A Small Minimal Aperiodic Reversible Turing machine

Julien Cassaigne (IML, CNRS, Marseille, France)
Nicolas Ollinger (LIFO, Univ. Orléans, France)
Rodrigo Torres (CMM, Univ. de Chile, Concepción, Chile)

Journées SDA2+Frac — April 9th, 2014
Motivation

Solve a conjecture that we had with J. Kari a few years ago:

Theorem  To find if a given complete reversible Turing machine admits a periodic orbit is $\Sigma_1$-complete.
1. Dynamics of Turing machines
Turing machines

Definition A Turing machine is a triple \((Q, \Sigma, \delta)\) where \(Q\) is the finite set of states, \(\Sigma\) is the finite set of tape symbols and 
\(\delta : Q \times \Sigma \to Q \times \Sigma \times \{\leftarrow, \rightarrow\}\) is the transition function.

Transition \(\delta(s, a) = (t, b, d)\) means:

“in state \(s\), when reading the symbol \(a\) on the tape, replace it by \(b\) move the head in direction \(d\) and enter state \(t\).”

Remark We do not care about blank symbol or initial and final states, we see Turing machines as dynamical systems.
Moving head dynamics

\[ X_H = Q \times \mathbb{Z} \times \Sigma^\mathbb{Z} \cup \Sigma^\mathbb{Z} \]

\[ T_H : X_H \rightarrow X_H \]

\[
\vdots
\begin{array}{ccccccccc}
\cdot & \cdot & \cdot & 000000 & b & 000000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0000001 & d & 000000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 000000 & b & 110000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0000001 & p & 100000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 00000010 & d & 00000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0000001b & 010000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 00000011 & d & 10000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0000001q & 110000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 000000b & 101000000 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0000001p & 010000000 & \cdot & \cdot \\
\vdots
\end{array}
\]


Long shot
Moving head dynamics

\[ X_H = Q \times \mathbb{Z} \times \Sigma^\mathbb{Z} \cup \Sigma^\mathbb{Z} \]

\[ T_H : X_H \rightarrow X_H \]


Long shot
Moving tape dynamics

\[ X_T = Q \times \Sigma^\mathbb{Z} \]
\[ T_T : X_T \to X_T \]

\[ \cdots 0000000b00000000 \cdots \]
\[ \cdots 0000001d00000000 \cdots \]
\[ \cdots 0000000b11000000 \cdots \]
\[ \cdots 0000001p10000000 \cdots \]
\[ \cdots 0000010d00000000 \cdots \]
\[ \cdots 0000001b01000000 \cdots \]
\[ \cdots 0000011d10000000 \cdots \]
\[ \cdots 0000001q11000000 \cdots \]
\[ \cdots 0000000b10100000 \cdots \]
\[ \cdots 0000001p01000000 \cdots \]
\[ \cdots \]


Tracking shot
Moving tape dynamics

\[ X_T = Q \times \Sigma^\mathbb{Z} \]
\[ T_T : X_T \rightarrow X_T \]


Tracking shot
Trace subshift

\[ S_T \subseteq (Q \times \Sigma)^\omega \]

0 0 1 1 0 0 1 1 1 0 
\[ b \quad d \quad b \quad p \quad d \quad b \quad d \quad q \quad b \quad p \quad \ldots \]


Point of view shot
Simple dynamical properties

Definition A point \( x \in X \) is periodic if it admits a period \( p > 0 \) such that \( T^p(x) = x \).

Definition A machine is periodic if every point is periodic.

Remark Periodicity implies uniform periodicity: \( T^p = \text{Id} \).

Theorem[KO08] The periodicity problem is \( \Sigma_1 \)-complete.

Definition A machine is aperiodic if it has no periodic point.
Partial vs complete machines

**Definition** A TM is **complete** if \( \delta \) is completely defined, otherwise **undefined transitions** of a partial \( \delta \) correspond to **halting configurations**.

**Definition** A point is **mortal** if it eventually **halts**.

**Thm**[Hooper66] The **immortality problem** is \( \Pi_1 \)-complete.

**Rk** Mortality is different from **totality** which is \( \Pi_2 \)-complete.

**Thm**[KO08] The result is the same for **reversible TM**.
2. Reversible Turing machines
Reversible Turing machines

Intuitively, a TM is **reversible** if there exists another TM to compute backwards: “$T_2 = T_1^{-1}$”. **Forget technical details**.

**Definition** A TM is **reversible** if $\delta$ can be decomposed as:

$$\delta(s,a) = (t,b,\rho(t)) \quad \text{where} \quad (t,b) = \sigma(s,a)$$

$$\rho : Q \rightarrow \{\leftarrow, \rightarrow\}$$

$$\sigma \in \mathcal{S}_{Q \times \Sigma}$$

**Remark** $\delta^{-1}(t,b) = (s,a,\blacklozenge(\rho(s)))$
A complete RTM

It is time-symmetric: its own inverse up to state/symbol permutation.

1 ⇔ 2
b ⇔ p
d ⇔ q

2. Reversible Turing machines
Searching for a reduction

We want to prove the following:

**Theorem** To find if a given complete reversible Turing machine admits a periodic orbit is $\Sigma_1$-complete.

In the partial case we use the following tool:

**Prop[KO08]** To find if a given (aperiodic) RTM can reach a given state $t$ from a given state $s$ is $\Sigma_1$-complete.
The partial case

**Principle of the reduction** Associate to an (aperiodic) RTM $M$ with given $s$ and $t$ a new machine with a periodic orbit if and only if $t$ is reachable from $s$.

We need to find a way to **complete** the constructed machine.
3. a SMART machine
Conj[Kůrka97] Every complete TM has a periodic point.

Thm[BCN02] No, here is an aperiodic complete TM.

Rk It relies on the bounded search technique [Hooper66].

In 2008, I asked J. Cassaigne if he had a reversible version of the BCN construction...

...he answered with a small machine $\mathcal{C}$ which is a reversible and (drastic) simplification of the BCN machine.
Symmetry

It is both space- and time-symmetric.

\[
\begin{align*}
\blacktriangleleft & \iff \blacktriangleright \\
\text{b} & \iff \text{d} \\
\text{p} & \iff \text{q}
\end{align*}
\]
Aperiodicity

Proposition  The machine C is aperiodic.

Idea of the proof
1. The behavior starting from a tape of 0 is aperiodic;
2. Every block of 0 eventually grows;
3. Thus C is aperiodic.
Minimality

The behavior of $C$ can be precisely described.

Proposition The behavior starting from a tape of 0 is dense.

Proposition The trace subshift of $C$ is minimal.

Proposition The trace subshift of $C$ is substitutive.
Substitutive trace subshift

\[
\begin{align*}
\varphi(0) & = 0011 & \varphi(0) & = 0011 \\
\varphi(b) & = bdbp & \varphi(d) & = dbdq \\
\varphi(x) & = x & \varphi(x) & = x \\
\varphi(b) & = b & \varphi(d) & = d \\
\varphi(0) & = 0021 & \varphi(0) & = 0021 \\
\varphi(p) & = pdbp & \varphi(q) & = qbdaq \\
\varphi(x) & = 0x2x & \varphi(x) & = 0x2x \\
\varphi(p) & = pdqp & \varphi(q) & = qbppq
\end{align*}
\]
4. Embedding the machine
Embedding trick

Use the transitions of the Cassaigne machine to connect the $u'_i$ to the $u_i$ and the $v'_i$ to the $v_i$.

4. Embedding the machine
1. Dynamics of Turing machines

2. Reversible Turing machines

3. a SMART machine

4. Embedding the machine