Results

Using approximation to relate computational classes over the reals

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Outline			







- Real Recursive Functions
- Polynomial differential equations



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Motivation			

Is $f : \mathbb{R} \to \mathbb{R}$ computable?

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Motivation			

Is $f : \mathbb{R} \to \mathbb{R}$ computable?

Several notions of computability for real functions:

- Turing machine approach: Computable Analysis
- Continuous time analog models
- BSS machines
- ...

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Motivation			

$$\mathbf{C}(\mathbb{R}) =$$
"Analog"

Analog models considered in this talk:

- Polynomial Differential Equations
- Real Recursive Functions

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Motivation			

- Church-Turing type thesis for computation on the reals:
 - There are many distinct models of computation on the reals: Computable Analysis, Real recursive Functions, General Purpose Analog Computer, Neural Networks, Dynamic Systems,...
 - How are the models distinct?
 - What kind of modifications make them equal?
- Applications in discrete complexity theory.
 - Can separation questions (e.g. P versus NP) be reduced to relevant questions in Analysis?
 - Can we transfer those questions into relevant questions in Analysis?
 - See work by Costa and Mycka (2006, 2007) in this direction.

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Computable Ana	alysis		

• $f(x) \in \mathbf{C}(\mathbb{R})$:

There is a computable function $F^{x}(n)$ with an oracle for the real number x such that $|f(x) - F^{x}(n)| \le 1/n$.

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Computable An	alysis		

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• **E**(**R**): like **C**(**R**), replacing computable by elementary computable.

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Approximation and Completion

Goal. $C(\mathbb{R}) = A(LIM)$, broken into 2 steps:

- (Approximation) $\mathbf{C}(\mathbb{R}) \approx A$.
- (Completion) $\mathbf{C}(\mathbb{R}) = A(LIM)$.

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Approximation Relation

Definition

 $\mathcal{A} \preceq^{\varepsilon}_{+} \mathcal{B}$ iff

For every $f(\bar{x}) \in A$ and every $\alpha(\bar{x}, \bar{y}) \in \varepsilon$ there is $f^*(\bar{x}, \bar{y}) \in \mathcal{B}$ such that $|f(\bar{x}) - f^*(\bar{x}, \bar{y})| \le \alpha(\bar{x}, \bar{y})$.

 $\mathcal{A} \approx^{\varepsilon} \mathcal{B} \text{ iff } \mathcal{A} \preceq^{\varepsilon}_{+} \mathcal{B} \text{ and } \mathcal{B} \preceq^{\varepsilon}_{+} \mathcal{A}.$

Lemma (transitivity)

Let ε be an error class. If $\mathcal{A} \preceq^{\varepsilon}_{+} \mathcal{B}$ and $\mathcal{B} \preceq^{\varepsilon}_{+} \mathcal{C}$ then $\mathcal{A} \preceq^{\varepsilon}_{+} \mathcal{C}$.

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Completion Ope	ration		

LIM is the operation:

- Input: $f(\bar{x}, t)$
- Output: $F(\bar{x}) = \lim_{t\to\infty} f(\bar{x}, t)$, if the limit exists and $F \leq 1/t f$, for positive t

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If OP is an operation and \mathcal{F} a set of functions, then $\mathcal{F}(OP)$ is:

 $\mathcal{F} \cup \{\mathsf{OP}(f) \mid f \in \mathcal{F}\}$

Thus $\mathcal{F}(LIM)$ is a "completion" of \mathcal{F} .

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If OP is an operation and \mathcal{F} a set of functions, then $\mathcal{F}(OP)$ is:

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Thus $\mathcal{F}(LIM)$ is a "completion" of \mathcal{F} .

Note. LIM is a weak kind of limit operation:

•
$$\mathbf{C}(\mathbb{R}) = \mathbf{C}(\mathbb{R})(\mathsf{LIM});$$

• $\mathbf{E}(\mathbb{R}) = \mathbf{E}(\mathbb{R})(\mathsf{LIM}).$

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Real Recursive Functions			
Function Algebra	ras		

Suppose \mathcal{B} is a set of functions (i.e. the basic functions) and \mathcal{O} is a set of operations. Then FA[\mathcal{B} ; \mathcal{O}] is the smallest set of functions containing \mathcal{B} and closed under \mathcal{O} .

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Function Algebras			

Suppose \mathcal{B} is a set of functions (i.e. the basic functions) and \mathcal{O} is a set of operations. Then FA[\mathcal{B} ; \mathcal{O}] is the smallest set of functions containing \mathcal{B} and closed under \mathcal{O} .

Basic Functions:

- Constant functions: $0, 1, -1, \pi$
- Projection functions "P" (example: P(x, y) = x)

•
$$\theta(x) = \begin{cases} 0, & x < 0; \\ x^3, & x \ge 0. \end{cases}$$

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The Operations			

Definition (COMP)

Input:
$$\overrightarrow{\mathbf{f}}$$
, $\overrightarrow{\mathbf{g}}$; Output: $\overrightarrow{\mathbf{h}} = \overrightarrow{\mathbf{g}} \circ \overrightarrow{\mathbf{f}}$.

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The Operations			

Definition (COMP)

Input:
$$\overrightarrow{\mathbf{f}}$$
, $\overrightarrow{\mathbf{g}}$; Output: $\overrightarrow{\mathbf{h}} = \overrightarrow{\mathbf{g}} \circ \overrightarrow{\mathbf{f}}$.

Definition (LI)

- Input: Functions: $\overrightarrow{\mathbf{g}}(\overline{x}), \ \overrightarrow{\mathbf{f}}(y, \overline{x})$.
- Output: $h_1(y, \bar{x})$ where (h_1, \ldots, h_n) is the solution to the IVP:

$$rac{\partial}{\partial y} \overrightarrow{\mathbf{h}} = \overrightarrow{\mathbf{f}}(y, \overline{x}) \overrightarrow{\mathbf{h}}(y, \overline{x})$$

 $\overrightarrow{\mathbf{h}}(0, \overline{x}) = \overrightarrow{\mathbf{g}}(\overline{x})$ (initial conditions)

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Real Recursive Functions			

Examples:

•
$$f(x, y) = x + y$$
 $f(x, 0) = x$
 $\partial_y f(x, y) = 1$ $f' = [1 \ 1]$

•
$$f(x,y) = xy$$
 $f(x,0) = 0$
 $\partial_y f(x,y) = x$ $f' = [y x]$

•
$$(\sin y, \cos y)$$
 $\begin{array}{c} f(0) = (0, 1) \\ \partial_y f(y) = (\cos y, -\sin y) \end{array} \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} f_2 \\ -f_1 \end{bmatrix}$

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Elementary Compu	utability.		

Let \mathcal{L} abbreviate FA[0, 1, -1, π , θ , P; comp, LI]. Let \mathcal{L}^{a} abbreviate FA[0, 1, -1, P; comp, LI].

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Elementary Comp	utability.		

Let \mathcal{L} abbreviate FA[0, 1, -1, π , θ , P; comp, LI]. Let \mathcal{L}^a abbreviate FA[0, 1, -1, P; comp, LI].

Theorem

- (Approximation) $\mathsf{E}(\mathbb{R}) \approx \mathcal{L} \approx \mathcal{L}^{\mathrm{a}}$
- (Completion) $\mathbf{E}(\mathbb{R}) = \mathcal{L}(LIM) = \mathcal{L}^{a}(LIM)$

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Elementary C	omputability.		

Let \mathcal{L} abbreviate FA[0, 1, -1, π , θ , P; comp, LI]. Let \mathcal{L}^{a} abbreviate FA[0, 1, -1, P; comp, LI].

Theorem

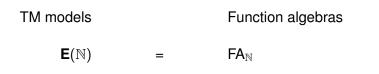
- (Approximation) $\mathsf{E}(\mathbb{R}) \approx \mathcal{L} \approx \mathcal{L}^{\mathrm{a}}$
- (Completion) $\mathbf{E}(\mathbb{R}) = \mathcal{L}(LIM) = \mathcal{L}^{a}(LIM)$
- (Alternative Completion) E(R) = L(dLIM) (similar to Bournez and Hainry 2004)

Definition

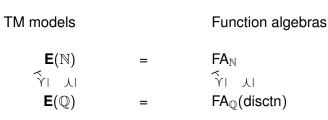
dLIM is the operation:

- Input: $f(t, \bar{x})$
- Output: $F(\bar{x}) = \lim_{t\to\infty} f(t,\bar{x})$, if $|\frac{\partial}{\partial t}f| \le 1/2^t$ for $t \ge 1$.

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Sketch of the pro	oof of $E(\mathbb{R}) pprox \mathcal{L}$		

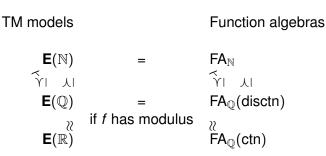


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Sketch of the pro	bof of $E(\mathbb{R}) pprox \mathcal{L}$		



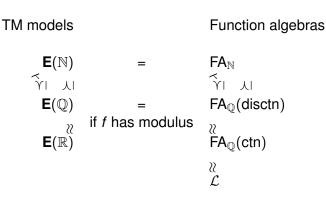
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Sketch of the proof of $E(\mathbb{R}) \approx \mathcal{L}$



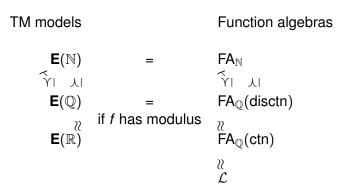
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Sketch of the proof of $\mathsf{E}(\mathbb{R}) \approx \mathcal{L}$



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(Campagnolo and Ojakian, Arch Math Logic, to appear)

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Real Recursive Functions			
Sketch of the proc	of of $\mathcal{L}pprox \mathcal{L}^{\mathrm{a}}$		

Recall that $\mathcal{L} = FA[0, 1, -1, \pi, \theta, P; comp, LI]$ and $\mathcal{L}^a = FA[0, 1, -1, P; comp, LI]$.

Goal: eliminate the non-analytic function θ

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Sketch of the proof	of $\mathcal{L} pprox \mathcal{L}^{\mathrm{a}}$		

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Goal: eliminate the non-analytic function θ

Show:

- $\theta, \pi \preceq \mathcal{L}^{a}$
- comp, $LI \preceq \mathcal{L}^a$

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Goal: eliminate the non-analytic function θ

Show:

- $\theta, \pi \preceq \mathcal{L}^{a}$
- comp, $LI \preceq \mathcal{L}^a$

General idea: using approximation and transitivity we can break down the proof of $E(\mathbb{R}) \approx \mathcal{L}^a$ into simpler pieces.

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Computability.			

Theorem (similar to Bournez and Hainry, 2005, 2006)

- (Approximation) $\mathbf{C}(\mathbb{R}) \approx FA[0, 1, \theta, P; comp, CLI, UMU]$
- (Completion)

$$\mathbf{C}(\mathbb{R}) = FA[0, 1, \theta, P; comp, CLI, UMU](LIM) = FA[0, 1, \theta, P; comp, CLI, UMU](dLIM)$$

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Polynomial differential equations			
Definition			

Note: The solutions of $\vec{\mathbf{y}}' = \vec{\mathbf{p}}(\vec{\mathbf{y}}, t)$) with initial condition $\vec{\mathbf{y}}(0) = \vec{\mathbf{y}}_0$ are exactly the functions generated by Shannon's General Purpose Analog Computer (Graça and Costa, 2003).

Definition

Let PI be the operation:

- Input: n − 1 polynomials: p
 [→](y, t), a polynomial q(x), and numbers α₁,..., α_{n−1} ∈ ℝ.
- Output: $y_1(t, x)$ where $(y_1, ..., y_n)$ is the solution of IVP: $\frac{\partial}{\partial t} \overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{p}}(\overrightarrow{\mathbf{y}}, t) \overrightarrow{\mathbf{y}}(0) = (\alpha_1, ..., \alpha_{n-1}, q(x))$

Definition

For $X \subseteq \mathbb{R}$, let GPAC_X be the set of functions generated by PI using polynomials with coefficients from X and initial conditions from X.

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Result			

Let \mathcal{CR} be the set of computable reals.

Theorem (Bournez, Campagnolo, Graça and Hainry, 2007)

- (Approximation) $\mathbf{C}(\mathbb{R}) \approx GPAC_{CR}$
- (Completion) $\mathbf{C}(\mathbb{R}) = GPAC_{CR}(\varepsilon LIM)$

Question. Is this true for $GPAC_{\mathbb{Q}}$ or even $GPAC_{\{0,1,-1\}}$?

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Computable Analysis can be caracterized with analog models.

Summary

- The connections can be organized using approximation and completion.
- New useful techniques: transitivity, eliminating non-analytic functions, lifting.

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Directions for res	earch		

- - Find simpler characterizations.
 - Use the same techniques to characterize complexity classes lower than the elementary.
 - Explore other kinds of "completion".
 - Are there characterizations of Computable Analysis, which naturally capture all of its functions, *without* a completion operation?

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Directions for research

- Find simpler characterizations.
- Use the same techniques to characterize complexity classes lower than the elementary.
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- Are there characterizations of Computable Analysis, which naturally capture all of its functions, *without* a completion operation?

Thanks!