A
Small Minimal Aperiodic Reversible Turing machine

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joint work with J. Cassaigne, R. Torres and A. Gajardo

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Motivation

Solve a conjecture that we had with J. Kari a few years ago:

Periodicity and Immortality in Reversible Computing

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Open Problems with conjectures

Conjecture 1 It is undecidable whether a given complete 2-RCM admits a periodic configuration. (proven if you remove complete or replace 2 by 3)

Conjecture 2 There exists a complete RTM without a periodic configuration. (known for DTM [BCN02])

Conjecture 3 It is undecidable whether a given complete RTM admits a periodic configuration. (known for DTM [BCN02])

Theorem To find if a given complete reversible Turing machine admits a periodic orbit is $\Sigma_1^0$-complete.
1. Dynamics of Turing machines
Turing machines

**Definition**  A **Turing machine** is a triple \((Q, \Sigma, \delta)\) where \(Q\) is the finite set of states, \(\Sigma\) is the finite set of tape symbols and \(\delta : Q \times \Sigma \rightarrow Q \times \Sigma \times \{\leftarrow, \rightarrow\}\) is the transition function.

Transition \(\delta(s, a) = (t, b, d)\) means:

"in state \(s\), when reading the symbol \(a\) on the tape, replace it by \(b\) move the head in direction \(d\) and enter state \(t\)."

**Remark**  We do not care about blank symbol or initial and final states, we see Turing machines as dynamical systems.
Moving head dynamics

\[ X_H = Q \times \mathbb{Z} \times \Sigma^\mathbb{Z} \cup \Sigma^\mathbb{Z} \]

\[ T_H : X_H \rightarrow X_H \]


Long shot
Moving tape dynamics

\[ X_T = Q \times \Sigma^\mathbb{Z} \]

\[ T_T : X_T \rightarrow X_T \]


Tracking shot
Trace subshift

\[ S_T \subseteq (Q \times \Sigma)^\omega \]


Point of view shot
Definition A point $x \in X$ is **periodic** if it admits a period $p > 0$ such that $T^p(x) = x$.

Definition A machine is **periodic** if every point is periodic.

Remark Periodicity implies uniform periodicity: $T^p = \text{Id}$.

Theorem[KO08] The **periodicity problem** is $\Sigma^0_1$-complete.

Definition A machine is **aperiodic** if it has no periodic point.
Partial vs complete machines

Definition  A TM is complete if \( \delta \) is completely defined, otherwise undefined transitions of a partial \( \delta \) correspond to halting configurations.

Definition  A point is mortal if it eventually halts.

Thm[Hooper66]  The immortality problem is \( \Pi^0_1 \)-complete.

Rk  Mortality is different from totality which is \( \Pi^0_2 \)-complete.

Thm[KO08]  The result is the same for reversible TM.
2. Reversible Turing machines
Intuitively, a TM is reversible if there exists another TM to compute backwards: “$T_2 = T_1^{-1}$”. Forget technical details... 

**Definition**  A TM is reversible if $\delta$ can be decomposed as:

$$
\delta(s, a) = (t, b, \rho(t)) \quad \text{where} \quad (t, b) = \sigma(s, a)
$$

$$
\rho: Q \rightarrow \{\leftarrow, \rightarrow\}
$$

$$
\sigma \in \mathcal{S}_{Q \times \Sigma}
$$

**Remark**  $\delta^{-1}(t, b) = (s, a, \Diamond(\rho(s)))$
A complete RTM

It is **time-symmetric**: its own inverse up to state/symbol permutation.

\[
\begin{align*}
1 & \leftrightarrow 2 \\
b & \leftrightarrow p \\
d & \leftrightarrow q
\end{align*}
\]
We want to prove the following:

**Theorem**  To find if a given complete reversible Turing machine admits a periodic orbit is $\Sigma_1^0$-complete.

In the partial case we use the following tool:

**Prop[KO08]**  To find if a given (aperiodic) RTM can reach a given state $t$ from a given state $s$ is $\Sigma_1^0$-complete.
The partial case

**Principle of the reduction** Associate to an (aperiodic) RTM $\mathcal{M}$ with given $s$ and $t$ a new machine with a periodic orbit if and only if $t$ is reachable from $s$.

We need to find a way to **complete** the constructed machine.
3. a SMART machine
Cassaigne machine

 Conj[Kůrka97] Every complete TM has a periodic point.

 Thm[BCN02] No, here is an aperiodic complete TM.

 Rk It relies on the bounded search technique [Hooper66].

In 2008, I asked J. Cassaigne if he had a reversible version of the BCN construction...

...he answered with a small machine C which is a reversible and (drastic) simplification of the BCN machine.
Proposition  The machine $\mathcal{C}$ is aperiodic.

Idea of the proof (R. Torres)

1. The behavior starting from a tape of 0 is aperiodic;
2. Every block of 0 eventually grows;
3. Thus $\mathcal{C}$ is aperiodic.
Minimality

The behavior of $\mathcal{C}$ can be precisely described (R. Torres).

**Proposition** The behavior starting from a tape of 0 is dense.

**Proposition** The trace subshift of $\mathcal{C}$ is minimal.

**Proposition** The trace subshift of $\mathcal{C}$ is substitutive.
4. Embedding the machine
Embedding trick

Use the transitions of the Cassaigne machine to connect the $u_i'$ to the $u_i$ and the $v_i'$ to the $v_i$. 
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