Aperiodic tilings and substitutions

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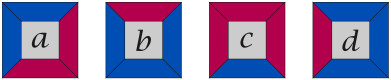
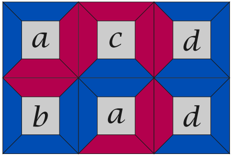
The Domino Problem (DP)

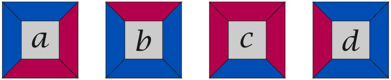
"Assume we are ***given a finite set of square plates*** *of the same size with* ***edges colored****, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are* ***not permitted to rotate or reflect a plate****. The question is to find an effective procedure by which we can* ***decide****, for each given finite set of plates,* ***whether we can cover up the whole plane*** *(or, equivalently, an infinite quadrant thereof)* ***with copies of the plates subject to the restriction that adjoining edges must have the same color****."*

(Wang, 1961)

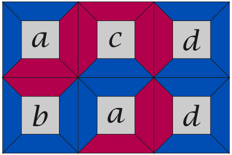




Wang tiles



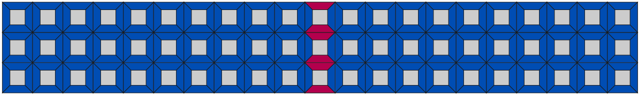
A **tile set** $\tau \subseteq \Sigma^4$ is a tile set with colored edges.



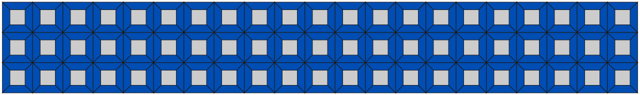
The set of **τ-tilings** $X_\tau \subseteq \tau^{\mathbb{Z}^2}$ is the set of colorings of $\mathbb{Z}^2$ by τ where colors match along edges.

Periodic Tilings

Definition A tiling is **periodic** with period p if it is invariant by a **translation** of vector p.



Lemma If a tile set admits a **periodic** tiling then it admits a **biperiodic** tiling.

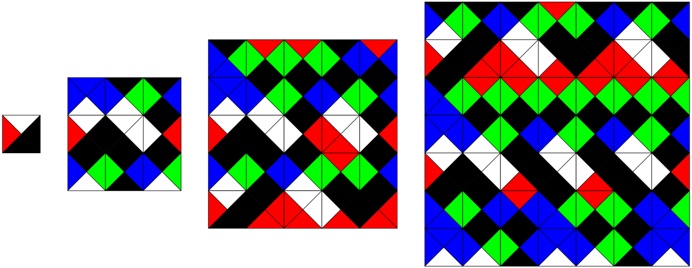


Lemma Tile sets tiling the plane biperiodically are **re** (**recursively enumerable**).

co-Tiling

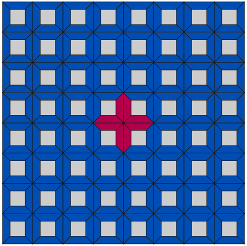
Lemma Tile sets tiling the plane are **co-re**.

Sketch of the proof Consider tilings of larger and larger square regions. If the set does not tile the plane, by compacity, there exists a size of square it cannot cover with tiles.



Aperiodicity

Definition A tiling is **aperiodic** if it admits no non-trivial period.



Definition A tile set is **aperiodic** if it admits a tiling and all its tilings are aperiodic.

Remark If there were **no aperiodic** tile set, the Domino Problem would be **decidable**.

Undecidability of DP

Theorem[Berger 1964] **DP** is **undecidable**.

Remark To prove it one needs **aperiodic** tile sets.

Seminal self-similarity based proofs *(reduction from **HP**)*:

- Berger, 1964 *(20426 tiles, a full PhD thesis)*
- Robinson, 1971 *(56 tiles, 17 pages, long case analysis)*
- Durand et al, 2007 *(Kleene's fixpoint existence argument)*

Tiling rows seen as transducer trace based proof:
Kari, 2007 *(affine maps, reduction from **IP**)*

And others!

- Mozes, 1990 *(non-deterministic substitutions)*
- Aanderaa and Lewis, 1980 *(1-systems and 2-systems)*

In this talk

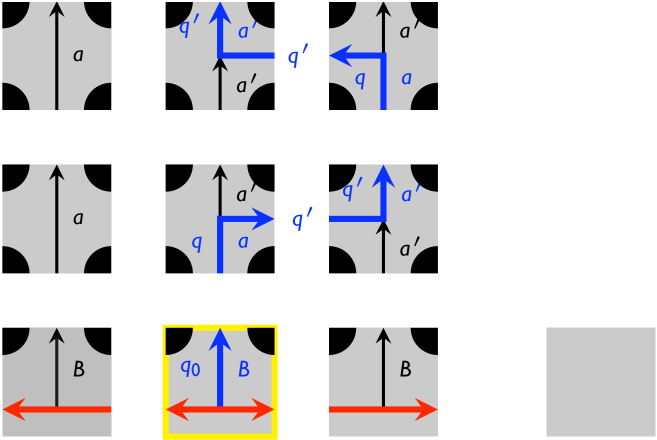
A simple original construction of an aperiodic tile set based on two-by-two substitution systems...

...and its application to an old historical construction.

This work combines tools and ideas from:

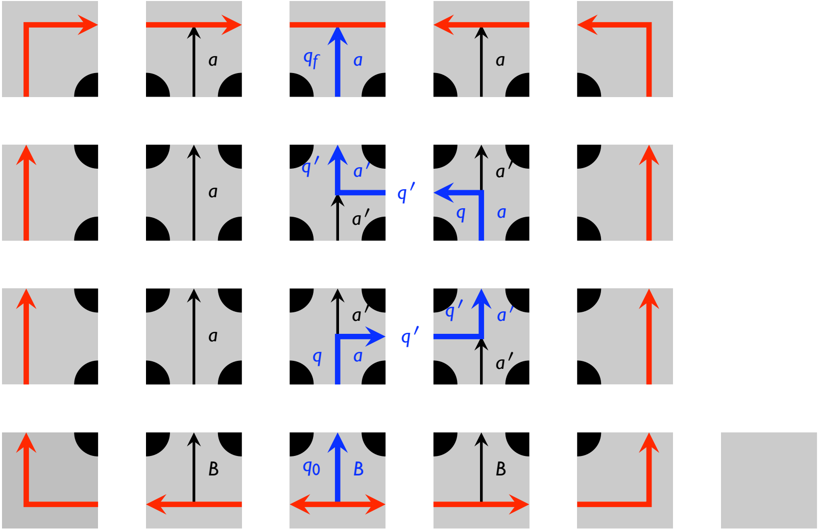
- **[Berger 64]** *The Undecidability of the Domino Problem*
- **[Robinson 71]** *Undecidability and nonperiodicity for tilings of the plane*
- **[Grünbaum Shephard 89]** *Tilings and Patterns, an introduction*
- **[Durand Levin Shen 05]** *Local rules and global order, or aperiodic tilings*

Tiling with a fixed tile



No **halting** tile.

Finite Tiling

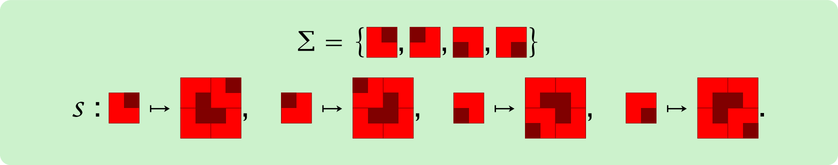


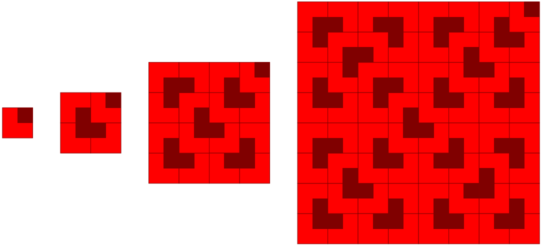
1. Two-by-two Substitution Systems

2. An Aperiodic Tile Set

3. Conclusion

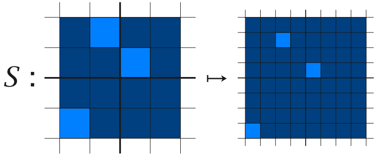
Substitutions





Two-by-two substitutions

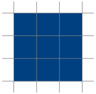
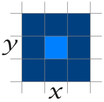
A **2x2 substitution** $s : \Sigma \to \Sigma^{\boxplus}$ maps letters to squares of letters on the same finite alphabet.



The substitution is extended as a **global map** $S : \Sigma^{\mathbb{Z}^2} \to \Sigma^{\mathbb{Z}^2}$ on colorings of the plane:

$$\forall z \in \mathbb{Z}^2, \forall k \in \boxplus, \quad S(c)(2z+k) = s(c(z))(k)$$

Limit set and history

$\Lambda_s = \Big\{$  $\Big\} \cup \Big\{$ 

$\Big\}_{x,y \in \mathbb{Z}^2}$

The **limit set** $\Lambda_s \subseteq \Sigma^{\mathbb{Z}^2}$ is the maximal attractor of S:

$$\Lambda_s = \bigcap_{t \in \mathbb{N}} \left\langle S^t \left(\Sigma^{\mathbb{Z}^2} \right) \right\rangle_\sigma$$

The limit set is the set of colorings admitting an **history** $(c_i)_{i \in \mathbb{N}}$ where $c_i = \sigma_{u_i}(S(c_{i+1}))$.

Unambiguous substitutions

A substitution is **aperiodic** if its limit set Λ_s is aperiodic.

A substitution is **unambiguous** if, for every coloring c from its limit set Λ_s, there exists a unique coloring c' and a unique translation $u \in \boxplus$ satisfying $c = \sigma_u(S(c'))$.

Proposition **Unambiguity** implies **aperiodicity**.

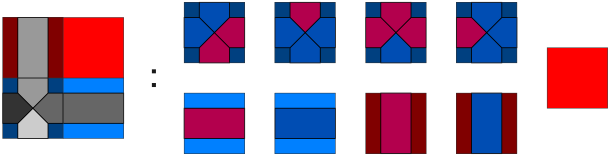
Sketch of the proof. Consider a periodic coloring with minimal period p, its preimage has period $p/2$. $\Diamond$

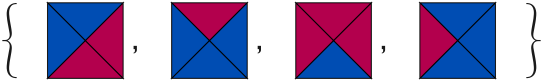
Idea. Construct a tile set whose tilings are in the limit set of an unambiguous substitution system.

Coding tile sets into tile sets

Definition A tile set τ' **codes** a tile set τ, according to a **coding rule** $t : \tau \to \tau'^{\boxplus}$ if t is injective and

$$X_{\tau'} = \{\sigma_u(t(c)) | c \in X_\tau, u \in \boxplus\}$$





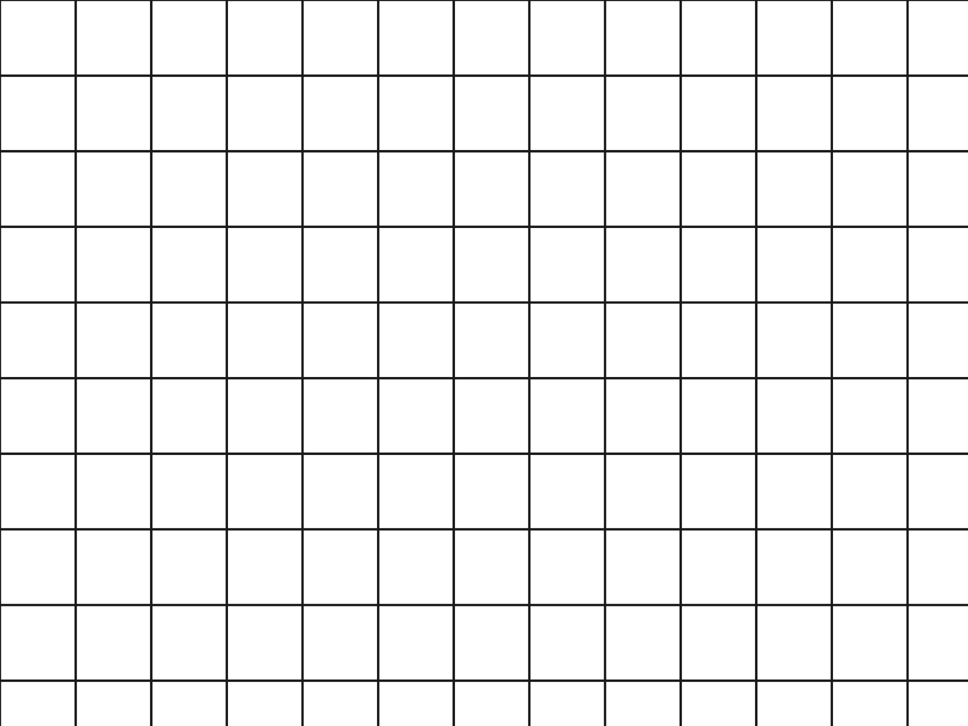
Unambiguous self-coding

Definition A tile set τ **codes** a substitution $s : \tau \to \tau^{\boxplus}$ if it codes itself according to the coding rule s.

Proposition A tile set both admitting a tiling and **coding** an **unambiguous** substitution is **aperiodic**.

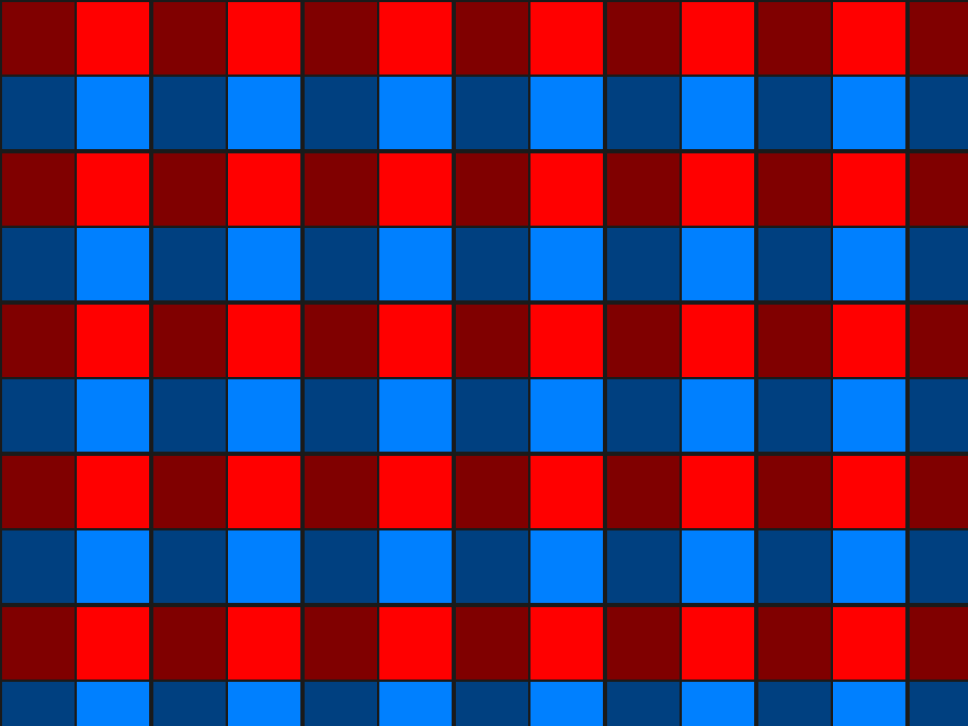
Sketch of the proof. $X_\tau \subseteq \Lambda_s$ and $X_\tau \neq \emptyset$. ◊

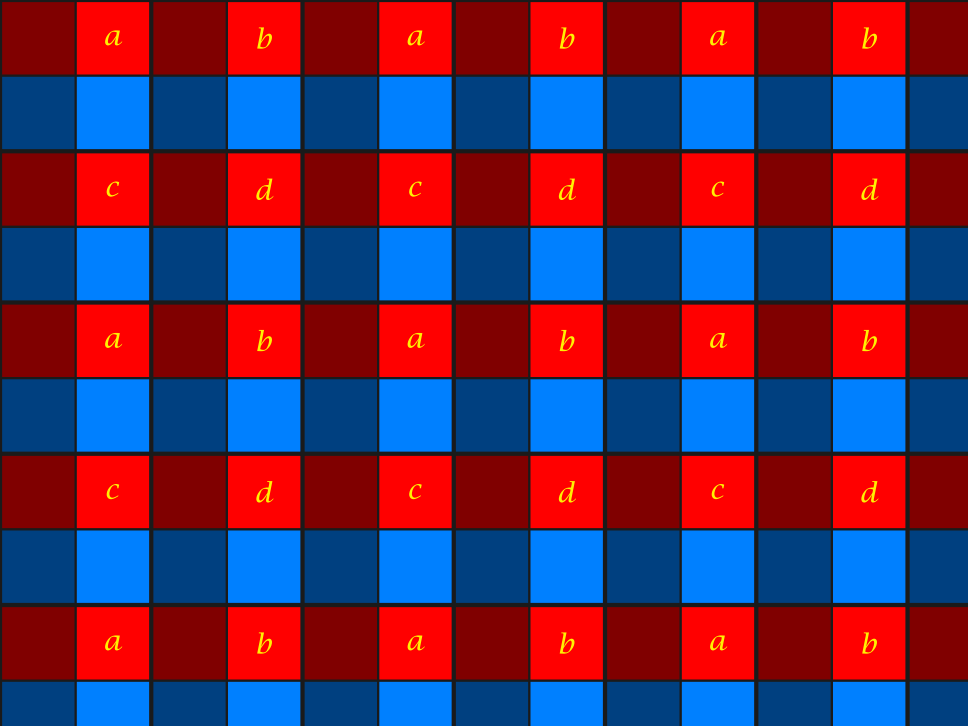
Idea. Construct a tile set whose tilings are in the limit set of a **locally checkable** unambiguous substitution embedding a whole history.



a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*

a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*




a b a b a b
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a b a b a b
c d c d c d
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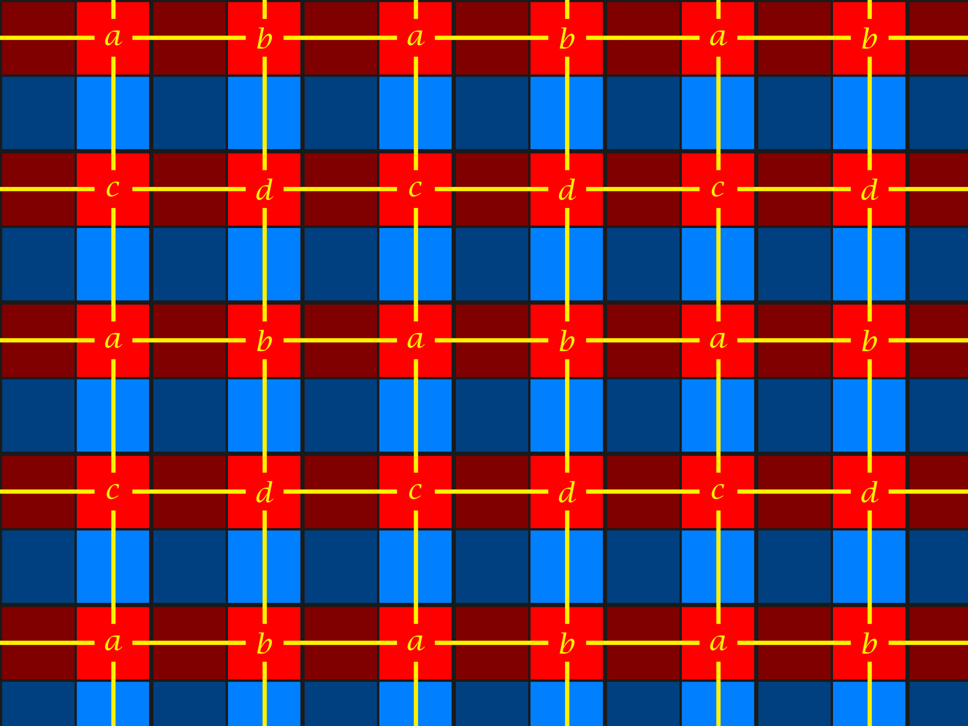
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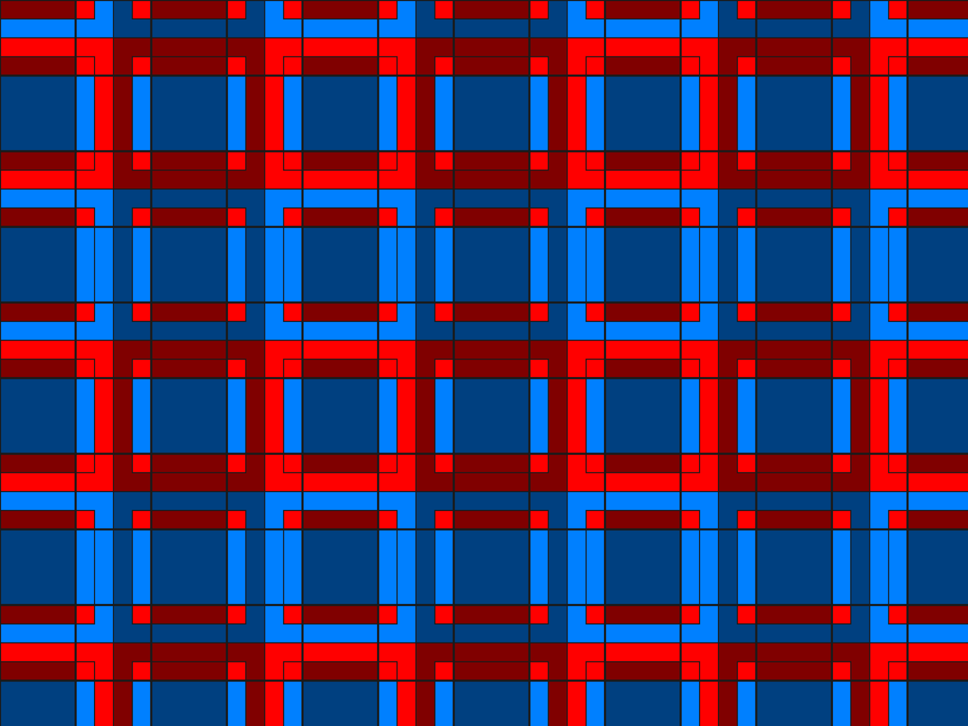

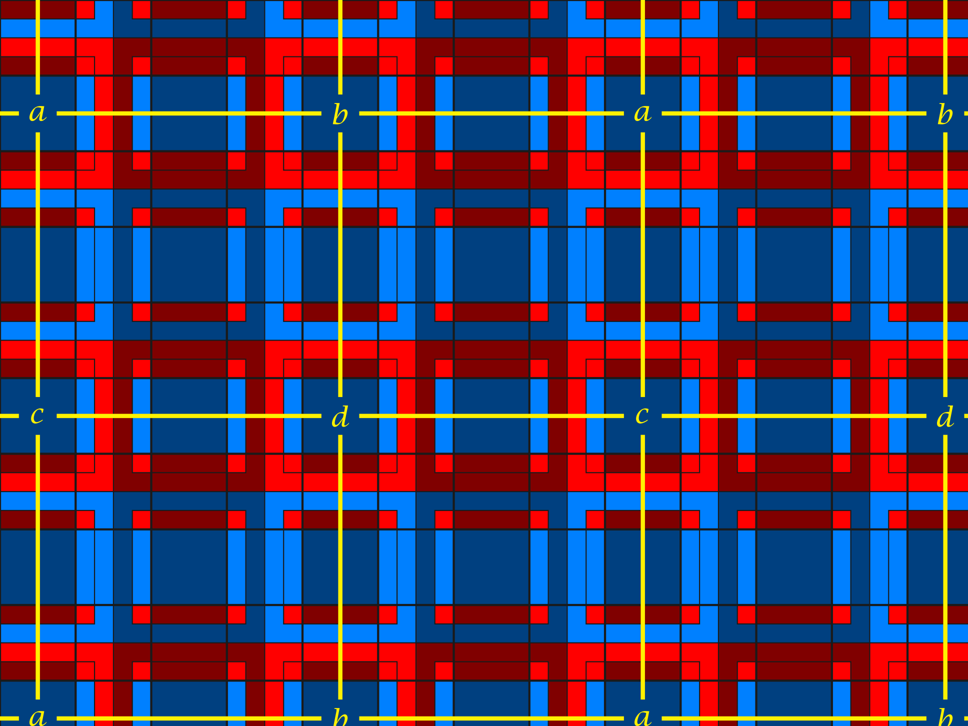
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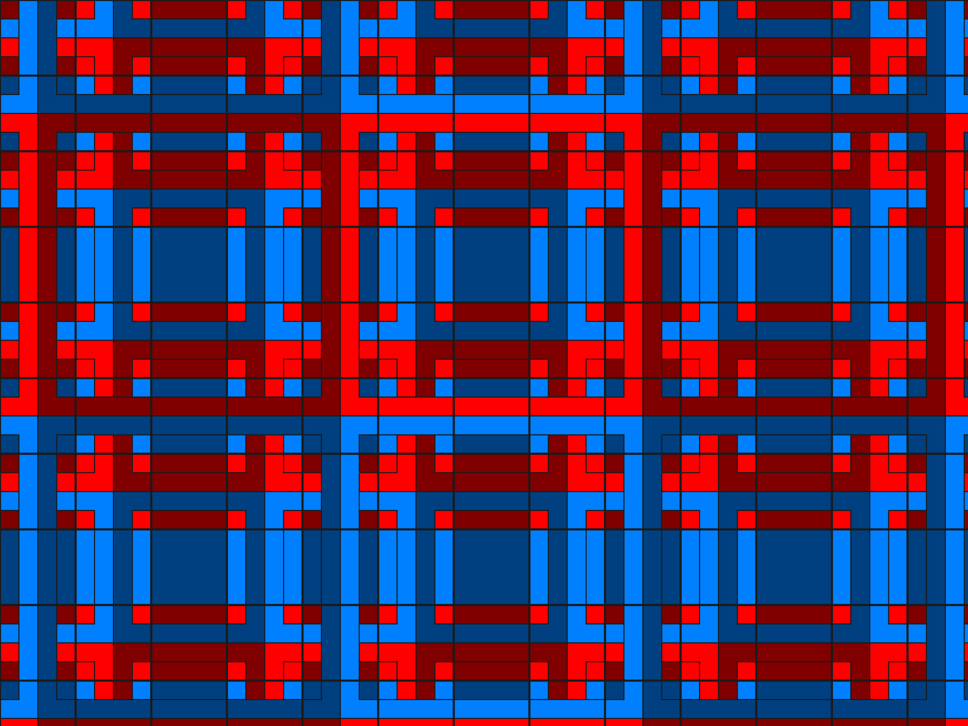
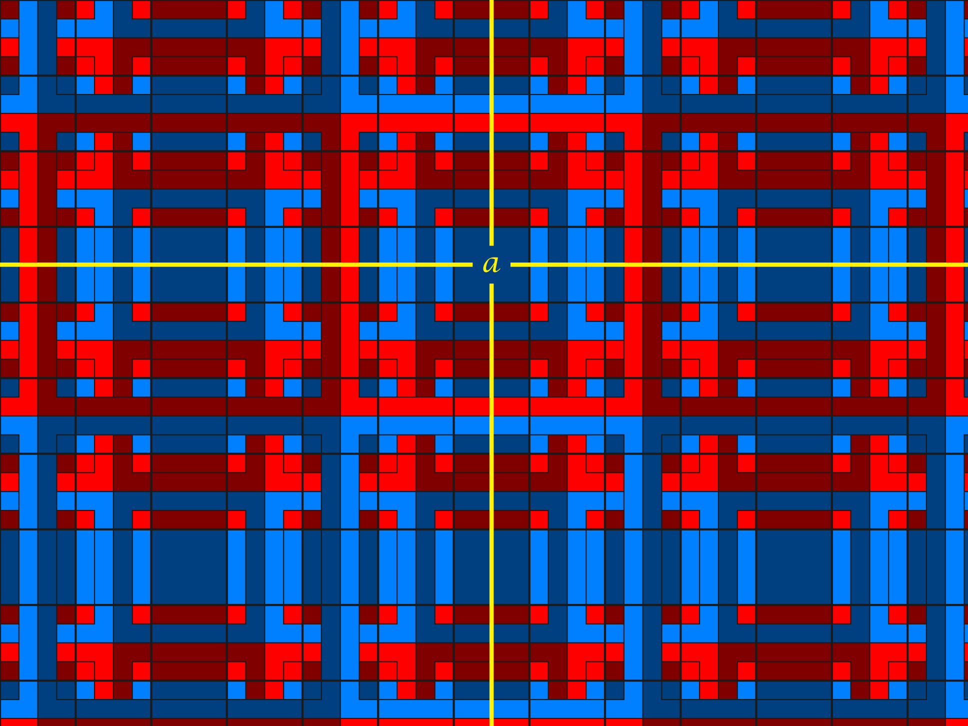
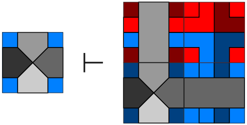



a

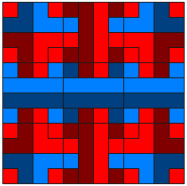
Is this self-encoding?

Iterating the coding rule one obtains 56 tiles.



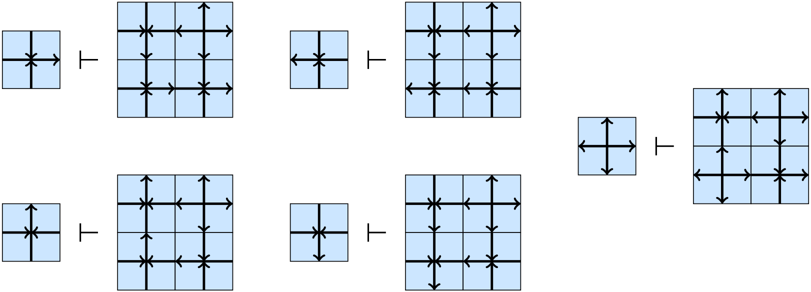
coding rule

Unfortunately, this tile set is **not self-coding**.



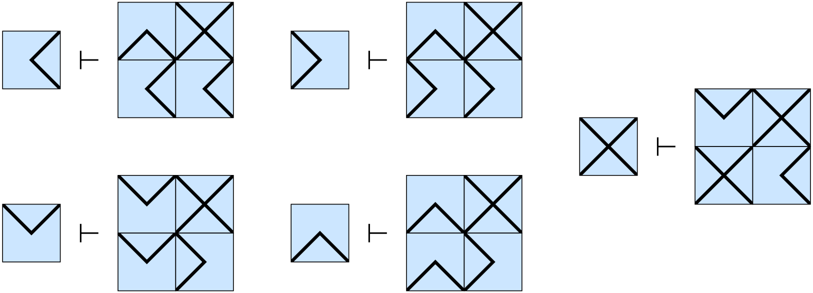
Idea Add a **synchronizing substitution** as a **third layer**.

à la Robinson

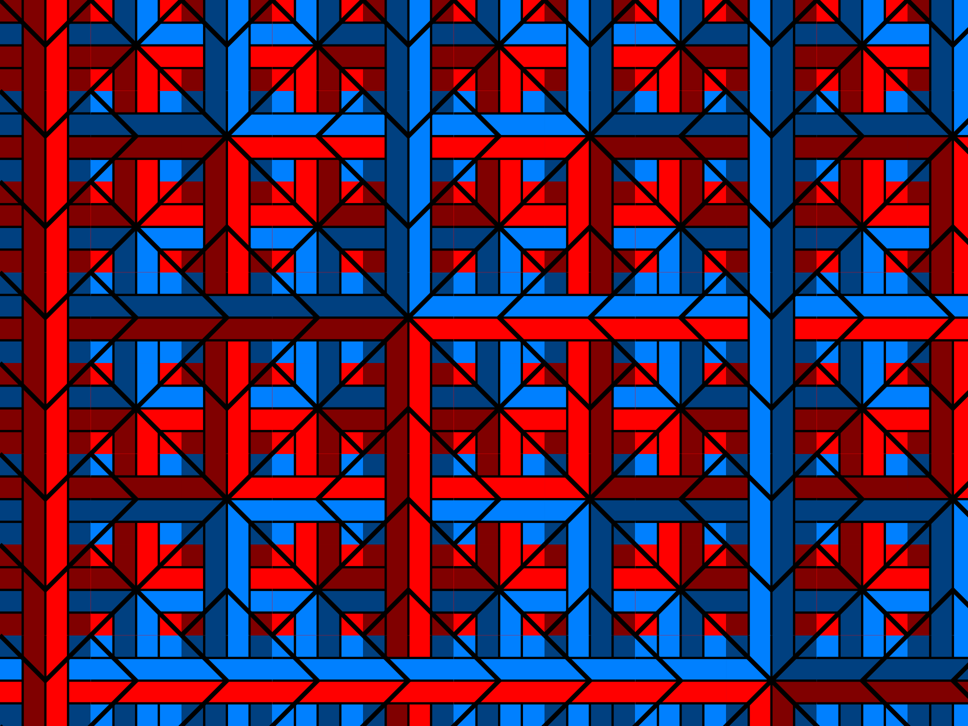


Proposition The associated tile set of **104 tiles** admits a tiling and **codes** an **unambiguous** substitution.

à la Robinson



Proposition The associated tile set of **104 tiles** admits a tiling and **codes** an **unambiguous** substitution.



Aperiodicity: sketch of the proof

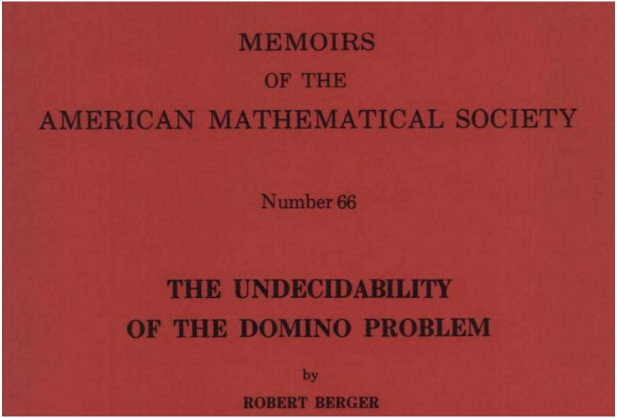
1. The tile set admits a tiling:
Generate a valid tiling by iterating the substitution rule:
$X_\tau \cap \Lambda_s \neq \emptyset$.

2. The substitution is unambiguous:
It is injective and the projectors have disjoined images.

3. The tile set codes the substitution:

(a) each tiling is an image of the canonical substitution
Consider any tiling, level by level, short case analysis.

(b) the preimage of a tiling is a tiling
Straightforward by construction (preimage remove constraints).


MEMOIRS
OF THE
AMERICAN MATHEMATICAL SOCIETY

Number 66

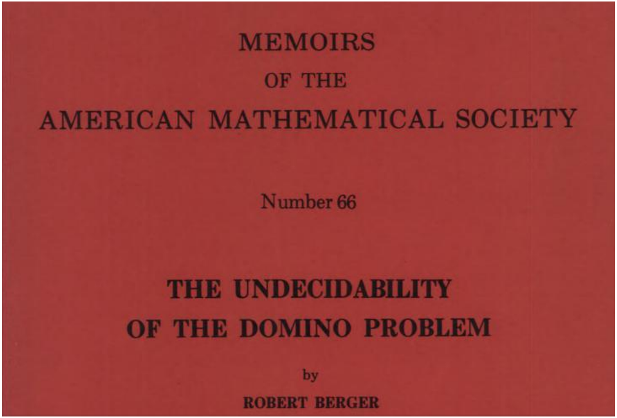
THE UNDECIDABILITY OF THE DOMINO PROBLEM

by

ROBERT BERGER

"Robert Berger (born 1938) is known for inventing the first aperiodic tiling using a set of 20,426 distinct tile shapes."

[Robert Berger Wikipedia entry]



"(. . .) In 1966 R. Berger discovered the first aperiodic tile set. It contains 20,426 Wang tiles, (. . .)
Berger himself managed to reduce the number of tiles to 104 and he described these in his thesis, though they were omitted from the published version (Berger [1966]). (. . .)" [GrSh, p.584]

THE UNDECIDABILITY OF THE DOMINO PROBLEM

A thesis presented

by

Robert Berger

to

The Division of Engineering and Applied Physics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Applied Mathematics

Harvard University

Cambridge, Massachusetts

July 1964

APPENDIX II

A SIMPLER SOLVABLE DOMINO SET WITH NO TORUS

The skeleton set, K, analyzed in PART 3, is a solvable domino set with no torus. Since it is designed to serve also as a base set for modeling of Turing machines, it is not surprising that simpler solvable, torus-less domino sets exist. One such set, call it Q, is specified by Tables 9-12. The first three tables show the base, skeleton, and parity prototypes of Q. Although these tables show symbols in the center of domino edges, the base, skeleton, and parity channels should be thought of as distinct. Table 12 serves the same function for Q as did Table 4 for K, namely that of specifying which products of prototypes are permitted. However, since Q is a fairly small set, it is not too cumbersome to enumerate only those dominoes which are actually used in solutions of Q, 104 in all. (No concerted attempt has been made to find the smallest solvable torus-less domino set.)

Figure 24 shows, separately, skeleton signals and parity signals in the same portion of a solution of Q. If Figure 24 is rotated one-eighth turn clockwise, its skeleton signals bear a strong resemblance to the CD-signals of K.

A person who understands the skeleton set should have no trouble convincing himself of the likelihood that all solutions of Q look line extensions of Figure 24. The following hints will help.

The skeleton set, K, analyzed in PART 3, is a solvable domino set with no torus. Since it is designed to serve also as a base set for modeling of Turing machines, it is not surprising that simpler solvable, torusless domino sets exist. One such set, call it Q, is specified by Tables

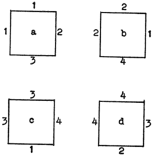
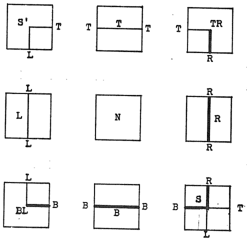


Table 9

Base Prototypes of Q



Note:

Use of same line weight for a horizontal and a (different) vertical signal introduces no ambiguity.

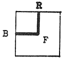


Table 10

Skeleton Prototypes of Q

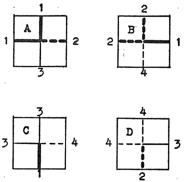
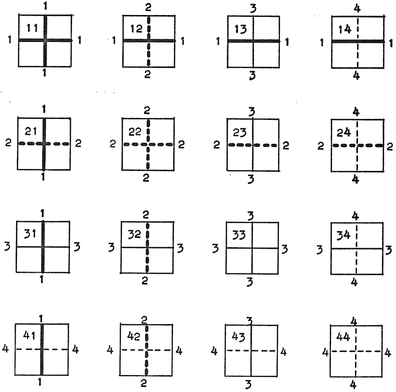


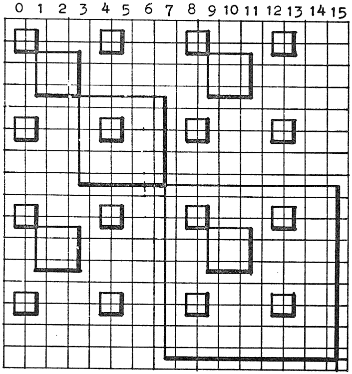
Table 11

Parity Prototypes of Q

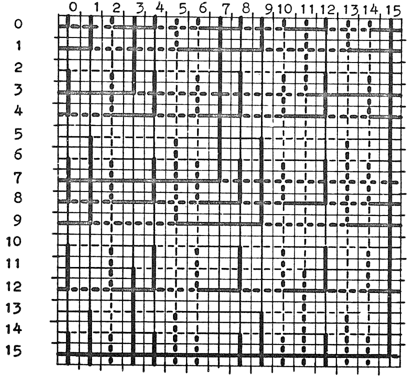


Base	Skeleton	A	B	C	D	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
a	S' S T	x																			
	L TR																				
	BL R B																				
	F N		x	x	x																
b	S' S T																				
	L TR							x		x	x	x	x			x				x	
	BL R B					x	x	x	x					x	x	x	x				
	F N					x	x	x	x					x	x	x	x	x	x		x
c	S' S T									x	x		x								
	L TR																				
	BL R B					x	x	x		x	x	x		x	x	x		x	x	x	
	F N					x	x		x	x	x			x	x		x	x	x		x
d	S' S T	x								x	x	x									
	L TR							x		x	x	x	x			x				x	
	BL R B					x x	x x	x x	x	x	x	x		x x	x x	x x	x	x	x	x	
	F N		x	x	x	x	x	x	x	x	x	■	■	x	x	x	x	x	x	■	x

Table 12 Prototype Products in Q



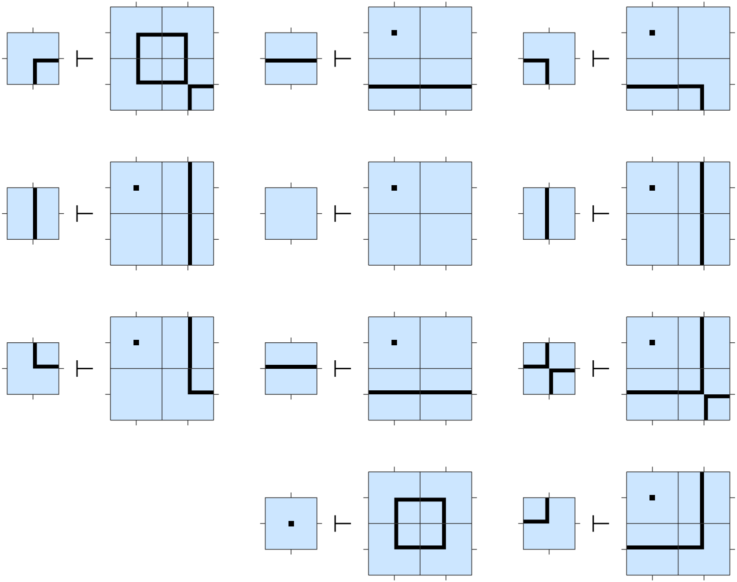
Skeleton Signals

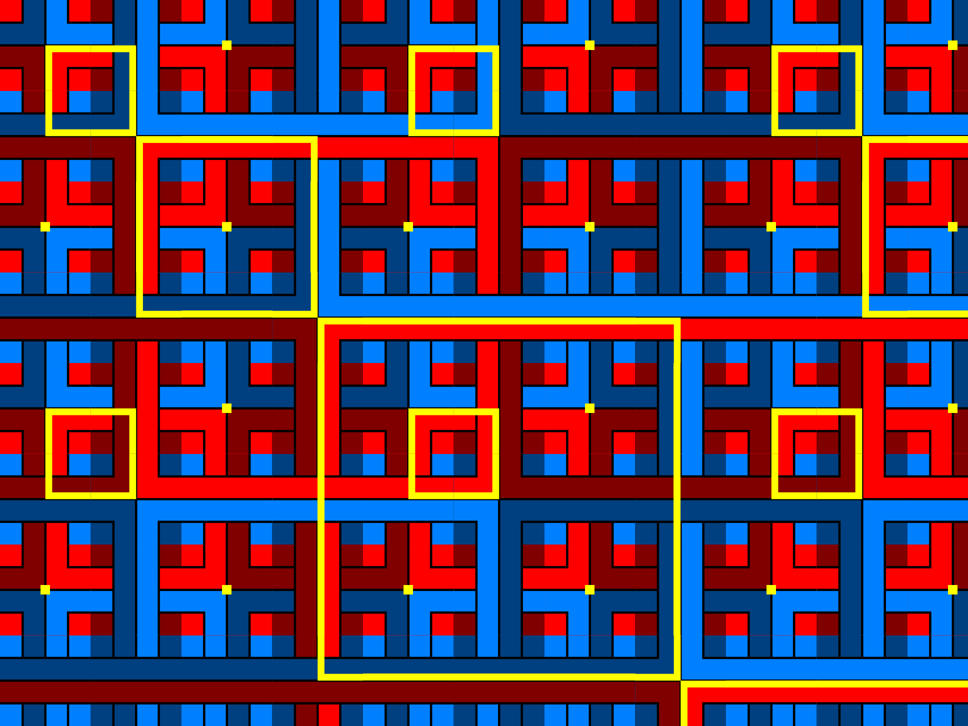


Parity Signals

Figure 24 Part of the Solution of Q

Berger's skeleton substitution





Berger's forgotten aperiodic tile set

Proposition The associated tile set of **103 tiles** admits a tiling and **codes** an **unambiguous** substitution.

Remark The number of tiles **does not grow monotonically** in the number of letters of the synchronizing layer.

5 letters	→	104 tiles
11 letters	→	103 tiles

1. Two-by-two Substitution Systems

2. An Aperiodic Tile Set

3. Conclusion

To continue...

Theorem The **limit set** of a 2x2 substitution is **sofic**.

Idea To encode Λ_s via **local matching rules**, decorate s into a **locally checkable** $s^\bullet$ embedding a whole history.

Corollary[Berger 1964] **DP** is **undecidable**.

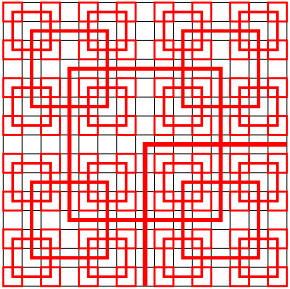
Idea Construct a **2x2 substitution** whose **limit set** contains everywhere **squares** of larger and larger size, insert **Turing computation** inside those squares.

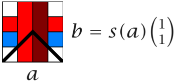
Enforcing substitutions *via* tilings

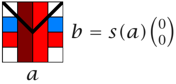
Let π map every tile of $\tau(s')$ to $s'(a)(u)$ where a and u are the letter and the value of ⊞ on layer 1.

Proposition. Let s' be any substitution system. The tile set $\tau(s')$ enforces s':
$\pi\left(X_{\tau(s')}\right) = \Lambda_{s'}$.

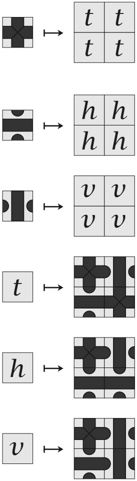
Idea. Every tiling of $\tau(s')$ codes an history of s' and every history of s' can be encoded into a tiling of $\tau(s')$.

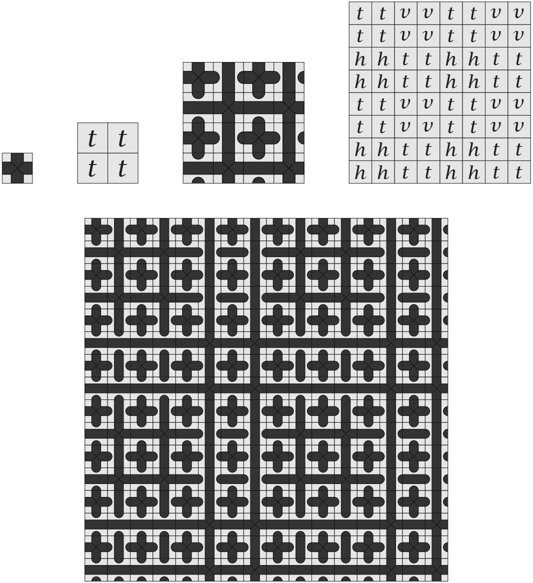




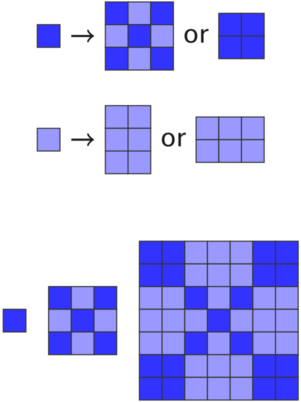


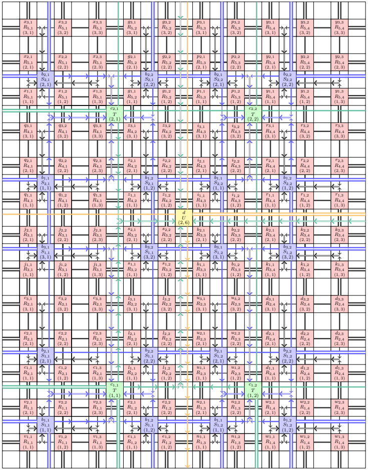
Squares everywhere





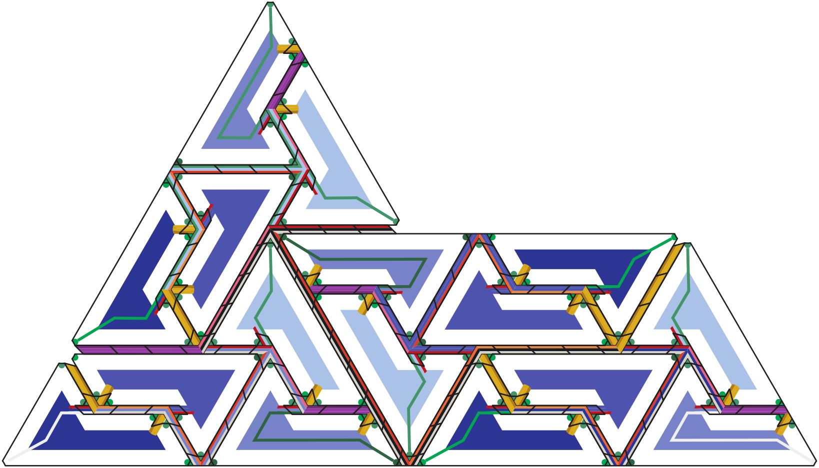
Mozes 1990





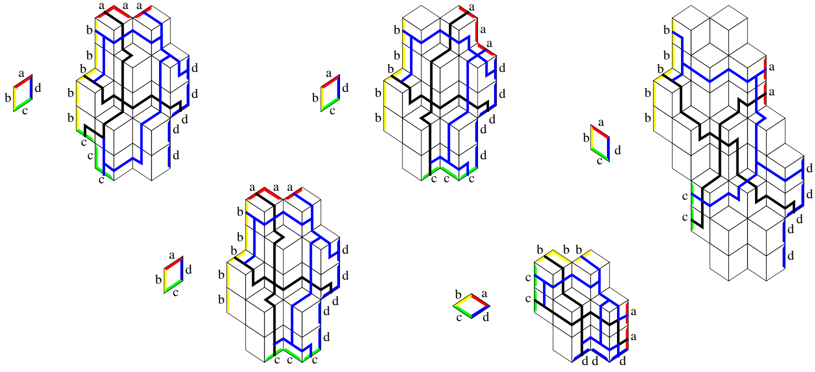
Theorem[Mozes 1990] The limit set of a **non-deterministic rectangular substitution** *(+ some hypothesis)* is sofic.

Goodman-Strauss 1998



Theorem[Goodman-Strauss 1998] The limit set of **homothetic substitution** *(+ some hypothesis)* is sofic.

Fernique-O 2010



Theorem[Fernique-O 2010] The limit set of a **combinatorial substitution** *(+ some hypothesis)* is sofic.