Satisfiability Parsimoniously Reduces to the Tantrix $^{\text{TM}}$ Rotation Puzzle Problem

Dorothea Baumeister and Jörg Rothe

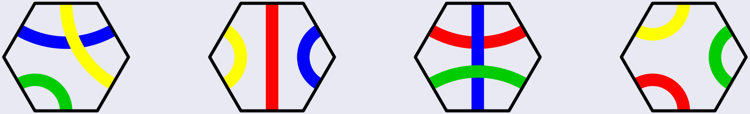
Institut für Informatik, Universität Düsseldorf

MCU Conference, September 2007

Outline

Tantrix ^{TM} Tiles and Rules

Different types of tiles



Sint Brid Chin Rond

4 colors and 4 types of tiles $\Rightarrow$ 56 different tiles

Golden rule

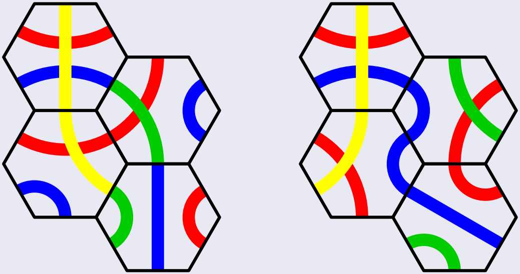
The colors of two joint edges must always be the same.

Rotation Puzzle

Aim

Rotate the tiles to fulfill the golden rule.

Example



Puzzle Solution

⇒ Only one valid solution

Definitions

Definition

NP is the class of problems solvable in nondeterministic polynomial time.

Definition (Papadimitriou, Yannakakis [PY84])

$\mathrm{DP} = \{A - B \mid A, B \in \mathrm{NP}\}$

Definition (Valiant [Val79])

Let $acc_M(x)$ denote the number of accepting computation paths of an NPTM M on input x, then define the function class $\#\mathrm{P} = \{acc_M \mid \ M \text{ is an NPTM}\}$.

Definition

For $f, g : \Sigma^* \to \mathbb{N}$, $f \leq^p_{par} g$ if there exists a polynomial-time computable function ρ such that for each $x \in \Sigma^*$, $f(x) = g(\rho(x))$.

Variants of the Tantrix TM Rotation Puzzle Problem

Let $\mathcal{A}$ be a (partial) function mapping the elements of $\mathbb{Z}^2$ to T, where T is the set of all Tantrix TM tiles. Let $\mathrm{Sol}_{\mathrm{TRP}}(\mathcal{A})$ denote the set of solutions of a given TRP instance $\mathcal{A}$.

Definition

Name: Tantrix TM Rotation Puzzle (TRP, for short).
Given: A finite shape function $\mathcal{A} : \mathbb{Z}^2 \to T$
Question: Is the rotation puzzle defined by $\mathcal{A}$ solvable?

Definition

The corresponding *counting* and *unique problems* are defined by

$$\#\mathrm{TRP}(\mathcal{A}) = \|\mathrm{Sol}_{\mathrm{TRP}}(\mathcal{A})\|$$

$$\texttt{Unique-TRP} = \{\mathcal{A} \mid \|\mathrm{Sol}_{\mathrm{TRP}}(\mathcal{A})\| = 1\}$$

Reduction from Holzer and Holzer [HH04]

Proofsketch

Reduction from boolean circuits with only AND and NOT gates

1. Build a planar circuit without wire-crossings
2. Substitute the circuit with corresponding Tantrix ™ subpuzzles

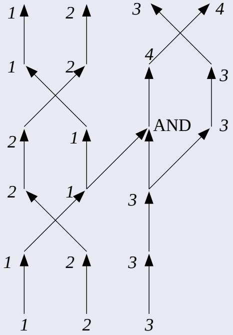
There is a satisfying assignment to the variables of the circuit

$$\Leftrightarrow$$

There is a solution to the resulting TRP instance

TRP Subpuzzles and Boolean Circuits

Example for a planar circuit



Planar circuit for $\alpha_4 = AND(1, 3)$

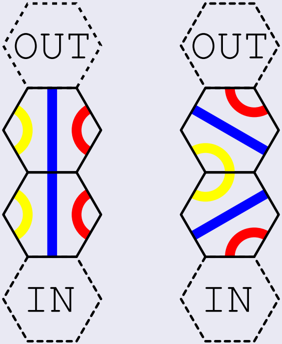
Subpuzzles

Circuit	Subpuzzle
Input variables	BOOL
Wires	WIRE MOVE COPY
Gates	AND NOT
Output	TEST

The color *blue* represents *true*, while *red* represents *false*.

Example (1)

WIRE subpuzzle from Holzer and Holzer [HH04]

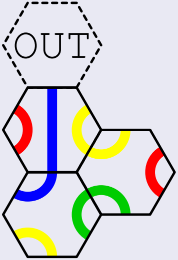


IN: true IN: false

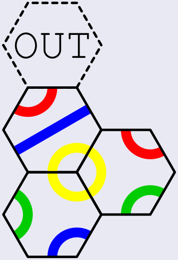
⇒ 4 possible solutions for each input color

Example (2)

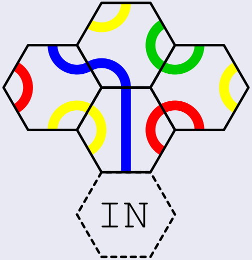
Solutions to the original BOOL and TEST subpuzzles from [HH04]



BOOL: true



BOOL: false



TEST

$\Rightarrow$ Already unique solutions

Parsimonious Reduction

Modification

Reducing the number of solutions for each subpuzzle to one.

The number of satisfying assignment to the variables of the circuit

=

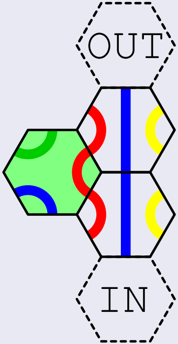
The number of solutions of the resulting TRP instance

Theorem

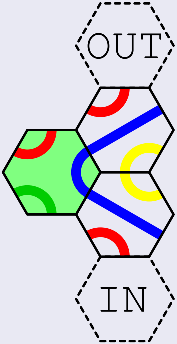
$\#SAT \leq^{p}_{par} \#TRP$

Example (1)

Modified WIRE subpuzzle



IN: true

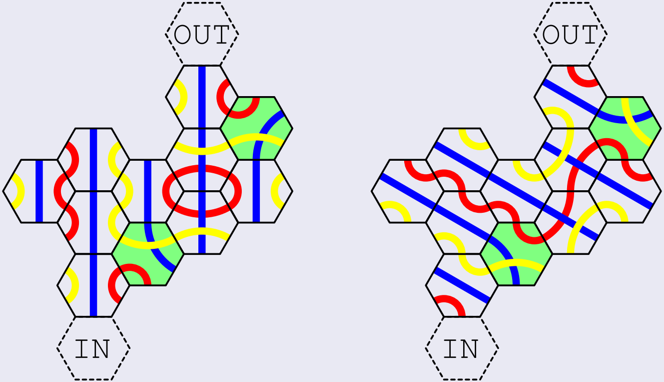


IN: false

⇒ Unique solutions for each input color

Overview (1)

Solutions to the modified MOVE subpuzzle

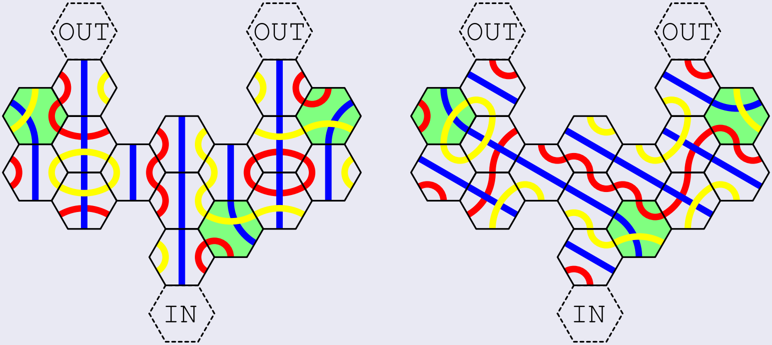


IN: true IN: false

⇒ Unique solutions for each input color

Overview (2)

Solutions to the modified COPY subpuzzle



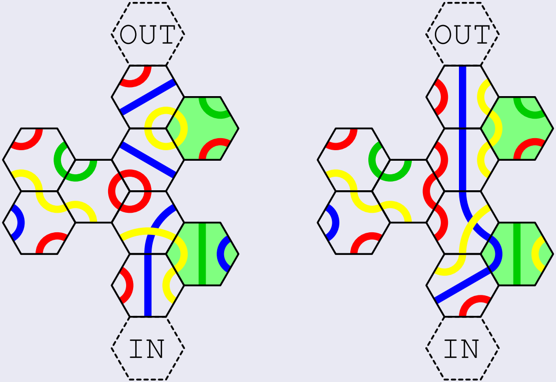
IN: true

IN: false

⇒ Unique solutions for each input color

Overview (3)

Solutions to the modified NOT subpuzzle

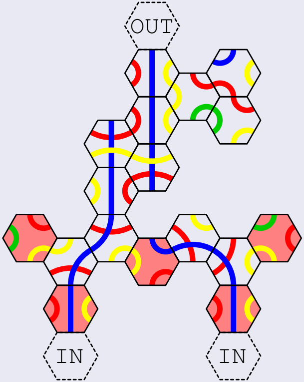


IN: true IN: false

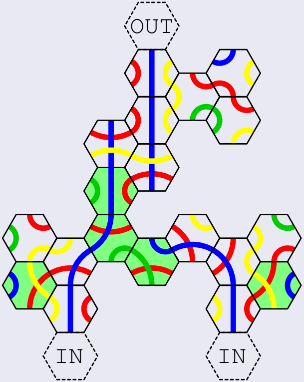
⇒ Unique solutions for each input color

Example (2)

AND subpuzzle with both inputs true



Original subpuzzle from [HH04]

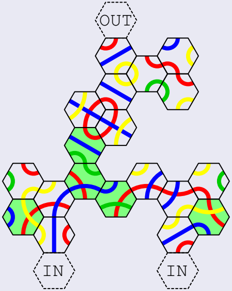


Modified subpuzzle

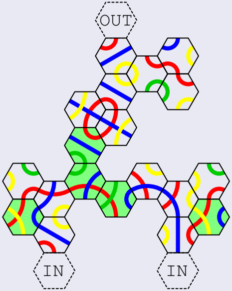
⇒ Modified subpuzzle: Unique solutions for each input color

Overview (4)

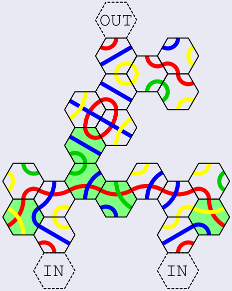
Solutions to the modified AND subpuzzle



IN: true, false



IN: false, true



IN: false, false

⇒ Unique solutions for each input color

Results

Theorem

1. $\#SAT \leq^{p}_{par} \#TRP$, *i.e. SAT parsimoniously reduces to TRP*
2. `Unique-SAT` $\leq^{p}_{par}$ `Unique-TRP`
3. `Unique-TRP` *is* DP*-complete*
   (under randomized polynomial-time reductions in the sense of Valiant and Vazirani [VV86])

Overview of Complexity Results for k-TRP

Definition

Name: k-Color Tantrix$^{\text{TM}}$ Rotation Puzzle (k-TRP, for short).
Instance: A finite shape function $\mathcal{A} : \mathbb{Z}^2 \to T_k$
Question: Is there a solution to the rotation puzzle defined by $\mathcal{A}$?

Results

Problem	Complexity (Source)	Parsimonious Reduction?
1-TRP	in P (trivial)	
2-TRP	NP-complete ([BR07])	yes ([BR07])
3-TRP	NP-complete ([BR07])	yes ([BR07])
4-TRP	NP-complete ([HH04])	yes, reduction shown above

References

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Discrete Applied Mathematics, 144(3):345–358, 2004.

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NP is as easy as detecting unique solutions.
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