# P vs. NP

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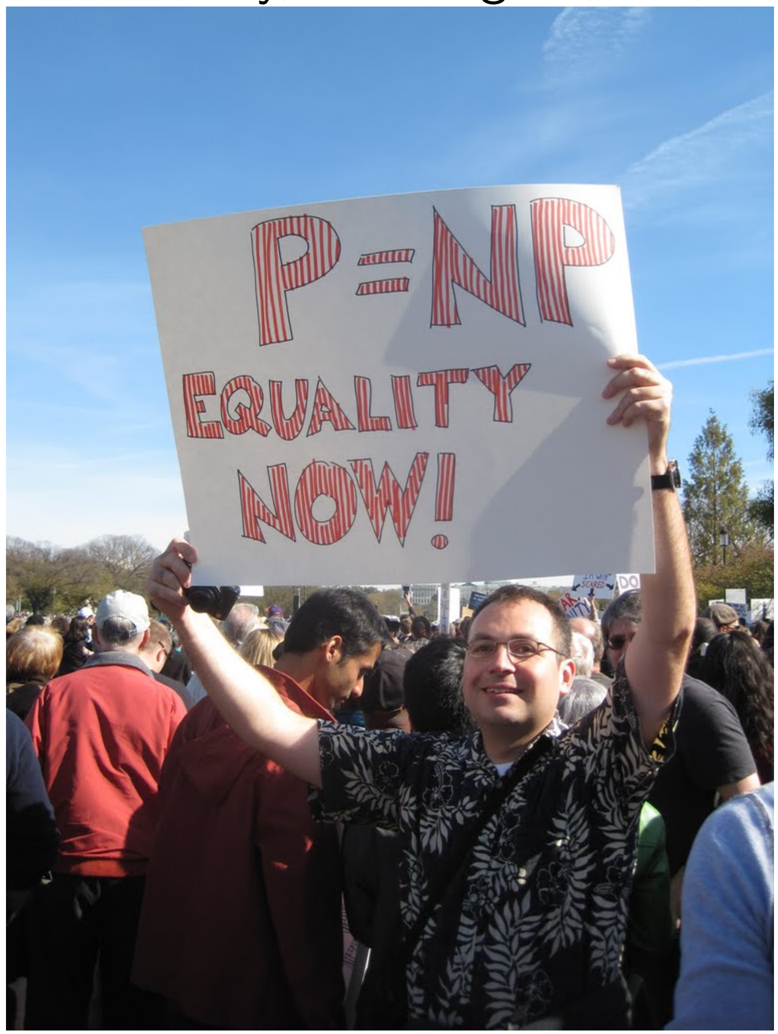
# An Outstanding Open Problem

Does $P = NP$?

- *The* major open problem in computer science
- A major open problem in mathematics
  - A Clay Institute Millennium Problem
  - Million dollar prize!
- On August 6, 2010, Vinay Deolalikar announced a proof (100-page manuscript) that $P \neq NP$.

What is this about? It is about computational complexity – how hard it is to solve computational problems.

Rally To Restore Sanity, Washington, DC, October 2010



# Computational Problems

**Example**: *Graph* – $G = (V, E)$

- $V$ – set of nodes
- $E$ – set of edges

**Two notions**:

- Hamiltonian Cycle: a cycle that visits every *node* exactly once.
- Eulerian Cycle: a cycle that visits every *edge* exactly once.

**Question**: How hard it is to find a Hamiltonian cycle? Eulerian cycle?

Figure 1: The Bridges of Königsberg

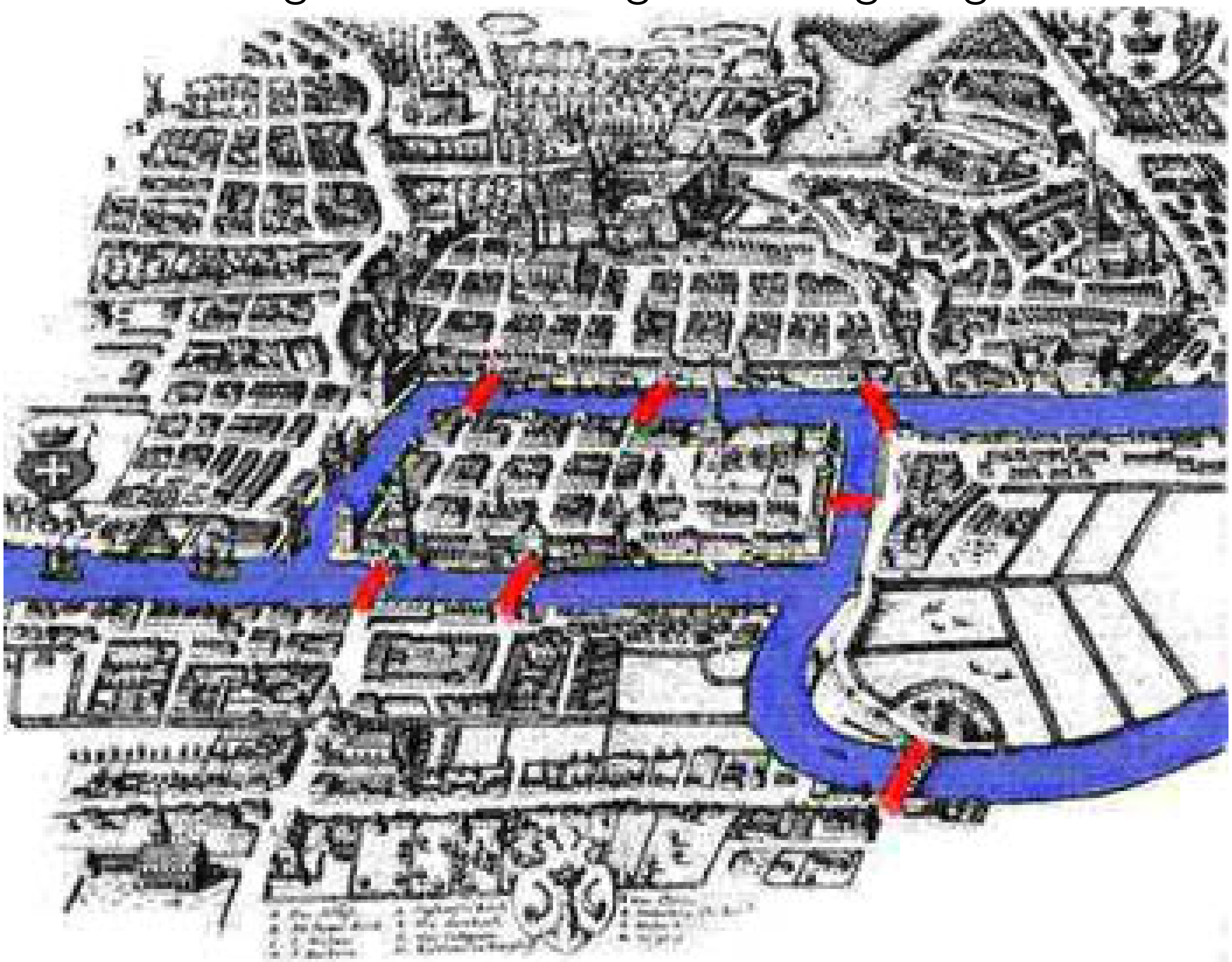


Figure 2: The Graph of The Bridges of Königsburg

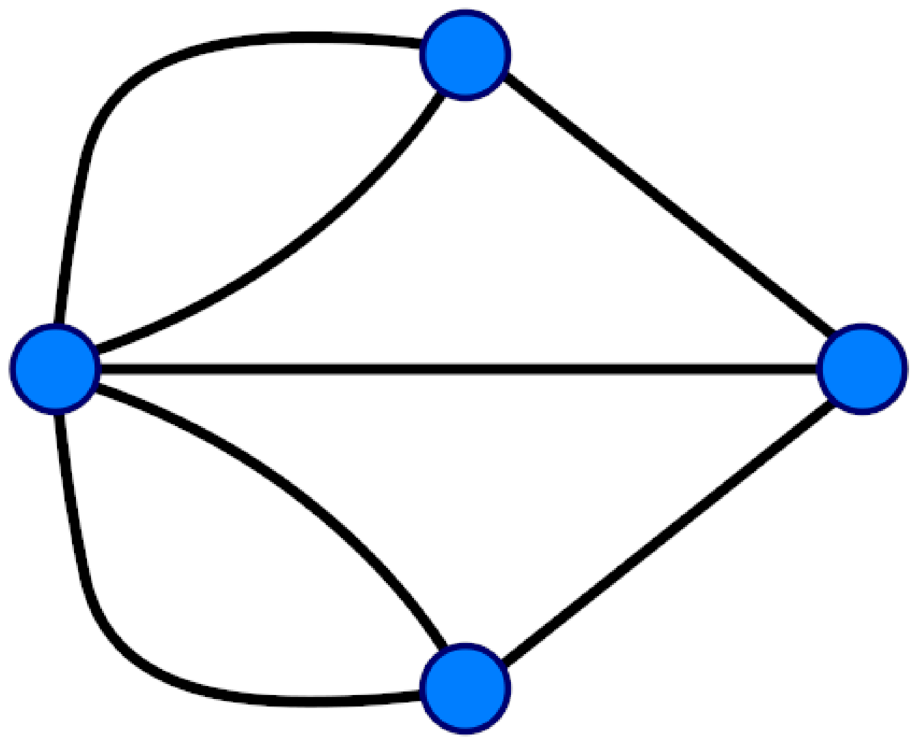
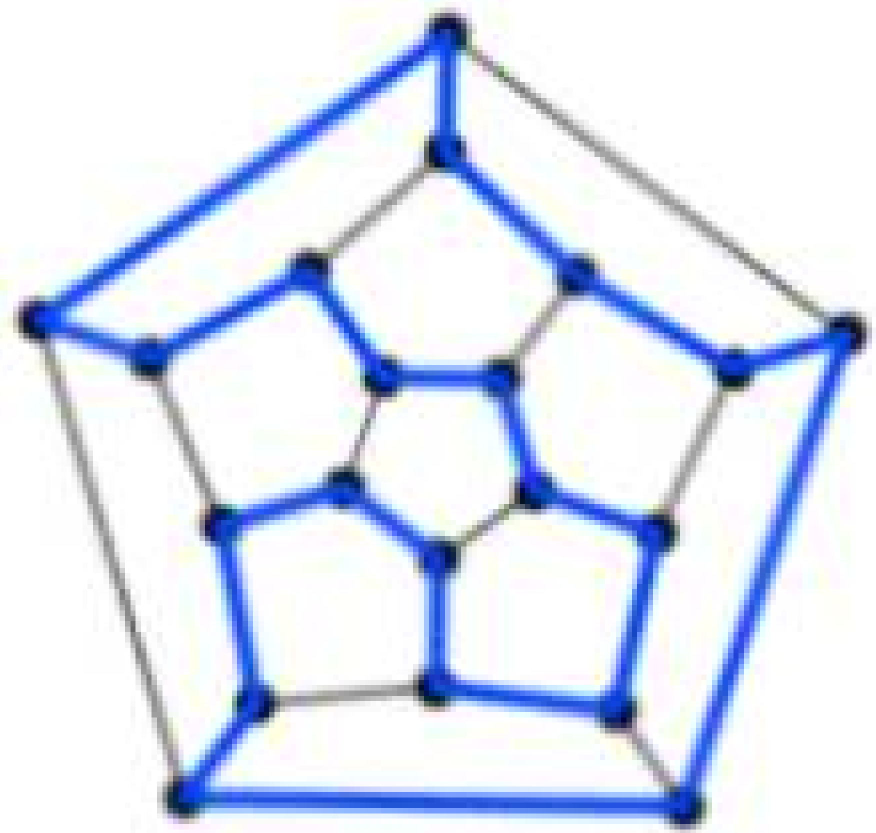


Figure 3: Hamiltonian Cycle



# Computational Complexity

**Measuring complexity**: How many (Turing machine) operations does it take to solve a problem of size $n$?

- *Size* of $(V, E)$: number of nodes plus number of edges.

**Complexity Class** $P$: problems that can be solved in *polynomial time* – $n^c$ for a *fixed* $c$

**Examples**:

- Is a number even?
- Is a number square?
- Does a graph have an Eulerian cycle?

*What about the Hamiltonian Cycle Problem?*

# Hamiltonian Cycle

- **Naive Algorithm**: Exhaustive search – run time is $n!$ operations
- **“Smart” Algorithm**: Dynamic programming – run time is $2^n$ operations

  **Note**: The universe is much younger than $2^{200}$ Planck time units!

**Fundamental Question**: Can we do better?

- Is HamiltonianCycle in $P$?

# Checking Is Easy!

**Observation**: Checking if a *given* cycle is a Hamiltonian cycle of a graph $G = (V, E)$ is *easy*!

**Complexity Class** $NP$: problems where solutions can be *checked* in polynomial time.

**Examples**:

- HamiltonianCycle
- Factoring numbers

**Significance**: Tens of thousands of optimization problems are in NP!!!

- CAD, flight scheduling, chip layout, protein folding, ...

# P vs. NP

- $P$: efficient *discovery* of solutions
- $NP$: efficient *checking* of solutions

**The Big Question**: Is $P = NP$ or $P \neq NP$?

- Is *checking* really easier than *discovering*?

**Intuitive Answer**: Of course, *checking* is easier than *discovering*, so $P \neq NP$!!!

- **Metaphor**: finding a needle in a haystack
- **Metaphor**: Sudoku
- **Metaphor**: mathematical proofs

**Alas**: We do not know how to prove that $P \neq NP$.

$$P \neq NP$$

**Consequences**:

- Cannot solve efficiently numerous important problems
- RSA encryption may be safe.

**Question**: Why is it so important to *prove* $P \neq NP$, if that is what is commonly believed?

**Answer**:

- If we cannot prove it, we do not really understand it.
- May be $P = NP$ and the "enemy" proved it and broke RSA!

$$P = NP$$

S. Aaronson, MIT: "If $P = NP$, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in 'creative leaps,' no fundamental gap between solving a problem and recognizing the solution once its found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss."

**Consequences**:

- Can solve efficiently numerous important problems.
- RSA encryption is not safe.

**Question**: Is it really possible that $P = NP$?

**Answer**: Yes! It'd require discovering a very clever algorithm, but it took 40 years to prove that LinearProgramming is in $P$.

# Sharpening The Problem

$NP$-**Complete Problems**: hardest problems is NP

- HamilatonianCycle is $NP$-complete!

**Corollary**: $P = NP$ if and only if HamiltonianCycle is in $P$

There are *thousands* of $NP$-complete problems. To resolve the $P = NP$ question, it'd suffice to prove that *one* of them is or is not in $P$.

# History

- 1950-60s: Futile effort to show hardness of search problems.
- Stephen Cook, 1971: Boolean Satisfiability is NP-complete.
- Richard Karp, 1972: 20 additional NP-complete problems– 0-1 Integer Programming, Clique, Set Packing, Vertex Cover, Set Covering, Hamiltonian Cycle, Graph Coloring, Exact Cover, Hitting Set, Steiner Tree, Knapsack, Job Scheduling, ...
  - *All* NP-complete problems are polynomially equivalent!
- Leonid Levin, 1973 (independently): Six NP-complete problems
- M. Garey and D. Johnson, 1979: "Computers and Intractability: A Guide to NP-Completeness" - hundreds of NP-complete problems.
- Clay Institute, 2000: $1M Award!

# Terminology

**Terminological Chaos**: The standard terminology did not converge until 1974.

Knuth, 1974, "A terminological Proposal"

- *Competing terms*: arduous, bad, costly, difficult, exorbitant, exparent, formidable, heavy, Herculean, impractical, interminable, intractable, obdurate, perarduous, polychronious, prodigious, Sisyphean, tricky.

- *Winning terms*: NP-hard and NP-complete.

# Logic and Complexity

Richard Lipton, Blog, Aug. 8, 2010:

> "At the highest level he is using the characterization of polynomial time via finite-model theory. His proof uses the beautiful result of Moshe Vardi (1982) and Neil Immerman (1986)."
>
> **Theorem**: On ordered structures, a relation is defined by a first-order formula plus the Least Fixed Point (LFP) operator if and only if it is computable in polynomial time.

**Paper**: "The complexity of relational query languages", 1982 $>$ 1100 citations.

**Terminology**:

- *Relation*: set of tuples of elements, e.g., $<$ is set of pairs
- *Model Theory*: logical theory of mathematical structures – branch of mathematical logic
- *Finite-Model Theory*: logical theory of *finite* mathematical structures – between mathematical logic and computer science

# The Language of Mathematics

G. Frege, Begriffsschrift, 1879: a universal mathematical language – *first-order logic*

- Objects, e.g., numbers
- Predicates (relationships), e.g., $2 < 3$
- Operations (functions), e.g., $2 + 3$
- Boolean operations: "and" ($\wedge$), "or" ($\vee$), "not" ($\neg$), "implies" ($\rightarrow$)
- Quantifiers: "for all" ($\forall x$), "there exists" ($\exists x$)

**Back to Aristotle**:

- "All men are mortal"
- "For all $x$, if $x$ is a man, then $x$ is mortal"
- $(\forall x)(Man(x) \rightarrow Mortal(x))$

# First-Order Logic on Graphs

**Syntax**:

- Variables: $x$, $y$, $z$, . . . (range over nodes)
- Atomic formulas: $E(x, y)$, $x = y$
- Formulas: Atomic Formulas $+$ Boolean Connectives $+$ First-Order Quantifiers

**Examples**:

- $\varphi_1$: "node $x$ has at least two distinct neighbors"

$$(\exists y)(\exists z)(\neg(y = z) \land E(x, y) \land E(x, z))$$

- $\varphi_2$: "nodes $x$ and $y$ are connected by a path of length two":

$$(\exists z)(E(x, z) \land E(z, y))$$

**Formulas as Queries**:

- $\varphi_1$ "computes" the set of nodes with at least two distinct neighbors.
- $\varphi_2$ "computes" the set of pairs of nodes connected by a path of length two.

# Logic and Complexity

**Theorem**: [Immerman-V.]: Polynomial time computability is equivalent to computability by iterating positive first-order queries.

**Significance**:

- Machine-free characterization of $P$
  - **Note**: No Turing machines, no polynomial, no time!
- Normal form for $P$

# Positivity

- *Positive*: $\varphi_2$: "nodes $x$ and $y$ are connected by a path of length two":

$$(\exists z)(E(x,z) \land E(z,y))$$

- *Non-Positive*: $\varphi_3$: "nodes $x$ and $y$ are connected by an incomplete triangle":

$$(\exists z)(E(x,y) \land E(x,z) \land \neg E(y,z))$$

**Significance of Positivity**: Iteration yields an increasing sequence of relations, guaranteeing convergence.

# Example: 2-Colorability

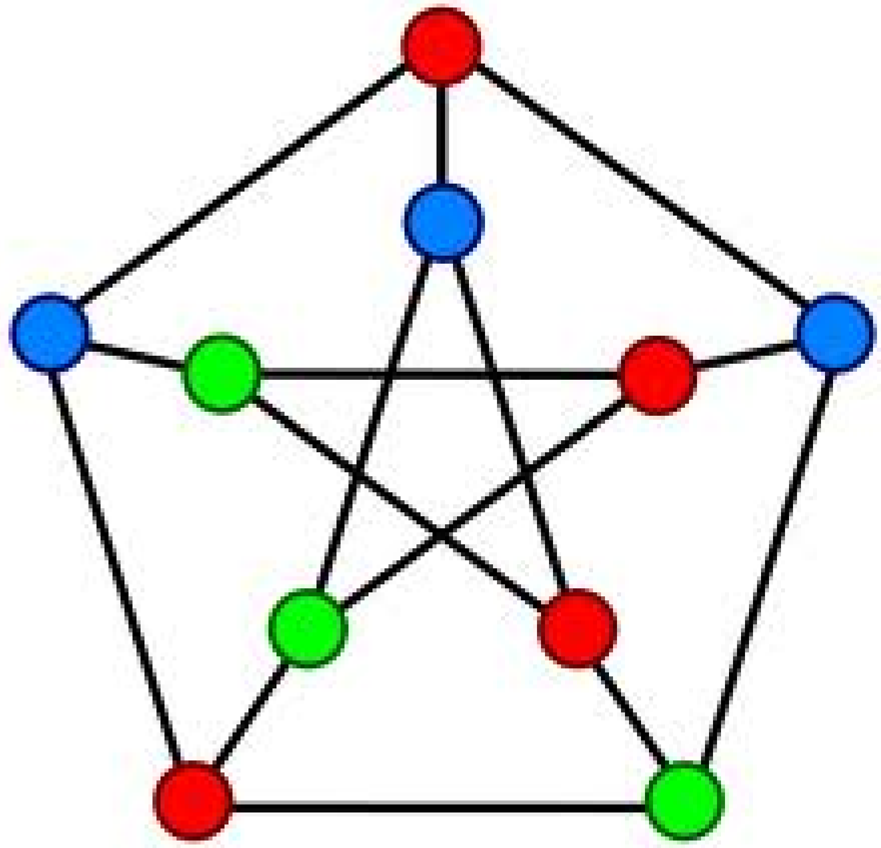
**Graph Coloring**:

- *Graph* – $G = (V, E)$
- $k$-*coloring*: $h : V \rightarrow \{1, \ldots, k\}$
- *Nonmonocromacity*: $h(u) \neq h(v))$ for all $(u, v) \in E$
- $k$-*Colorability*: Does $G$ have $k$-coloring?

**Complexity**:

- 3-Colorability is NP-complete.
- 2-Colorability is in PTIME.

Figure 4: 3-Coloring



# 2-Colorability

**Fact**: A graph is 2-colorable iff it has no cycle of odd length.

**Example**: Logical characterization of non-2-colorability

$$
\begin{aligned}
O(X,Y) &\leftarrow \quad E(X,Y) \\
O(X,Y) &\leftarrow \quad O(X,Z), E(Z,W), E(W,Y) \\
Not2Colorable &\leftarrow O(X,X)
\end{aligned}
$$

## Another Connection between Logic and Complexity

**Boolean Satisfiability (SAT)**; Given a Boolean expression in the form of “and of ors”, is there a *satisfying* solution (an assignment of 0’s and 1’s to the variables that makes the expression equal 1)?

**Example**:

$$(\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3 \vee x_4) \wedge (x_3 \vee x_1 \vee x_4)$$

**Solution**: $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$

Cook-Levin Theorem: SAT is NP-complete

# A Physics Perspective

- *Literal*: Positive or negative variable – $x_1, \neg x_2$
- *Clause*: Disjunction (or) of literals – $(\neg x_1 \vee x_2 \vee x_3)$

**Energy State**:

- Satisfied clause: $0$
- Unsatisfied clause: $1$
- Total energy: sum of clausal energies=number of *unsatisfied* clauses

**Physics Perspective**: Does expression have a zero-energy state?

- Formula satisfied $\Leftrightarrow$ zero-energy state

# $k$-SAT

$k$-**SAT**:

- Each clause contains precisely $k$ literals.
- $2$-SAT is in $P$.

$$(\neg x_1 \vee x_2) \wedge (\neg x_2 \vee \neg x_3) \wedge (x_1 \vee x_4)$$

- $k$-SAT is $NP$-complete for $k > 2$.

# Random $k$-SAT

**Random $k$-SAT**:

- Parameters:
  - number of variables – $n$,
  - number of clauses – $m$
- $m/n$=Number of clauses divided by number of variables: **density** – fixed!
- Choose clauses at random, uniformly
- Limit: $n, m \to \infty$

# Evolution of Random $k$-SAT

**Intuition**: Density analogous to temperature

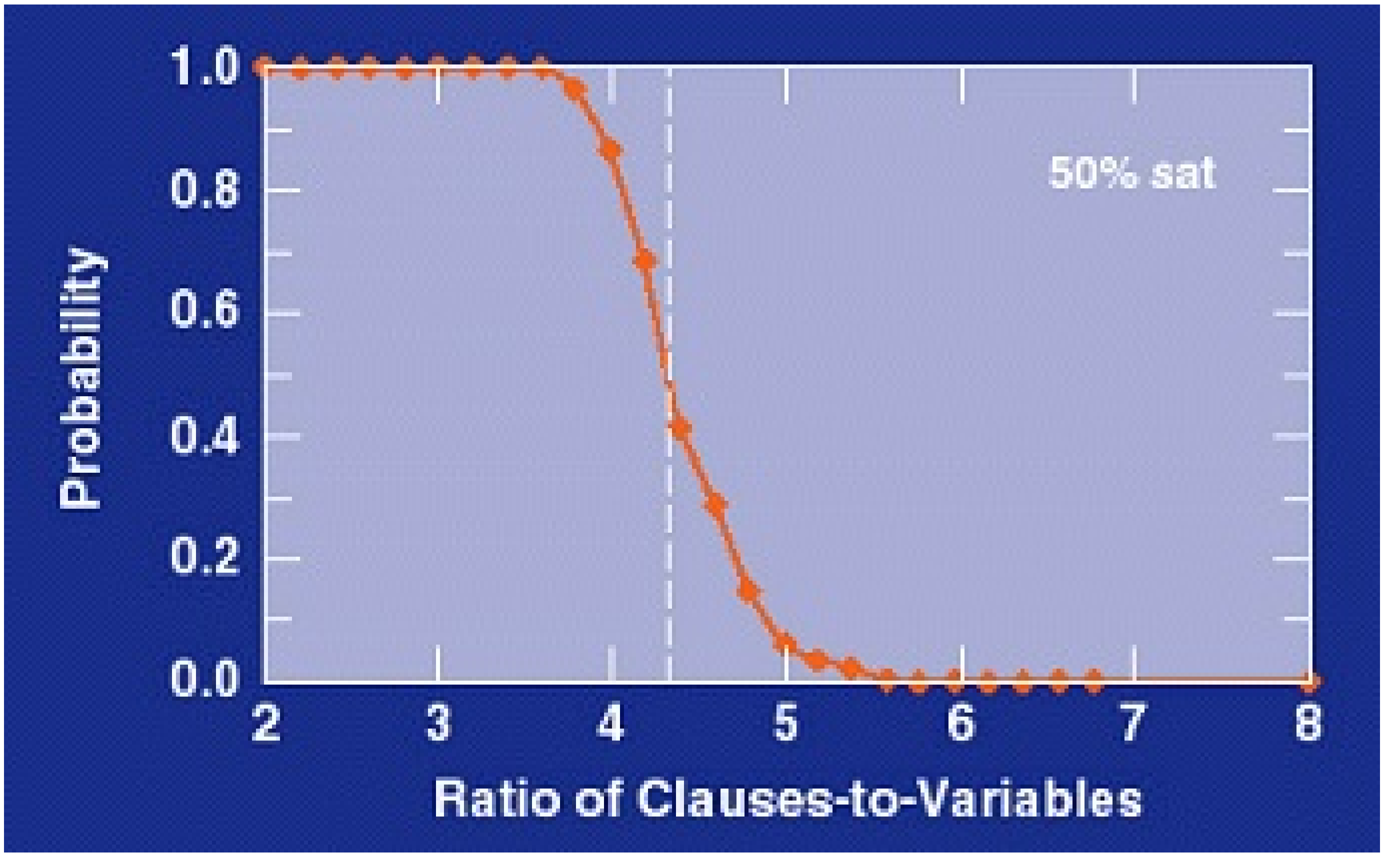
- *Low density*: low energy state – high probability of satisfiability – limit$= 1$
- *High density*: high energy state – low probability of satisfiability – limit$= 0$

**Empirical Observation**: Phase transition – limit probability drops from $1$ to $0$

- $2$-SAT: phase transition at density $1$ (also proved formally)
- $3$-SAT: phase transition at density $4.26$

1991-2010: Extensive research on statistical behavior of Random $k$-SAT

## Phase Transition of 3-SAT



# Essence of V.D.'s Proof

**Crux**: $9$-SAT can not be in $P$!

- If $9$-SAT is in $P$, then it can be expressed in FO+LFP, by the Immerman-V. Theorem.
- But, the FO+LFP normal form is inconsistent with what is known about statistical behavior of random $9$-SAT.

# Reaction to Proof Announcement

A huge buzz!!!

**Why**?

- People announce solutions of the problem all the time.
- Every few months paper posted on arXiv.org.

**But**:

- V.D. is a Principal Research Scientist at HP.
- Stephen Cook (founding figure in complexity theory): "This appears to be a relatively serious claim"
- Nice connection of complexity, logic, and physics!
- Richard Lipton (senior complexity theorist and influential blogger): Blog item on August 8, 2010, slashdotted

# Proof Checking at The Internet Age

"*Ten Days of Fame*": Proof discredited in ten days!

- Aug. 6: Manuscript sent to 22 people and put on web page
- Aug. 7: First blog post [Greg Baker]
- Aug. 8: Second blog post [Richard Lipton], Slashdot
  - extensive commentary
- Aug. 9: Wikipedia article about V.D. (deleted later)
- Aug. 10: Wiki for technical discussion established
  - hundreds of edits
  - Fields medalists involved
- Aug. 15: CACM blogpost by Lipton
- Aug.16: New York Times article

## The Flaw

**A major problem**: V.D.'s proof does not seem to distinguish between intractable and tractable cases of $k$-SAT.

**Cause**: Misuse of the Immerman-V. Theorem.

# A Tractable Fragment of SAT

**Affine Boolean Satisfiability (Affine SAT)**: Given a Boolean expression in the form of "and of xors", is there a *satisfying* solution (an assignment of 0's and 1's to the variables that makes the expression equal 1)?

**Example**:

$$(\neg x_1 \oplus x_2 \oplus x_3) \wedge (\neg x_2 \oplus \neg x_3 \oplus x_4) \wedge (x_3 \oplus x_1 \oplus x_4)$$

**In essence**: Linear equations modulo 2

- Solve using Gaussian elimination

**But**: Random $k$-SAT and Random Affine $k$-SAT are quite similar statistically!

## Revision at the Internet Age

- First draft, Aug. 6
- Second draft Aug. 9–11
- Third draft, Aug. 11–17
- All drafts removed after Aug 17

**Consensus**: The P vs. NP problem withstood another challenge and remained wide open!

- Wikipedia: "However, the general consensus amongst theoretical computer scientists is now that the attempted proof is not correct, nor even a significant advancement in our understanding of the problem."

## No Concession!

From V.D.'s website:

"The preliminary version was meant to solicit feedback from a few researchers as is customarily done. It illustrated the interplay of principles from various areas, which was the major effort in constructing the proof. I have fixed all the issues that were raised about the preliminary version in a revised manuscript; clarified some concepts; and obtained simpler proofs of several claims. Once I hear back from the journal as part of due process, I will put up the final version on this website."

# Reflection on P vs. NP

**Old Cliché** "What is the difference between theory and practice? In theory, they are not that different, but in practice, they are quite different."

**P vs. NP in practice**:

- P=NP: Conceivably, NP-complete problems can be solved in polynomial time, but the polynomial is $(10n)^{1000}$ – *impractical*!

- P≠NP: Conceivably, NP-complete problems can be solved by $n^{\log\log\log n}$ operations – *practical*!

**Conclusion**: No guarantee that solving P vs. NP would yield practical benefits.

## Theory, Practice &Programming

- Theory: You know something, but it doesn't work.
- Practice: Something works, but you don't know why
- Programming: Combine theory and practice: Nothing works, and we don't know why!

## Are NP-Complete Problems Really Hard?

- When I was a graduate student, SAT was a “scary” problem, not to be touched with a 10-foot pole.
- Indeed, there are SAT instances with a few hundred variables that cannot be solved by any extant SAT solver.
- But today’s SAT solvers, which enjoy wide industrial usage, routinely solve real-life SAT instances with over one million variables!

**Conclusion** We need a richer and broader complexity theory, a theory that would explain both the difficulty and the easiness of problems like SAT.