} \;\middle|\; \forall r \geqslant 0, \forall m \in S^{[-r,r]^d}, m \prec c \Rightarrow m \in L \right\}$$

Warning It might be that $L(\overline{L}) \supsetneq L$.

Forbidden words

$\begin{matrix} a & b & b \\ a & a & b \\ a & a & b \end{matrix}$

Proposition A subshift is characterized by the set of its **forbidden words**: the complementary of its language.

Proposition Subshifts are in bijection with **minimal sets of forbidden words** (for set inclusion).

Ex $X = S_{\{bb\}}$

SFT, tilings, soficity

Definition A **subshift of finite type** (**SFT**) is defined by a finite set of forbidden words.

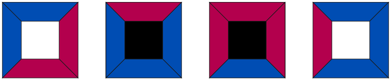
Remark **SFT** correspond to **tilings**: colorings with local uniform constraints.

Definition A **sofic subshift** is the image of a SFT by a CA.

Proposition **1D sofic subshifts** are subshifts that admit a **regular language** of forbidden words.

2D sofic subshifts

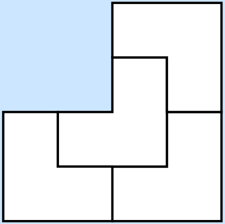
Proposition **2D sofic subshifts** are tile-by-tile projections of tilings by **Wang tiles**.



Goal Provide tools to manipulate and characterize **2D sofic subshifts**:

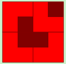
- constructions to characterize soficity;
- tools to prove non soficity.

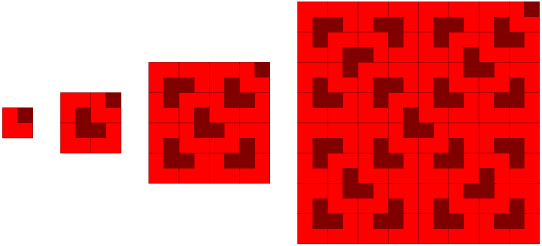
What is a **2D rational language**?



Subsitutions

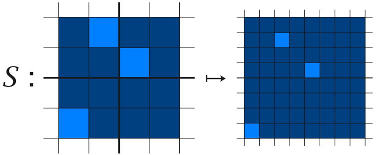
$\Sigma = \{$ , , ,  $\}$

$s :$  $\mapsto$  *+ rotations*



Two-by-two substitutions

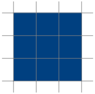
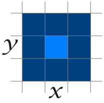
A **2x2 substitution** $s : \Sigma \to \Sigma^{\boxplus}$ maps letters to squares of letters on the same finite alphabet.



The substitution is extended as a **global map** $S : \Sigma^{\mathbb{Z}^2} \to \Sigma^{\mathbb{Z}^2}$ on colorings of the plane:

$$\forall z \in \mathbb{Z}^2, \forall k \in \boxplus, \quad S(c)(2z + k) = s(c(z))(k)$$

Limit set and history

$$\Lambda_S = \left\{ \right.$$

$$\left. \right\} \cup \left\{ \right.$$


$$\left. \right\}_{x,y \in \mathbb{Z}^2}$$

The **limit set** $\Lambda_s \subseteq \Sigma^{\mathbb{Z}^2}$ is the maximal attractor of S:

$$\Lambda_s = \bigcap_{t \in \mathbb{N}} \left\langle S^t \left(\Sigma^{\mathbb{Z}^2} \right) \right\rangle_\sigma$$

The limit set is the set of colorings admitting an **history** $(c_i)_{i \in \mathbb{N}}$ where $c_i = S(c_{i+1}) \cdot u_i$.

Unambiguous substitutions

A substitution is **aperiodic** if its limit set Λ_S is aperiodic.

A substitution is **unambiguous** if, for every coloring C from its limit set Λ_S, there exists a unique coloring C' and a unique translation $u \in \boxplus$ satisfying $C = u \cdot S(C')$.

Proposition **Unambiguity** implies **aperiodicity**.

Sketch of the proof. Consider a periodic coloring with minimal period p, its preimage has period $p/2$. $\Diamond$

Idea. Construct a tile set whose tilings are in the limit set of an unambiguous substitution system.

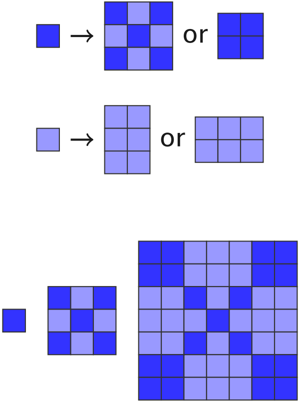
Main result

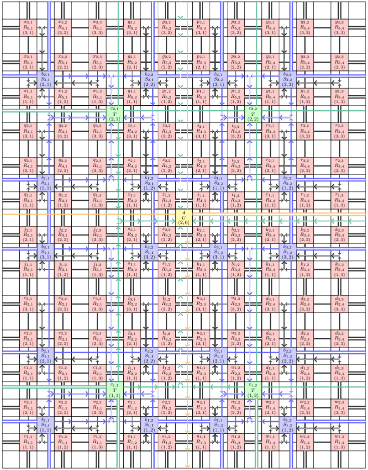
Theorem The **limit set** of a 2x2 substitution is **sofic**.

Idea To encode Λ_s via **local matching rules** decorate s into a **locally checkable** $s^\bullet$ embedding a whole history.

Remark The **key step** is to construct an **aperiodic tile set**.

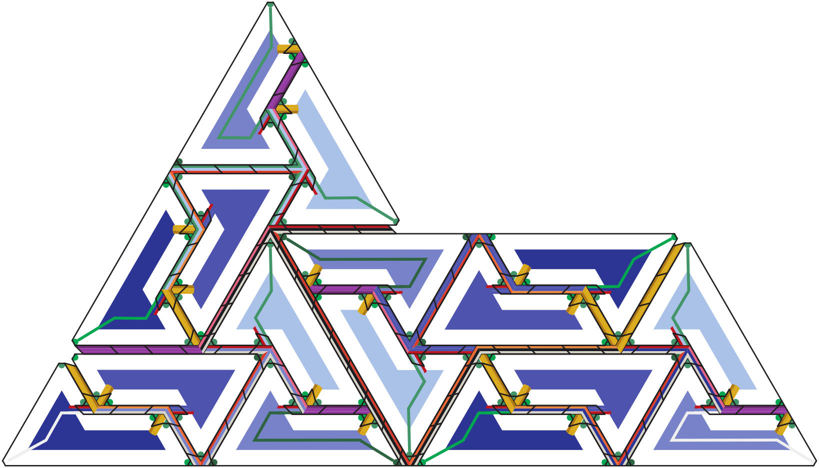
Mozes 1990





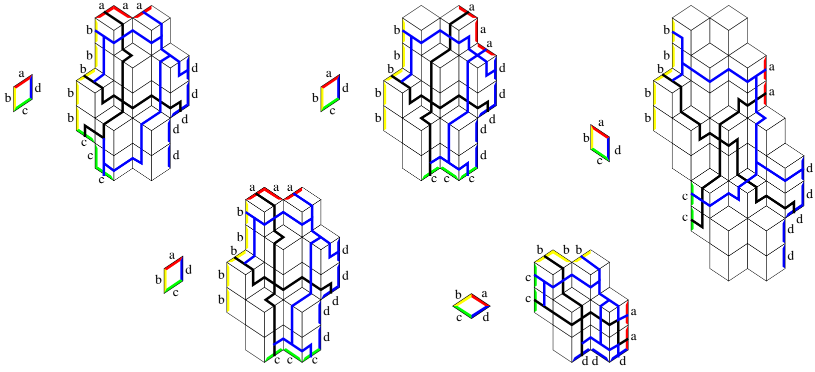
Theorem[Mozes 1990] The limit set of a **non-deterministic rectangular substitution** *(+ some hypothesis)* is sofic.

Goodman-Strauss 1998



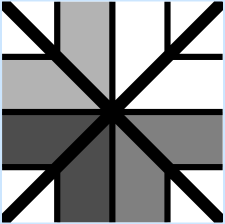
Theorem[Goodman-Strauss 1998] The limit set of **homothetic substitution** *(+ some hypothesis)* is sofic.

Fernique-O 2010



Theorem[Fernique-O 2010] The limit set of a **combinatorial substitution** *(+ some hypothesis)* is sofic.

1. Tilings
2. Soficity
3. Substitutions
4. **104**
5. Conclusion

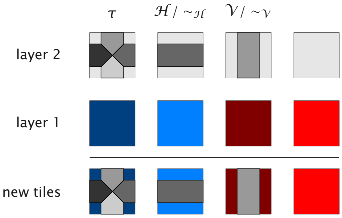


Coding tile sets into tile sets

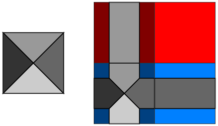
A **tile set** τ is a triple $(T, \mathcal{H}, \mathcal{V})$ where $\mathcal{H}$ and $\mathcal{V}$ define horizontal and vertical matching constraints.

Definition A tile set $(T', \mathcal{H}', \mathcal{V}')$ **codes** a tile set $(T, \mathcal{H}, \mathcal{V})$, according to a **coding rule** $t : T \to T'^{\boxplus}$ if t is injective and

$$X_{\tau'} = \{u \cdot t(C) | C \in X_\tau, u \in \boxplus\}$$



coding tile set


coding rule

Unambiguous self-coding

Definition A tile set $(T, \mathcal{H}, \mathcal{V})$ **codes** a substitution $s : T \to T^{\boxplus}$ if it codes itself according to the coding rule s.

Proposition A tile set both admitting a tiling and **coding** an **unambiguous** substitution is **aperiodic**.

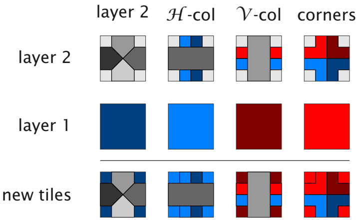
Sketch of the proof. $X_\tau \subseteq \Lambda_S$ and $X_\tau \neq \emptyset$. ◊

A coding scheme with fixpoint?

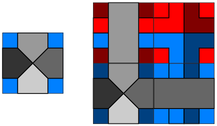
Better scheme: **not strictly increasing** the number of tiles.

Problem It **cannot encode any layered tile set**, constraints between layer 1 and layer 2 are checked edge by edge.

Patch Add a **third layer** with one bit of information per edge.



coding tile set



coding rule

Canonical substitution

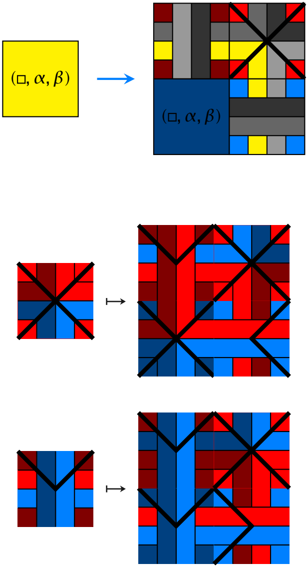
Copy the tile in the SW corner but for layer 1.

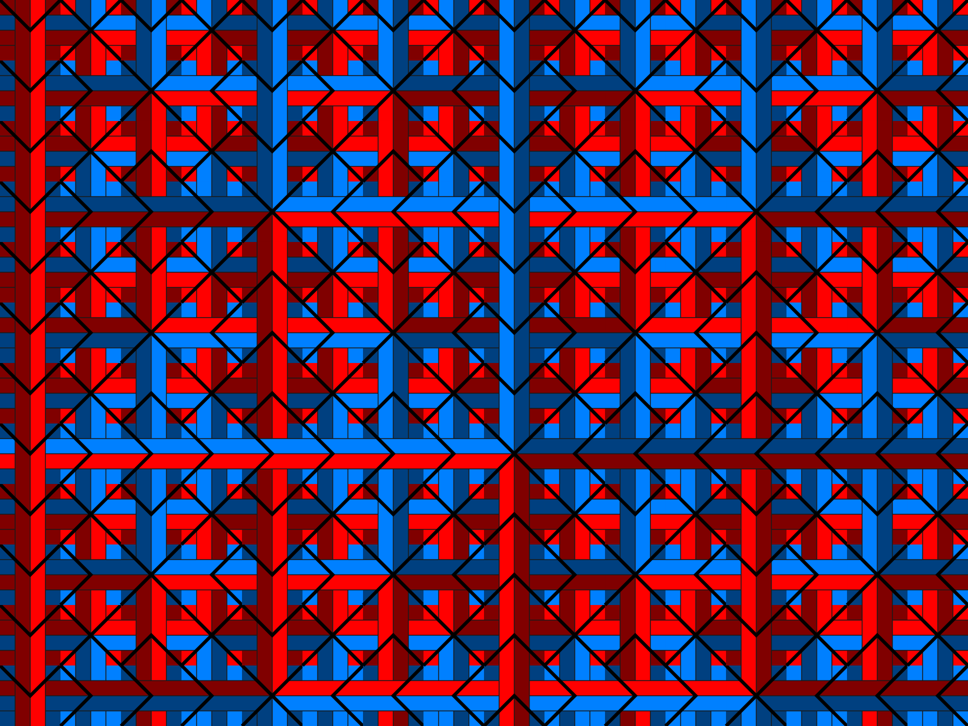
Put the only possible X in NE that carry layer 1 of the original tile on SW wire.

Propagate wires colors.

Let H et V tile propagate layer 3 arrows.

The substitution is injective.





Aperiodicity: sketch of the proof

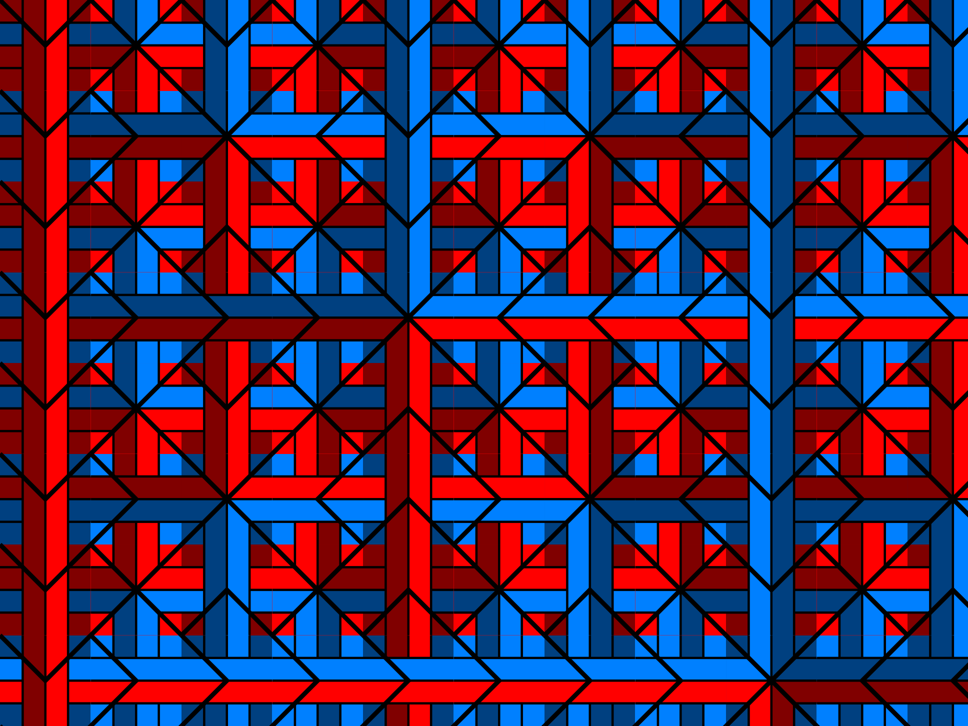
1. The tile set admits a tiling:
Generate a valid tiling by iterating the substitution rule:
$X_\tau \cap \Lambda_S \neq \emptyset$.

2. The substitution is unambiguous:
It is injective and the projectors have disjoined images.

3. The tile set codes the substitution:

(a) each tiling is an image of the canonical substitution
Consider any tiling, level by level, short case analysis.

(b) the preimage of a tiling is a tiling
Straightforward by construction (preimage remove constraints).

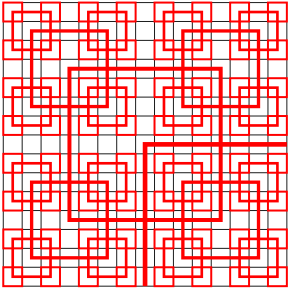


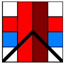
Enforcing substitutions *via* tilings

Let π map every tile of $\tau(s')$ to $s'(a)(u)$ where a and u are the letter and the value of ⊞ on layer 1.

Proposition. Let s' be any substitution system. The tile set $\tau(s')$ enforces s':
$\pi\left(X_{\tau(s')}\right) = \Lambda_{S'}$.

Idea. Every tiling of $\tau(s')$ codes an history of S' and every history of S' can be encoded into a tiling of $\tau(s')$.





a

$b = s(a)\binom{1}{1}$

a

$b = s(a)\binom{0}{0}$

1. Tilings
2. Soficity
3. Substitutions
4. 104
5. **Conclusion**

???

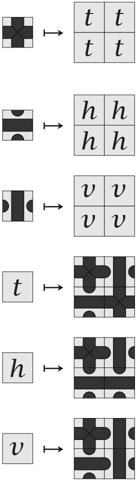
Back to DP

Theorem The **limit set** of a 2x2 substitution is **sofic**.

Theorem[Berger 1964] **DP** is **undecidable**.

Idea Construct a **2x2 substitution** whose **limit set** contains everywhere **squares** of larger and larger size, insert **Turing computation** inside those squares.

To conclude



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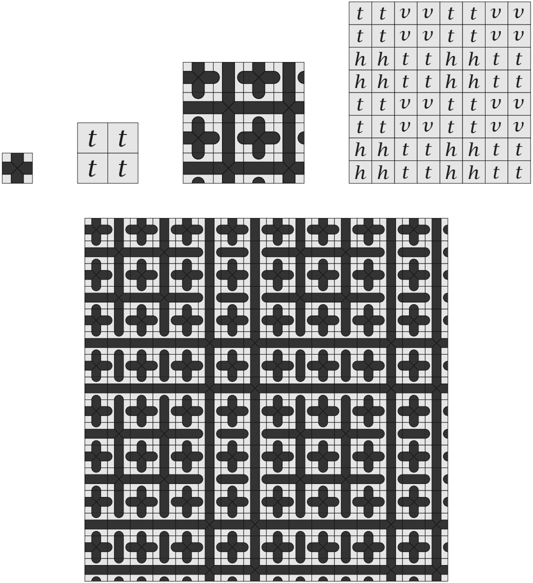

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Encart publicitaire

Substitutions et pavages II : soficité directionnelle

Bastien Le Gloannec

LIFO, Université d'Orléans

GdT GAMoC — 10 novembre 2011