Using approximation to relate computational classes over the reals

Manuel Campagnolo[1] Kerry Ojakian[2]

[1]DM/ISA, Technical University of Lisbon and SQIG/IT Lisbon
mlc@math.isa.utl.pt

[2]SQIG/IT Lisbon and IST, Technical University of Lisbon
ojakian@math.ist.utl.pt

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Outline

Motivation

Is $f : \mathbb{R} \to \mathbb{R}$ computable?

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Several notions of computability for real functions:

- Turing machine approach: Computable Analysis
- Continuous time analog models
- BSS machines
- ...

Motivation

$\mathbf{C}(\mathbb{R})$ = *"Analog"*

Analog models considered in this talk:

- Polynomial Differential Equations
- Real Recursive Functions

Motivation

- Church-Turing type thesis for computation on the reals:
  - There are many distinct models of computation on the reals: Computable Analysis, Real recursive Functions, General Purpose Analog Computer, Neural Networks, Dynamic Systems,...
  - How are the models distinct?
  - What kind of modifications make them equal?
- Applications in discrete complexity theory.
  - Can separation questions (e.g. P versus NP) be reduced to relevant questions in Analysis?
  - Can we transfer those questions into relevant questions in Analysis?
  - See work by Costa and Mycka (2006, 2007) in this direction.

Computable Analysis

Definition

- $f(x) \in \mathbf{C}(\mathbb{R})$:

  There is a computable function $F^x(n)$ with an oracle for the real number x such that $|f(x) - F^x(n)| \leq 1/n$.

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- $\mathbf{E}(\mathbb{R})$: like $\mathbf{C}(\mathbb{R})$, replacing computable by elementary computable.

Approximation and Completion

Goal. $\mathbf{C}(\mathbb{R}) = \text{A(LIM)}$, broken into 2 steps:

- (Approximation) $\mathbf{C}(\mathbb{R}) \approx \text{A}$.
- (Completion) $\mathbf{C}(\mathbb{R}) = \text{A(LIM)}$.

Approximation Relation

Definition

$\mathcal{A} \preceq_{+}^{\varepsilon} \mathcal{B}$ iff

> *For every $f(\bar{x}) \in \mathcal{A}$ and every $\alpha(\bar{x}, \bar{y}) \in \mathcal{E}$ there is $f^*(\bar{x}, \bar{y}) \in \mathcal{B}$ such that $|f(\bar{x}) - f^*(\bar{x}, \bar{y})| \leq \alpha(\bar{x}, \bar{y})$.*

$\mathcal{A} \approx^{\varepsilon} \mathcal{B}$ iff $\mathcal{A} \preceq_{+}^{\varepsilon} \mathcal{B}$ and $\mathcal{B} \preceq_{+}^{\varepsilon} \mathcal{A}$.

Lemma (transitivity)

Let $\mathcal{E}$ be an error class. If $\mathcal{A} \preceq_{+}^{\varepsilon} \mathcal{B}$ and $\mathcal{B} \preceq_{+}^{\varepsilon} \mathcal{C}$ then $\mathcal{A} \preceq_{+}^{\varepsilon} \mathcal{C}$.

Completion Operation

Definition

LIM is the operation:

- Input: $f(\bar{x}, t)$
- Output: $F(\bar{x}) = \lim_{t \to \infty} f(\bar{x}, t)$, if the limit exists and $F \preceq^{1/t} f$, for positive t

Definition

If OP is an operation and $\mathcal{F}$ a set of functions, then $\mathcal{F}(\mathrm{OP})$ is:

$$\mathcal{F} \;\cup\; \{\mathrm{OP}(f) \mid f \in \mathcal{F}\}$$

Thus $\mathcal{F}(\mathrm{LIM})$ is a “completion” of $\mathcal{F}$.

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Note. LIM is a weak kind of limit operation:

- $\mathbf{C}(\mathbb{R}) = \mathbf{C}(\mathbb{R})(\mathrm{LIM})$;
- $\mathbf{E}(\mathbb{R}) = \mathbf{E}(\mathbb{R})(\mathrm{LIM})$.

Function Algebras

Definition

Suppose $\mathcal{B}$ is a set of functions (i.e. the basic functions) and $\mathcal{O}$ is a set of operations. Then FA$[\mathcal{B}; \mathcal{O}]$ is the smallest set of functions containing $\mathcal{B}$ and closed under $\mathcal{O}$.

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Basic Functions:

- Constant functions: $0, 1, -1, \pi$
- Projection functions "P" (example: $\mathrm{P}(x, y) = x$)
- $\theta(x) = \begin{cases} 0, & x < 0; \\ x^3, & x \geq 0. \end{cases}$

The Operations

Definition (COMP)

Input: $\overrightarrow{\mathbf{f}}$, $\overrightarrow{\mathbf{g}}$; Output: $\overrightarrow{\mathbf{h}} = \overrightarrow{\mathbf{g}} \circ \overrightarrow{\mathbf{f}}$.

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Definition (LI)

- Input: Functions: $\overrightarrow{\mathbf{g}}(\bar{x})$, $\overrightarrow{\mathbf{f}}(y, \bar{x})$.
- Output: $h_1(y, \bar{x})$ where $(h_1, \ldots, h_n)$ is the solution to the IVP:

$$\frac{\partial}{\partial y}\overrightarrow{\mathbf{h}} = \overrightarrow{\mathbf{f}}(y, \bar{x})\, \overrightarrow{\mathbf{h}}(y, \bar{x})$$
$$\overrightarrow{\mathbf{h}}(0, \bar{x}) = \overrightarrow{\mathbf{g}}(\bar{x}) \qquad \textit{(initial conditions)}$$

Examples:

- $f(x,y) = x + y$ $\quad f(x,0) = x$, $\partial_y f(x,y) = 1$ $\quad f' = [1\ 1]$

- $f(x,y) = xy$ $\quad f(x,0) = 0$, $\partial_y f(x,y) = x$ $\quad f' = [y\ x]$

- $(\sin y, \cos y)$ $\quad f(0) = (0,1)$, $\partial_y f(y) = (\cos y, -\sin y)$ $\quad \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} f_2 \\ -f_1 \end{bmatrix}$

Elementary Computability.

Let $\mathcal{L}$ abbreviate FA[0, 1, -1, π, θ, P; comp, LI].
Let $\mathcal{L}^{\text{a}}$ abbreviate FA[0, 1, -1, P; comp, LI].

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Theorem

- *(Approximation)* $\mathbf{E}(\mathbb{R}) \approx \mathcal{L} \approx \mathcal{L}^{\text{a}}$
- *(Completion)* $\mathbf{E}(\mathbb{R}) = \mathcal{L}(LIM) = \mathcal{L}^{\text{a}}(LIM)$

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- *(Completion)* $\mathbf{E}(\mathbb{R}) = \mathcal{L}(LIM) = \mathcal{L}^{\mathrm{a}}(LIM)$
- *(Alternative Completion)* $\mathbf{E}(\mathbb{R}) = \mathcal{L}(dLIM)$ *(similar to Bournez and Hainry 2004)*

Definition

dLIM is the operation:

- Input: $f(t, \bar{x})$
- Output: $F(\bar{x}) = \lim_{t\to\infty} f(t, \bar{x})$, if $|\frac{\partial}{\partial t} f| \leq 1/2^t$ for $t \geq 1$.

Sketch of the proof of $\mathbf{E}(\mathbb{R}) \approx \mathcal{L}$

TM models		Function algebras
$\mathbf{E}(\mathbb{N})$	=	$\mathrm{FA}_{\mathbb{N}}$

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$\mathbf{E}(\mathbb{Q})$	= if f has modulus	$\mathrm{FA}_{\mathbb{Q}}(\mathrm{disctn})$
$\approx$		$\approx$
$\mathbf{E}(\mathbb{R})$		$\mathrm{FA}_{\mathbb{Q}}(\mathrm{ctn})$
		$\approx$
		$\mathcal{L}$

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$\not\preceq \quad \succeq$		$\not\preceq \quad \succeq$
$\mathbf{E}(\mathbb{Q})$	=	$FA_{\mathbb{Q}}(\text{disctn})$
$\approx$	if f has modulus	$\approx$
$\mathbf{E}(\mathbb{R})$		$FA_{\mathbb{Q}}(\text{ctn})$
		$\approx$
		$\mathcal{L}$

(Campagnolo and Ojakian, Arch Math Logic, to appear)

Sketch of the proof of $\mathcal{L} \approx \mathcal{L}^{\mathrm{a}}$

Recall that $\mathcal{L} = \mathrm{FA}[0, 1, -1, \pi, \theta, \mathrm{P}; \mathrm{comp}, \mathrm{LI}]$ and $\mathcal{L}^{\mathrm{a}} = \mathrm{FA}[0, 1, -1, \mathrm{P}; \mathrm{comp}, \mathrm{LI}]$.

Goal: eliminate the non-analytic function θ

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Show:

- $\theta, \pi \preceq \mathcal{L}^{\mathrm{a}}$
- $\text{comp}, \text{LI} \preceq \mathcal{L}^{\mathrm{a}}$

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- $\mathrm{comp}, \mathrm{LI} \preceq \mathcal{L}^{\mathrm{a}}$

General idea: using approximation and transitivity we can break down the proof of $\mathbf{E}(\mathbb{R}) \approx \mathcal{L}^{\mathrm{a}}$ into simpler pieces.

Computability.

Theorem (similar to Bournez and Hainry, 2005, 2006)

- *(Approximation)* $\mathbf{C}(\mathbb{R}) \approx FA[0, 1, \theta, P; comp, CLI, UMU]$
- *(Completion)*

$$\begin{aligned} \mathbf{C}(\mathbb{R}) &= FA[0, 1, \theta, P; comp, CLI, UMU](LIM) \\ &= FA[0, 1, \theta, P; comp, CLI, UMU](dLIM) \end{aligned}$$

Definition

Note: The solutions of $\vec{\mathbf{y}}' = \vec{\mathbf{p}}(\vec{\mathbf{y}}, t))$ with initial condition $\vec{\mathbf{y}}(0) = \vec{\mathbf{y}}_0$ are exactly the functions generated by Shannon's General Purpose Analog Computer (Graça and Costa, 2003).

Definition

Let PI be the operation:

- Input: $n-1$ polynomials: $\vec{\mathbf{p}}(y, t)$, a polynomial $q(x)$, and numbers $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$.
- Output: $y_1(t, x)$ where $(y_1, ..., y_n)$ is the solution of IVP: $\frac{\partial}{\partial t}\vec{\mathbf{y}} = \vec{\mathbf{p}}(\vec{\mathbf{y}}, t)$ $\vec{\mathbf{y}}(0) = (\alpha_1, ..., \alpha_{n-1}, q(x))$

Definition

For $X \subseteq \mathbb{R}$, let GPAC_X be the set of functions generated by PI using polynomials with coefficients from X and initial conditions from X.

Result

Let $\mathcal{CR}$ be the set of computable reals.

Theorem (Bournez, Campagnolo, Graça and Hainry, 2007)

- *(Approximation)* $\mathbf{C}(\mathbb{R}) \approx GPAC_{\mathcal{CR}}$
- *(Completion)* $\mathbf{C}(\mathbb{R}) = GPAC_{\mathcal{CR}}(\varepsilon - LIM)$

Question. Is this true for $\text{GPAC}_{\mathbb{Q}}$ or even $\text{GPAC}_{\{0,1,-1\}}$?

Summary

- Computable Analysis can be caracterized with analog models.
- The connections can be organized using approximation and completion.
- New useful techniques: transitivity, eliminating non-analytic functions, lifting.

Directions for research

- Find simpler characterizations.
- Use the same techniques to characterize complexity classes lower than the elementary.
- Explore other kinds of “completion”.
- Are there characterizations of Computable Analysis, which naturally capture all of its functions, *without* a completion operation?

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Thanks!