The Transitivity Problem of Turing machines

(hal-01145799)

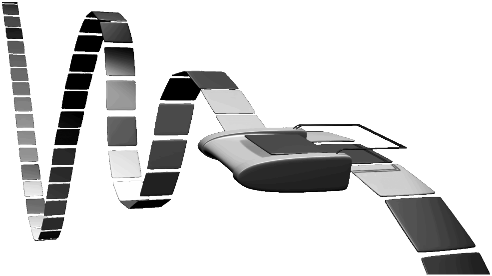
Anahí Gajardo (CMM, Univ. de Chile, Concepción, Chile)
Nicolas Ollinger (LIFO, Univ. Orléans, France)
Rodrigo Torres (CMM, Univ. de Chile, Concepción, Chile)

Loria, FM seminar — March 21st, 2016

Turing machines

The classical **Turing machine**: finitely many **states**, a (bi-)infinite **tape**, a mobile i/o **head** pointing on a cell

(optionally: blank symbol, starting and halting states).



Halting Problem[Σ_1^0-comp.] Given a TM and a finite starting configuration, decide if a halting state is eventually reached.

Reachability and similar questions

Reachability Problem[Σ_1^0-comp.] Given a TM and two states s and t, decide if state t is reachable from state s.

Totality Problem[Π_2^0-comp.] Given a TM, decide if it eventually halts starting from any **finite configuration**.

Mortality Problem[Σ_1^0-comp.] Given a TM, decide if it eventually halts starting from any configuration.

Periodicity Problem[Σ_1^0-comp.] Given a TM, decide if every configuration eventually loops by reaching itself again.

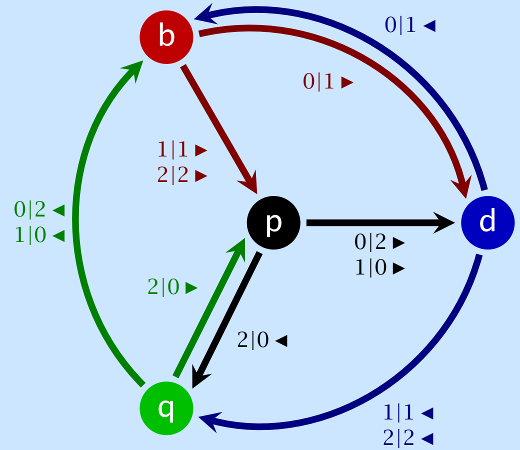
The Transitivity Problem

Transitivity Problem Given a TM, decide if every **partial configuration** is reachable from every **partial configuration** (completed into a proper configuration).

A **partial configuration** specifies only a finite segment of the tape plus the state and head position.

The Transitivity Problem is in Π_2^0. We prove that it is Π_1^0-hard.

Question Can you construct a transitive TM?



1. Dynamics of Turing machines

Turing machines

A **Turing machine** is a triple (Q, Σ, δ) where Q is the finite set of states, Σ is the finite set of tape symbols and $\delta : Q \times \Sigma \to Q \times \Sigma \times \{-1, 0, 1\}$ is the partial transition function.

A transition $\delta(s, a) = (t, b, d)$ means:
"in state s, when reading the symbol a on the tape,
replace it by b move the head in direction d and enter state t."

Configurations are triples $(s, c, p) \in Q \times \Sigma^{\mathbb{Z}} \times \mathbb{Z}$.

A **transition** transforms (s, c, p) into $(t, c', p + d)$ where $\delta(s, c(p)) = (t, b, d)$ and $c' = c$ everywhere but $c'(p) = b$.

Notation $(s, c, p) \vdash (t, c', p + d)$ and closures $\vdash^+$ and $\vdash^*$

Definitions

A configuration (s, c, p) is:

- **halting** if $\delta(s, c(p))$ is undefined, $(s, c(p))$ is a **halting pair**
- **periodic** if $(s, c, p) \vdash^{+} (s, c, p)$

A TM (Q, Σ, δ) is:

- **complete** if δ is complete
- **aperiodic** if it has no periodic configuration
- **surjective** if every configuration has a preimage
- **injective** if every configuration has at most one preimage

For complete machines surjective is equivalent to injective.

Reversibility

Injective TM are in fact reversible TM.

Definition A **reversible** TM $M = (Q, \Sigma, \delta)$ is characterized by a partial injective map ρ and a map μ such that $\delta(s, a) = (t, b, \mu(t))$ where $\rho(s, a) = (t, b)$.

The **reverse** of M is M^{-1} where $\delta^{-1}(t, b) = (s, a, -\mu(s))$.

$$(s, c, p) \vdash_M (t, c', p + \mu(t)) \implies (t, c', p) \vdash_{M^{-1}} (s, c, p - \mu(s))$$

A **starting pair** is a halting pair of the reverse.

A **starting configuration** is a halting config of the reverse.

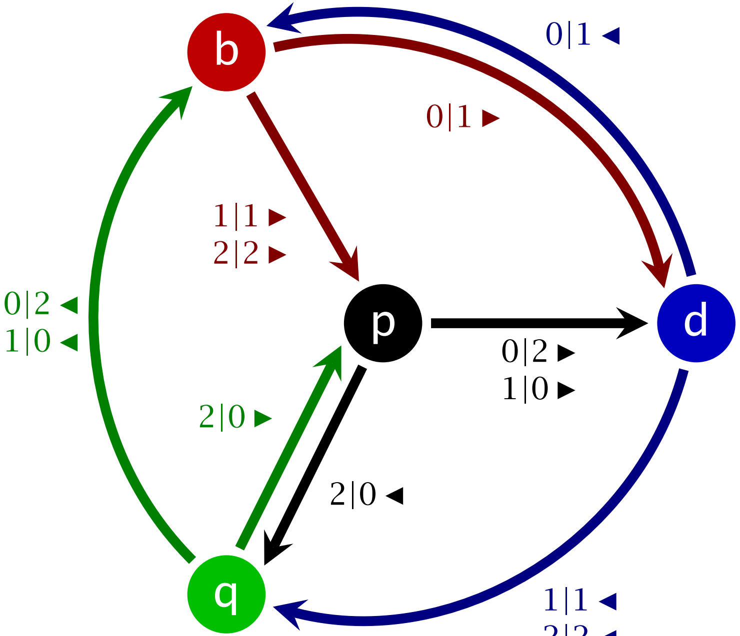
Naive dynamics

A **topological dynamical system** is a pair (X, T) where the topological space X is the **phase space** and the continuous function $T : X \to X$ is the **global transition function**.

The **orbit** of $x \in X$ is $\mathcal{O}(x) = (T^n(x))_{n\in\mathbb{N}}$.

Using the **product topology** one obtains a **topological dynamical system** (X, T) for a TM where the phase space is $X = Q \times \Sigma^{\mathbb{Z}} \times \mathbb{Z}$ and the transition function T is continuous.

Unfortunately, X is not **compact**, we follow Kůrka's alternative compact dynamical models TMH and TMT.

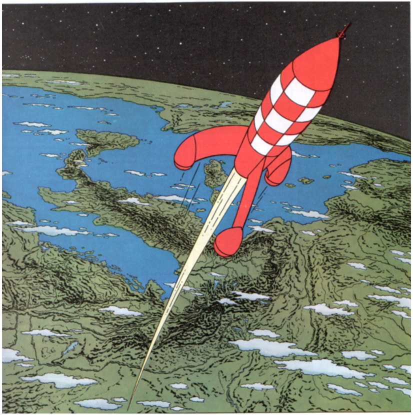

b
d
p
q
0|1 ◀
0|1 ▶
1|1 ▶
2|2 ▶
0|2 ◀
1|0 ◀
0|2 ▶
1|0 ▶
2|0 ▶
2|0 ◀
1|1 ◀
2|2 ◀

Moving head dynamics (TMH)

$$X_h \subset (Q \cup \Sigma)^{\mathbb{Z}}$$
$$T_h : X_h \to X_h$$

$\cdots$ 000000**b**000000000 $\cdots$
$\cdots$ 0000001**d**00000000 $\cdots$
$\cdots$ 000000**b**110000000 $\cdots$
$\cdots$ 0000001**p**10000000 $\cdots$
$\cdots$ 00000010**d**0000000 $\cdots$
$\cdots$ 0000001**b**01000000 $\cdots$
$\cdots$ 00000011**d**1000000 $\cdots$
$\cdots$ 0000001**q**11000000 $\cdots$
$\cdots$ 000000**b**101000000 $\cdots$
$\cdots$ 0000001**p**01000000 $\cdots$
$\vdots$

Hergé. *On a marché sur la lune*. Casterman, 1954.



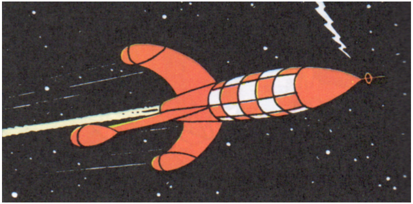
Long shot

Moving tape dynamics (TMT)

$$\begin{aligned} X_t &= {}^{\omega}\Sigma \times Q \times \Sigma^{\omega} \\ T_t &: X_t \to X_t \end{aligned}$$

$\cdots 0000000\mathbf{b}00000000 \cdots$
$\cdots 0000001\mathbf{d}00000000 \cdots$
$\cdots 0000000\mathbf{b}11000000 \cdots$
$\cdots 0000001\mathbf{p}10000000 \cdots$
$\cdots 0000010\mathbf{d}00000000 \cdots$
$\cdots 0000001\mathbf{b}01000000 \cdots$
$\cdots 0000011\mathbf{d}10000000 \cdots$
$\cdots 0000001\mathbf{q}11000000 \cdots$
$\cdots 0000000\mathbf{b}10100000 \cdots$
$\cdots 0000001\mathbf{p}01000000 \cdots$
$\vdots$

Hergé. *On a marché sur la lune*. Casterman, 1954.



Tracking shot

Trace-shift (ST)

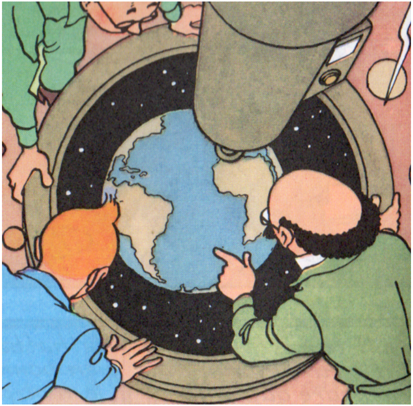
$$S_T \subseteq (Q \times \Sigma)^\omega$$

$$\sigma : S_T \to S_T$$

The column shift of TMT

0 0 1 1 0 0 1 1 1 0
b d b p d b d q b p . . .

Hergé. *On a marché sur la lune*. Casterman, 1954.



Point of view shot

Topological transitivity

Definition A dynamical system (X, T) is **transitive** is it admits a **transitive point** x such that $\overline{\mathcal{O}(x)} = X$.

Proposition (X, T) is **transitive** iff for every pair of open sets $U, V \subseteq X$, there exists t such that $T^t(U) \cap V \neq \emptyset$.

TMH $\forall u,v,u',v' \; \exists w,z,w',z',n \;\; T_h^n(wu.vz) = w'u'.v'z'$

TMT $\forall u,v,\alpha,u',v',\beta \; \exists w,z,w',z',n \; T_t^n(wu, \alpha, vz) = (w'u', \beta, v'z')$

ST $\forall u,v \in S_T \;\; \exists w \in S_T \quad uwv \in S_T$

TMH transitive $\Longrightarrow$ TMT transitive $\Longrightarrow$ ST transitive.

TMH transitive implies **complete**, **reversible** and **aperiodic**

Transitivities

Definition A point $x \in X$ is **periodic** if it admits a **period** $p > 0$ such that $T^p(x) = x$.

Proposition A TM with a **periodic** point is **not ST transitive**.

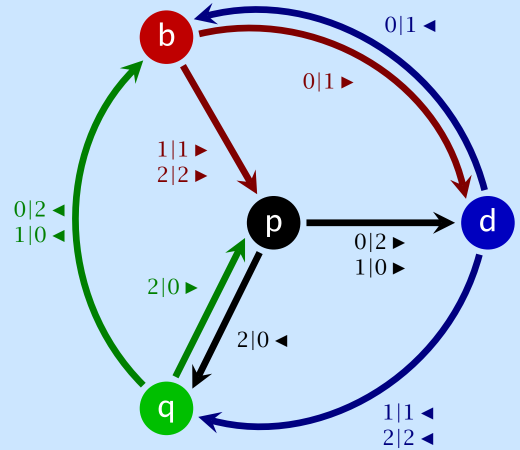
The single-state **shift** TM is **TMT transitive** but **not TMH**.

$$\delta(q, x) = (q, x, \blacktriangleright)$$

The single-state **eraser** TM is **ST transitive** but **not TMT**.

$$\delta(q, x) = (q, 0, \blacktriangleright)$$

Question Can you construct a TMH transitive TM?



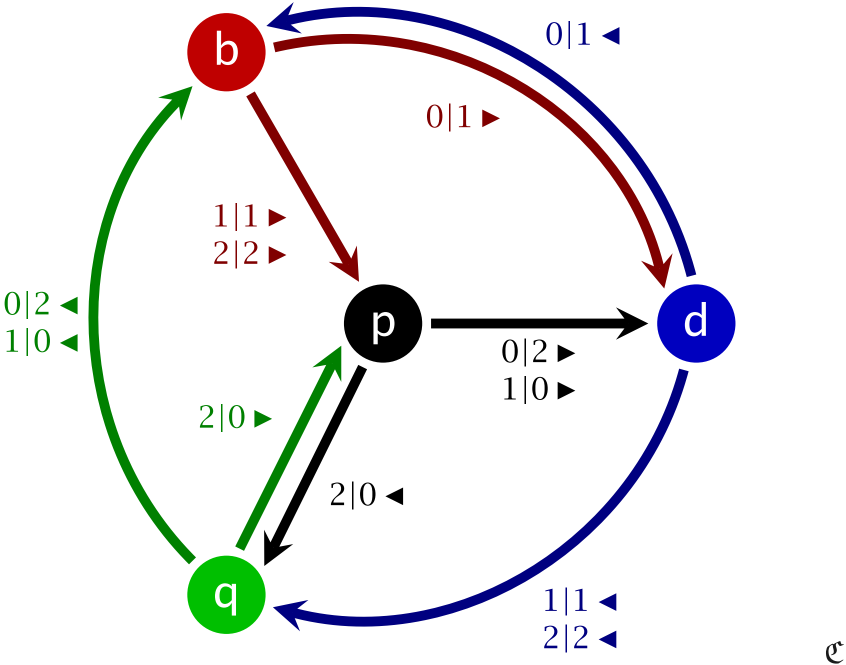
2. a SMART machine

A Small Minimal Aperiodic Reversible Turing machine

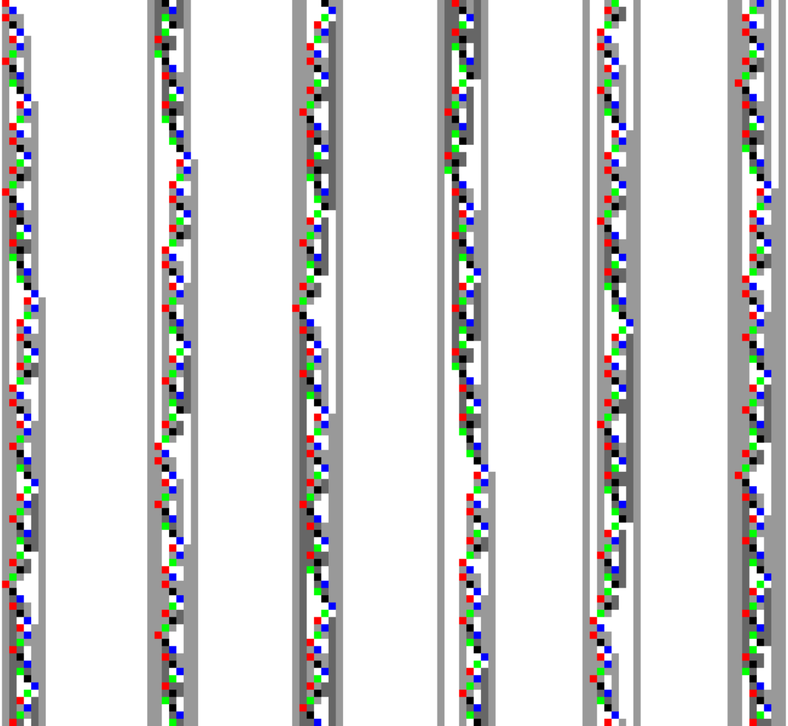
(hal-00975244)

Julien Cassaigne (IML, CNRS, Marseille, France)
Nicolas Ollinger (LIFO, Univ. Orléans, France)
Rodrigo Torres (CMM, Univ. de Chile, Concepción, Chile)

Journées SDA2+Frac — April 9th, 2014



b
p
d
q
0|1 ◀
0|1 ▶
1|1 ▶
2|2 ▶
0|2 ◀
1|0 ◀
0|2 ▶
1|0 ▶
2|0 ▶
2|0 ◀
1|1 ◀
2|2 ◀



The SMART machine $\mathfrak{C}$

A 4-state 3-symbols TM with nice properties:

complete no halting configuration

reversible reversed by a TM...

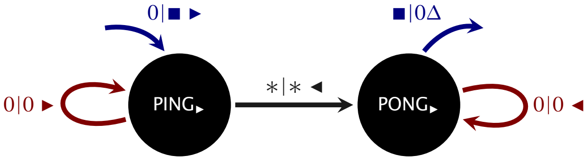
time-symetric ...essentially itself (up to details)

aperiodic no time periodic orbit

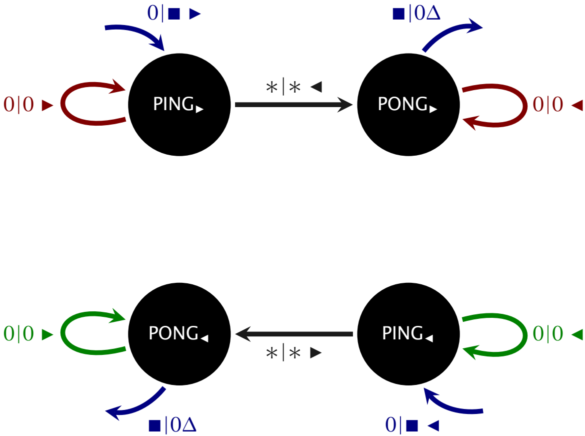
substitutive substitution-generated trace-shift language

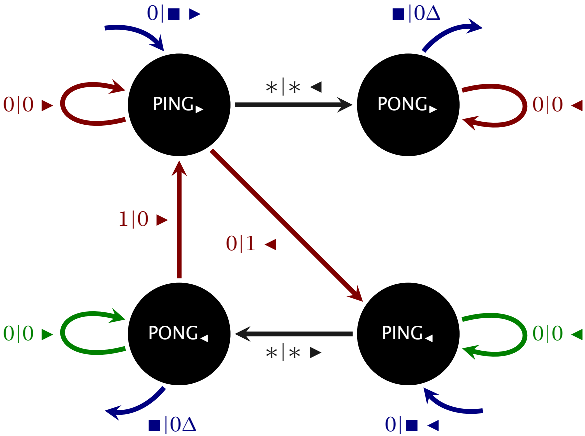
TMT-minimal every orbit is dense with moving tape

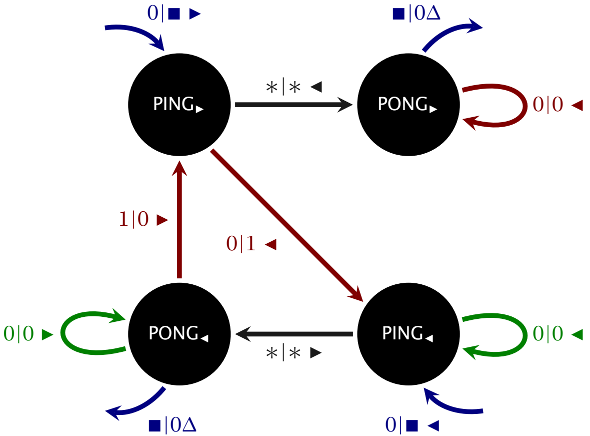
How does it work?

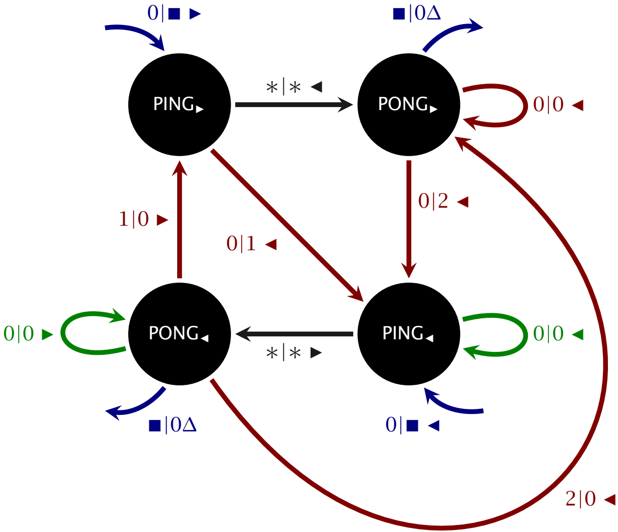


0|■ ▶
■|0Δ
0|0 ▶
PING▸
∗|∗ ◀
PONG▸
0|0 ◀

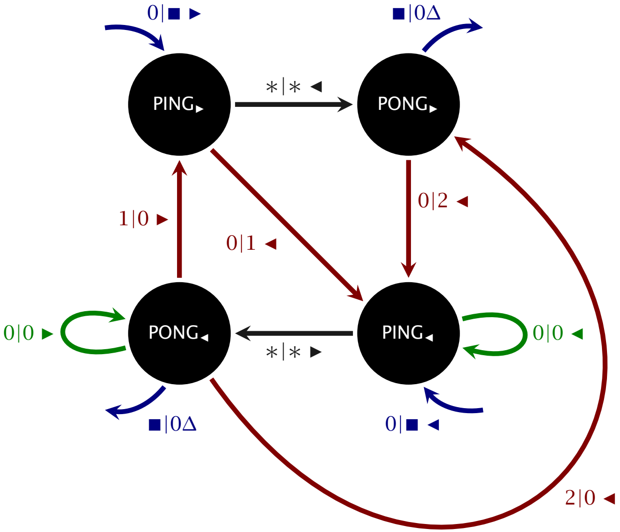

0|■ ▶
■|0Δ
0|0 ▶
PING▶
∗|∗ ◀
PONG▶
0|0 ◀
0|0 ▶
PONG◀
PING◀
0|0 ◀
∗|∗ ▶
■|0Δ
0|■ ◀

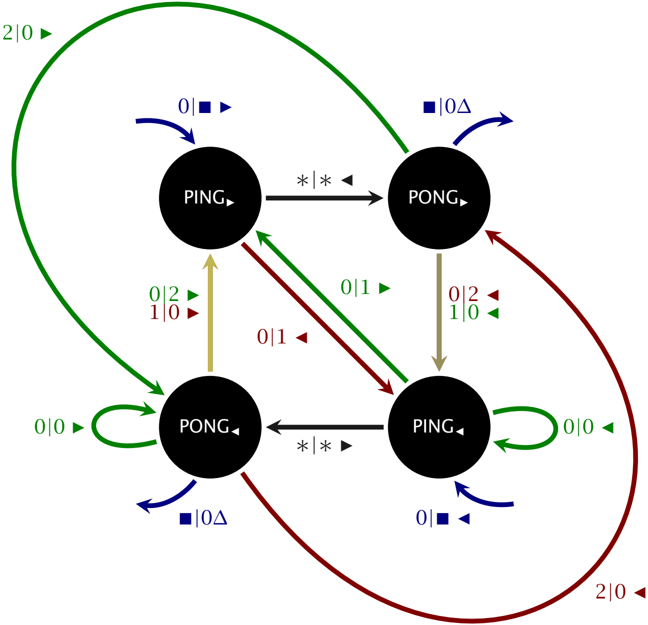

0|■ ▶
■|0Δ
0|0 ▶
PING▸
∗|∗ ◀
PONG▸
0|0 ◀
1|0 ▶
0|1 ◀
0|0 ▶
PONG◂
PING◂
0|0 ◀
∗|∗ ▶
■|0Δ
0|■ ◀


0|■ ▶
■|0Δ
PING▸
| ◀
PONG▸
0|0 ◀
1|0 ▶
0|1 ◀
0|0 ▶
PONG◂
PING◂
0|0 ◀
| ▶
■|0Δ
0|■ ◀

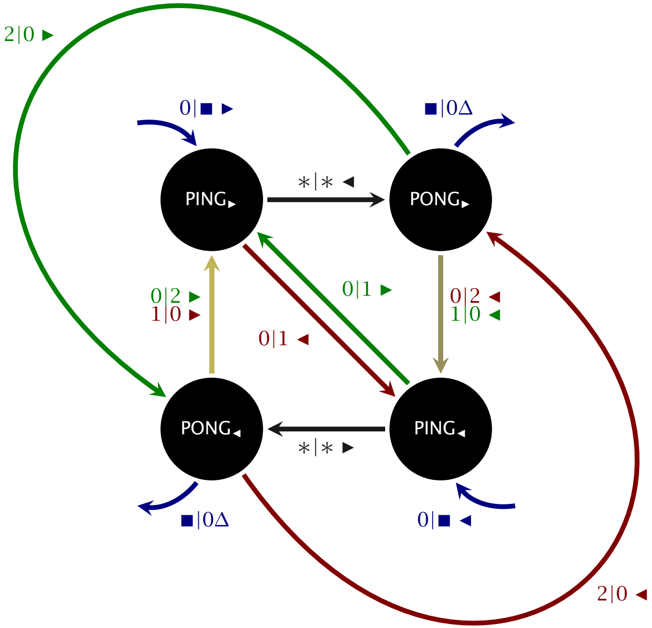


0|■ ▶
■|0Δ
PING▶
| ◀
PONG▶
0|0 ◀
0|2 ◀
1|0 ▶
0|1 ◀
0|0 ▶
PONG◀
PING◀
0|0 ◀
| ▶
■|0Δ
0|■ ◀
2|0 ◀

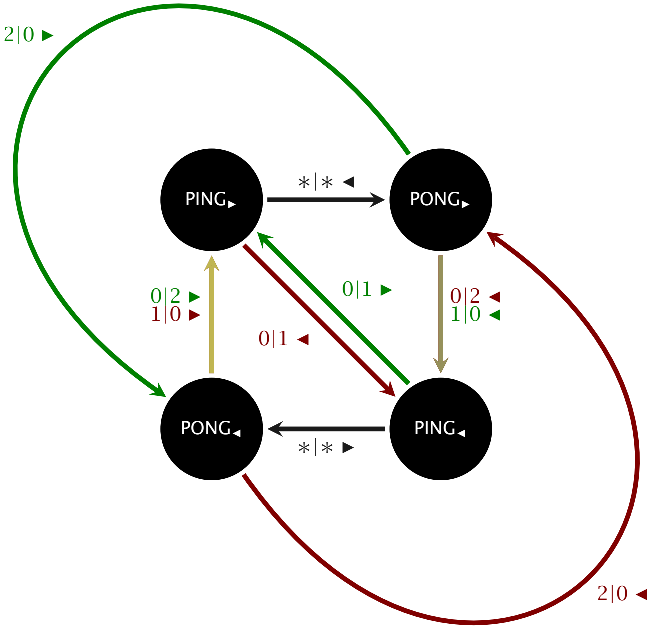

0|■ ▶
■|0Δ
∗|∗ ◀
PING▶
PONG▶
1|0 ▶
0|1 ◀
0|2 ◀
0|0 ▶
PONG◀
PING◀
0|0 ◀
∗|∗ ▶
■|0Δ
0|■ ◀
2|0 ◀



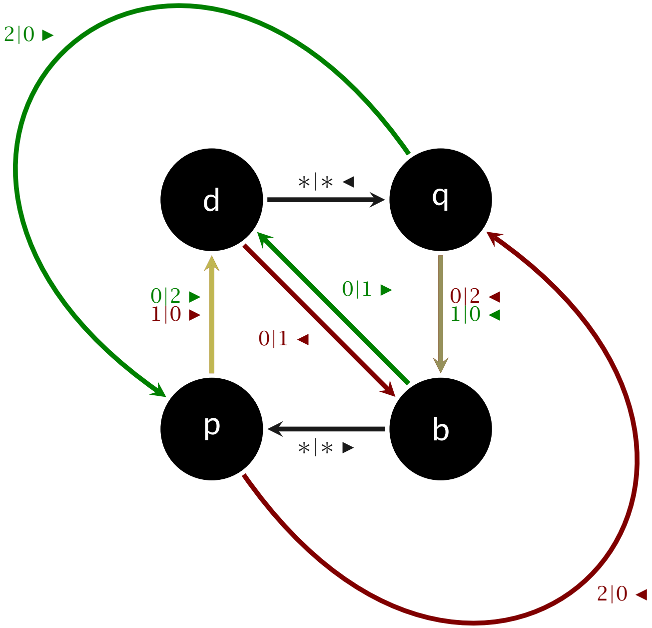
2|0 ▶
0|■ ▶
■|0Δ
PING▸
| ◀
PONG▸
0|1 ▶
0|2 ▶
1|0 ▶
0|2 ◀
1|0 ◀
0|1 ◀
0|0 ▶
PONG◂
PING◂
0|0 ◀
| ▶
■|0Δ
0|■ ◀
2|0 ◀



2|0 ▶
0|■ ▶
■|0Δ
PING▸
∗|∗ ◀
PONG▸
0|1 ▶
0|2 ▶
1|0 ▶
0|1 ◀
0|2 ◀
1|0 ◀
PONG◂
PING◂
∗|∗ ▶
■|0Δ
0|■ ◀
2|0 ◀



2|0 ▶
PING▸
| ◀
PONG▸
0|2 ▶
1|0 ▶
0|1 ▶
0|2 ◀
1|0 ◀
0|1 ◀
PONG◂
PING◂
| ▶
2|0 ◀



2|0 ▶
d
∗|∗ ◀
q
0|2 ▶
1|0 ▶
0|1 ▶
0|1 ◀
0|2 ◀
1|0 ◀
p
b
∗|∗ ▶
2|0 ◀

Recursive behavior

PING$_\blacktriangleright(n)$:
    for i=1 to n:
        d. 0|1, b ◀
        PING$_\blacktriangleleft(i-1)$
    d. $x|x$, q ◀
    for i=n downto 1:
        q. 0|2, b ◀
        PING$_\blacktriangleleft(i-1)$
    q. $y|0$, $\alpha(y)$ $\tau(y)$

PING$_\blacktriangleleft(n)$:
    for i=1 to n:
        b. 0|1, d ▶
        PING$_\blacktriangleright(i-1)$
    b. $x|x$, p ▶
    for i=n downto 1:
        p. 0|2, d ▶
        PING$_\blacktriangleright(i-1)$
    p. $y|0$, $\alpha'(y)$ $\tau'(y)$

$$\begin{cases} f(0) & = 2 \\ f(n+1) & = 3f(n) \end{cases}$$

Substitutive trace subshift

$$\varphi\begin{pmatrix}0\\ \mathbf{b}\end{pmatrix} = \begin{matrix}0&0&1&1\\ \mathbf{b}&\mathbf{d}&\mathbf{b}&\mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}0\\ \mathbf{d}\end{pmatrix} = \begin{matrix}0&0&1&1\\ \mathbf{d}&\mathbf{b}&\mathbf{d}&\mathbf{q}\end{matrix}$$

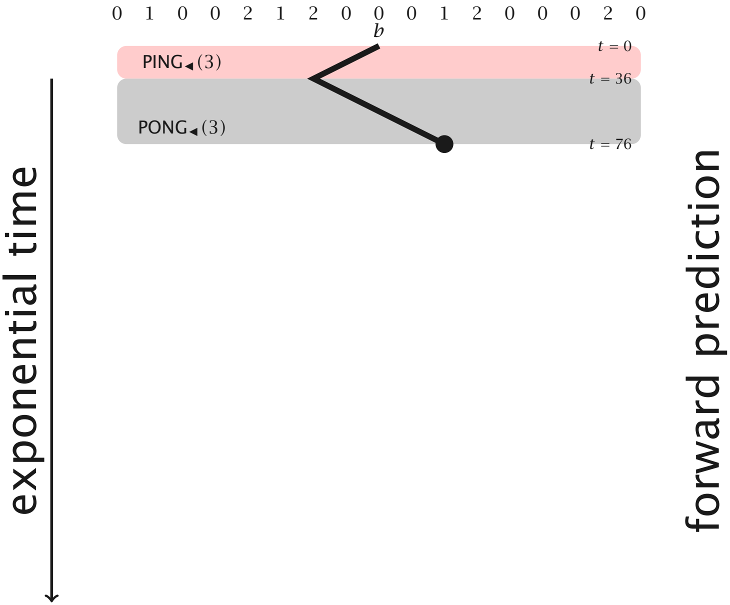
$$\varphi\begin{pmatrix}\mathrm{x}\\ \mathbf{b}\end{pmatrix} = \begin{matrix}\mathrm{x}\\ \mathbf{b}\end{matrix} \qquad \varphi\begin{pmatrix}\mathrm{x}\\ \mathbf{d}\end{pmatrix} = \begin{matrix}\mathrm{x}\\ \mathbf{d}\end{matrix}$$

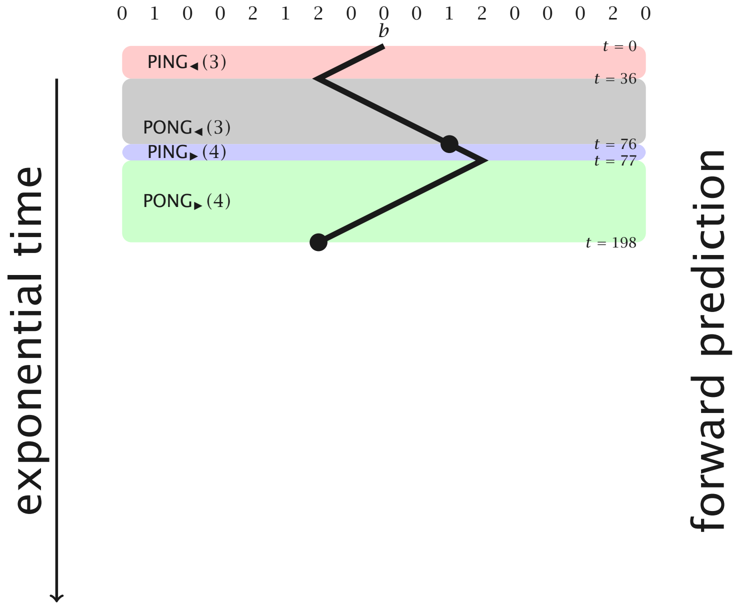
$$\varphi\begin{pmatrix}0\\ \mathbf{p}\end{pmatrix} = \begin{matrix}0&0&2&1\\ \mathbf{p}&\mathbf{d}&\mathbf{b}&\mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}0\\ \mathbf{q}\end{pmatrix} = \begin{matrix}0&0&2&1\\ \mathbf{q}&\mathbf{b}&\mathbf{d}&\mathbf{q}\end{matrix}$$

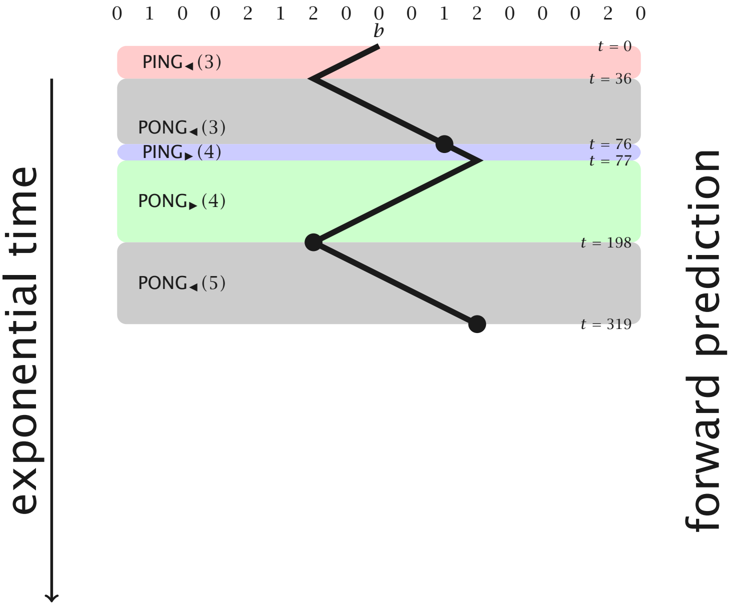
$$\varphi\begin{pmatrix}\mathrm{x}\\ \mathbf{p}\end{pmatrix} = \begin{matrix}0&\mathrm{x}&2&\mathrm{x}\\ \mathbf{p}&\mathbf{d}&\mathbf{q}&\mathbf{p}\end{matrix} \qquad \varphi\begin{pmatrix}\mathrm{x}\\ \mathbf{q}\end{pmatrix} = \begin{matrix}0&\mathrm{x}&2&\mathrm{x}\\ \mathbf{q}&\mathbf{b}&\mathbf{p}&\mathbf{q}\end{matrix}$$

0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0

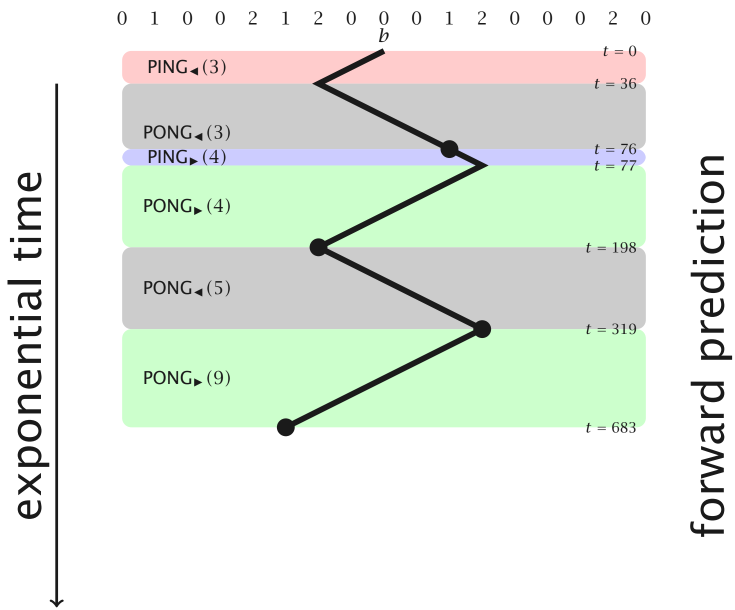
b


0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
exponential time
forward prediction

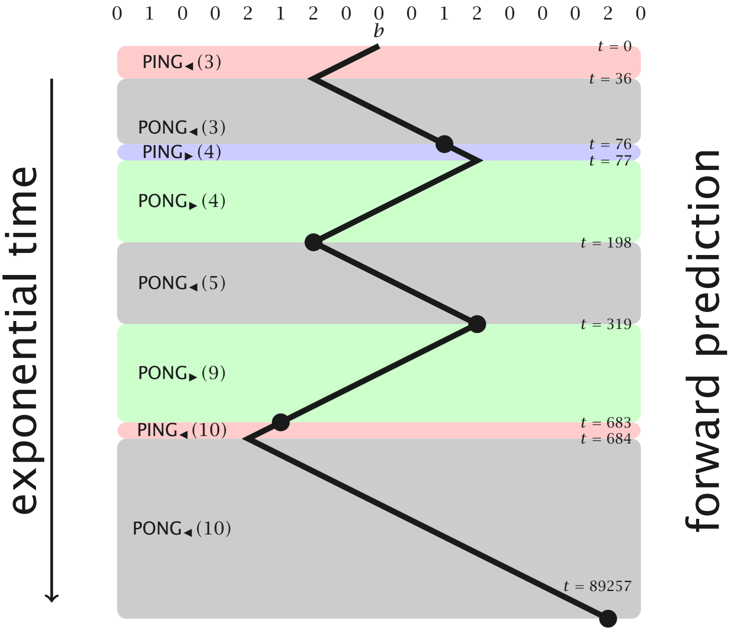

0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
exponential time
forward prediction

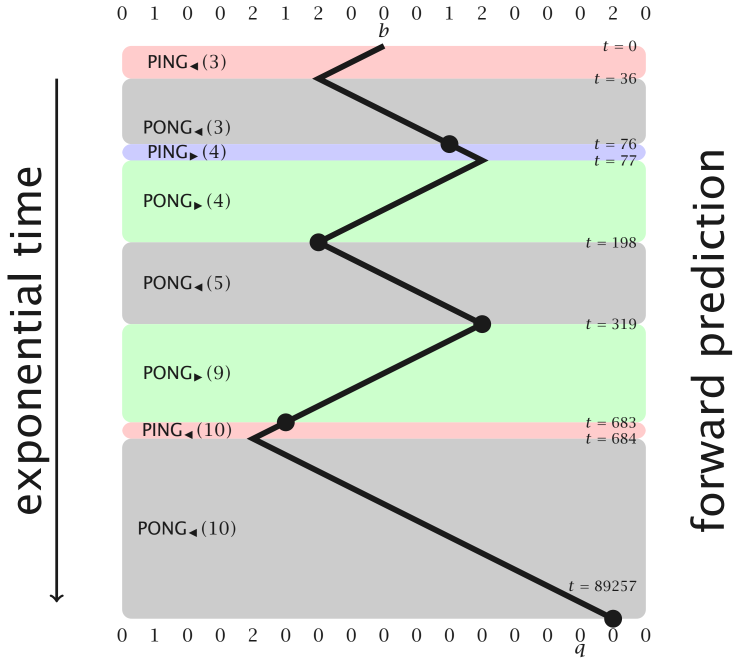


0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
PONG◀(5)
t = 319
exponential time
forward prediction

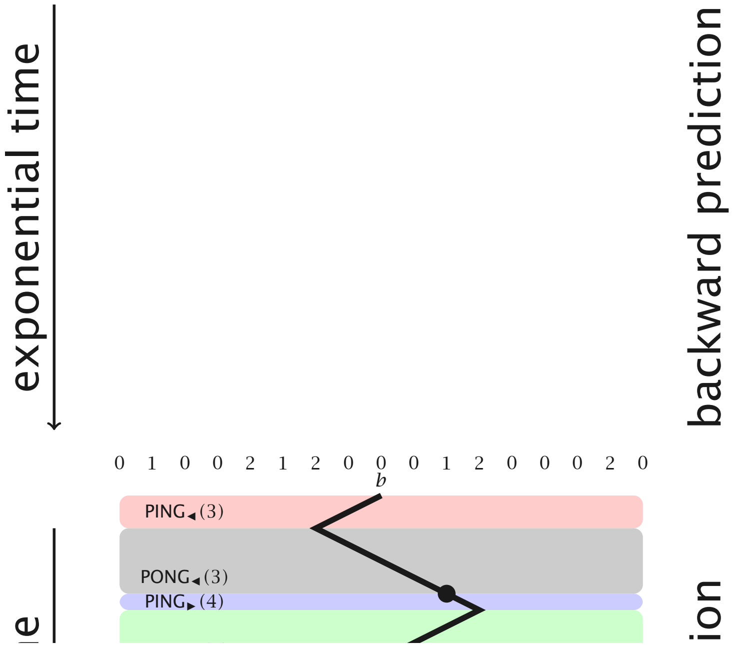


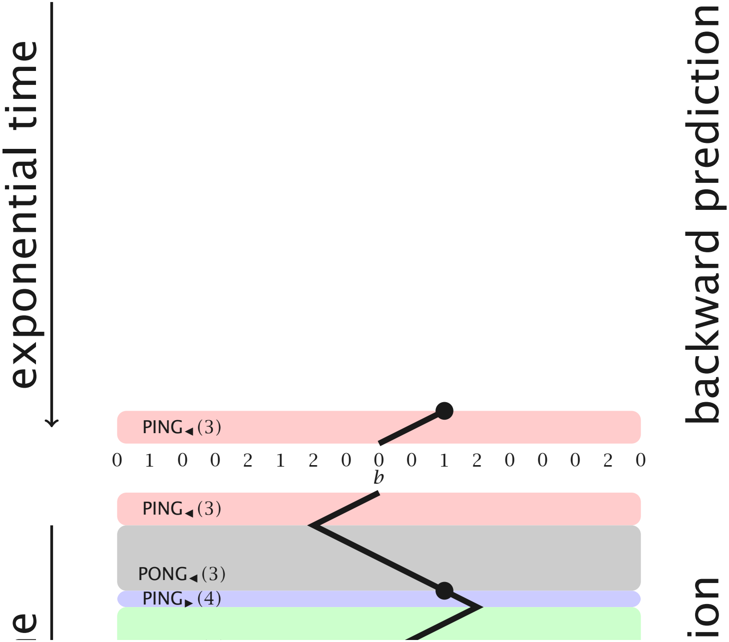
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
PONG◀(5)
t = 319
PONG▶(9)
t = 683
exponential time
forward prediction


0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
exponential time
forward prediction
PING◂(3)
PONG◂(3)
PING▸(4)
PONG▸(4)
PONG◂(5)
PONG▸(9)
PING◂(10)
PONG◂(10)
t = 0
t = 36
t = 76
t = 77
t = 198
t = 319
t = 683
t = 684
t = 89257

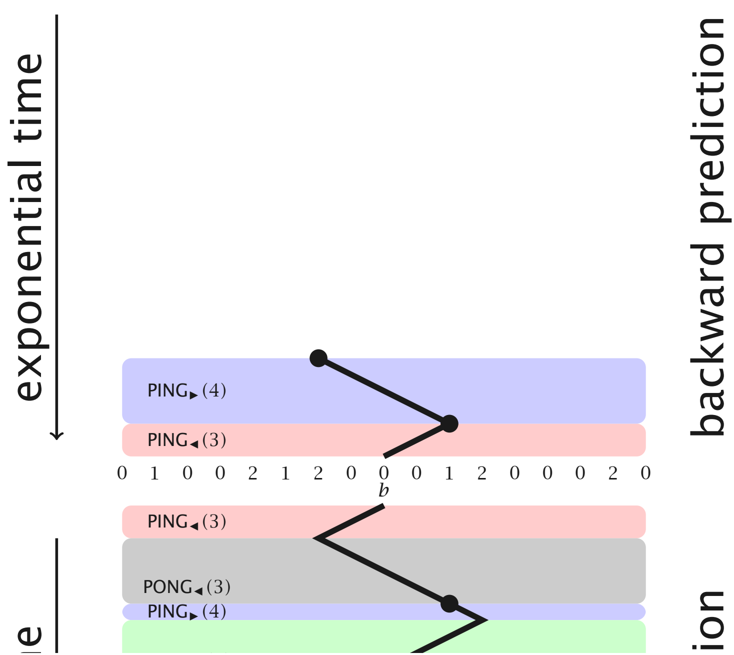


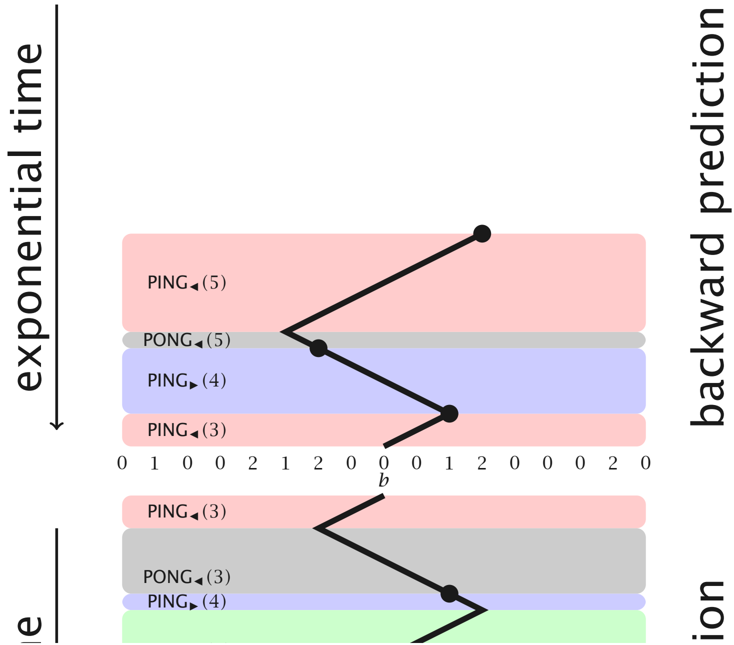
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
t = 0
PING◀(3)
t = 36
PONG◀(3)
t = 76
PING▶(4)
t = 77
PONG▶(4)
t = 198
PONG◀(5)
t = 319
PONG▶(9)
t = 683
PING◀(10)
t = 684
PONG◀(10)
t = 89257
0 1 0 0 2 0 0 0 0 0 0 0 0 0 0 0 0
q
exponential time
forward prediction

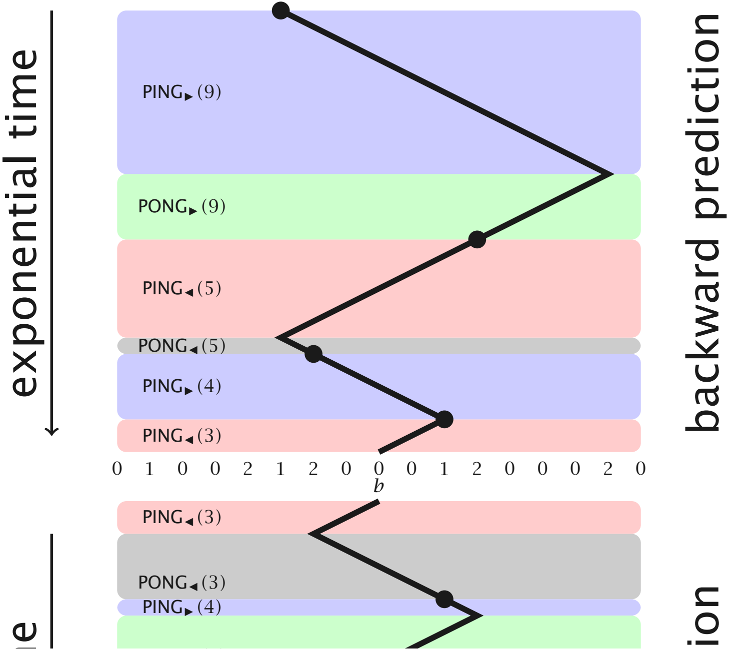

exponential time
backward prediction
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◀(3)
PONG◀(3)
PING▶(4)

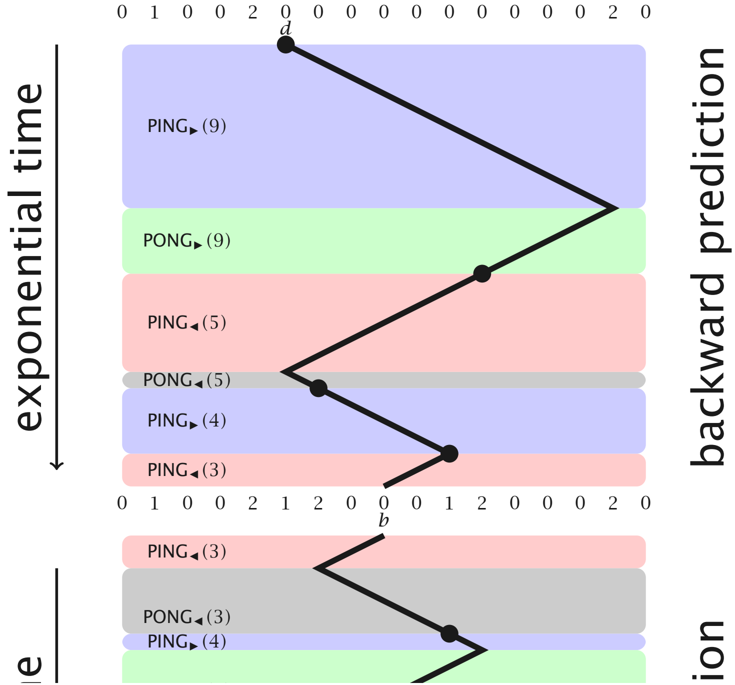


exponential time
backward prediction
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)


exponential time
backward prediction
PING▶(4)
PING◀(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◀(3)
PONG◀(3)
PING▶(4)


exponential time
backward prediction
PING◂(5)
PONG◂(5)
PING▸(4)
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)


exponential time
backward prediction
PING▸(9)
PONG▸(9)
PING◂(5)
PONG◂(5)
PING▸(4)
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)


0 1 0 0 2 0 0 0 0 0 0 0 0 0 0 2 0
d
exponential time
backward prediction
PING▸(9)
PONG▸(9)
PING◂(5)
PONG◂(5)
PING▸(4)
PING◂(3)
0 1 0 0 2 1 2 0 0 0 1 2 0 0 0 2 0
b
PING◂(3)
PONG◂(3)
PING▸(4)
ion

SMART is transitive in TMH, TMT and ST

Proposition $\left({}^{\omega}2 \cdot \underset{p}{2}\ 2^{\omega} \right)$ is a **transitive point**.

Proof

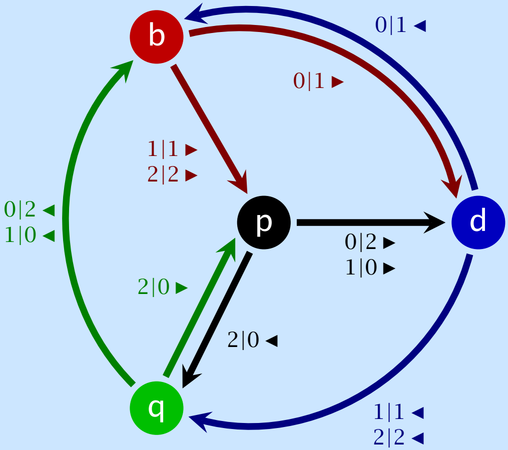
(Forward) For all $k \geqslant 0$:

$$\left({}^{\omega}2 \cdot \underset{p}{2}\ 2^{\omega} \right) \vdash^* \left({}^{\omega}2\ \underset{q}{2}\ 0^k \cdot 0\ 0^k\ 2^{\omega} \right) \quad .$$

(Backward) For every partial configuration $(\underset{\leftarrow}{u}\ \underset{\alpha}{\cdot}\ \underset{\rightarrow}{v})$, there exist $w, w' \in \{0,1,2\}^*$ and $k > 0$ big enough such that

$$\left({}^{\omega}2\ \underset{q}{2}\ 0^k \cdot 0\ 0^k\ 2^{\omega} \right) \vdash^* \left({}^{\omega}2\ w\ \underset{\leftarrow}{u}\ \underset{\alpha}{\cdot}\ \underset{\rightarrow}{v}\ w'\ 2^{\omega} \right) \quad .$$

∎



3. The complexity of transitivity

Reversing time

Combine Turing machines to construct bigger ones.

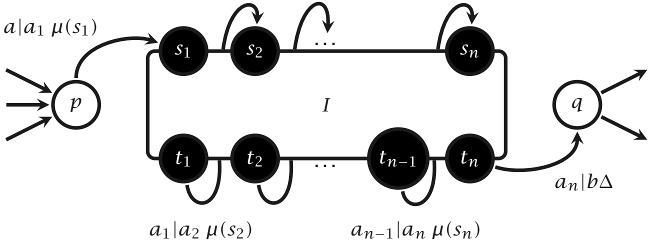
Reversing the time Given a reversible TM $M = (Q, \Sigma, \delta)$, construct $M_+ = (Q \times \{+\}, \Sigma, \delta^+)$ and $M_- = (Q \times \{-\}, \Sigma, \delta^-)$ where $(s, +)$ encodes M in state s running **forward** and $(s, -)$ running **backward**.

A typical use connects halting pairs from one machine to the corresponding starting pair of the other.

Embedding technique

A TM I with starting pairs $(s_1, a_1), \ldots, (s_n, a_n)$ and halting pairs $(t_1, a_1), \ldots, (t_n, a_n)$ is **innocuous** if starting from $(s_i, c, p + \mu(s_i))$ where $c(p) = a_i$ the machine might only halt in (t_i, c, p).

The **embedding** H^I of an **invited** innocuous TM I inside a **host** TM H is the TM containing a copy of both I and H where one transition $\delta(p, a) = (q, b, \Delta)$ from H is replaced by



Undecidability of transitivity

BRA Reachability Problem[Σ_1^0-comp. too] Given a binary reversible aperiodic TM, a starting pair (s, a) and a halting pair (t, b), decide if (t, b) is reachable from (s, a).

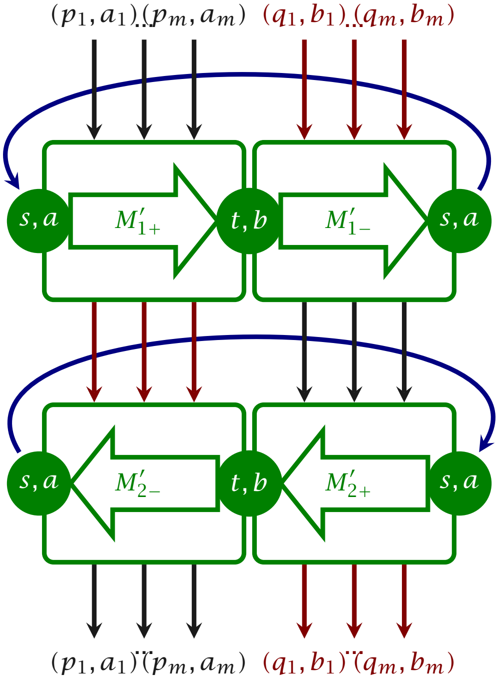
Theorem $\overline{\text{BRA Reachability Problem}} \leqslant_m$ Transitivity Problem

Proof

Let M, (s, a), (t, b) be an instance of the BRA Reachability Problem and M' be a copy of M with a third symbol $\$$.

Apply *Reversing time* to 2 copies of M' to construct an innocuous TM I as follows.

SMART^I is transitive iff (t, b) is not reachable from (s, a). ∎



(p_1, a_1) ... (p_m, a_m)
(q_1, b_1) ... (q_m, b_m)
s, a
M'_{1+}
t, b
M'_{1-}
s, a
s, a
M'_{2-}
t, b
M'_{2+}
s, a
(p_1, a_1) ... (p_m, a_m)
(q_1, b_1) ... (q_m, b_m)

Conclusion

Theorem **Transitivity** is Π_1^0-hard in TMH, TMT and ST.

Theorem **Minimality** is Σ_1^0-hard in TMT and ST.

What is the exact complexity of both these properties?

Is there some kind of Rice theorem for dynamical properties?

Table of contents