Study of Limits of Solvability in Tag Systems

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Introduction

- Tag systems – a quick tour
- Outline proof of the solvability of the halting and reachability problem for a specific class of tag systems:

  a. General structure and method

  b. Some basic techniques and cases
- Discussion: Some open questions

Definition of Tag systems

- Invented by Emil Leon Post in 1921 and shown to be Turing complete (universal) in 1961 by Minsky
- A tag system T, consists of a finite alphabet $\Sigma = \{a_0, a_1, ..., a_{\mu-1}\}$ of μ symbols, a deletion number $v \in \mathbb{N}$ and a finite set of μ words, $w_0, w_1, ..., w_{\mu-1}$ over the alphabet, including the empty word ϵ. Each of these words corresponds with one of the letters from the alphabet as follows:

$$
\begin{array}{lll}
a_0 & \rightarrow & a_{0,1}a_{0,2}...a_{0,n_0} \\
... & ... & ... \\
a_{\mu-1} & \rightarrow & a_{\mu-1,1}a_{\mu-1,2}...a_{\mu-1,n_{\mu-1}}
\end{array}
$$

  where each $a_{i,j} \in \Sigma, 0 \leq i < \mu$. Given an initial string A_0, the tag system tags the word associated with the leftmost letter of A_0 at the end of A_0, and deletes the first v symbols of A_0.

Further definitions and notational conventions

Definition 1 *The halting problem for tag systems is the problem to determine for a given tag system and any initial string* A_0 *whether the tag system will halt.*

Definition 2 *The reachability problem for tag systems is the problem to determine for a given tag system T, a fixed initial string* A_0 *and any arbitrary string A over the alphabet* Σ*, whether T will ever produce A when started with* A_0.

Definition 3 *Let T be a tag system with a deletion number v with* μ *symbols and words* $w_0, w_1, ..., w_{\mu-1}$. *Then:*

a. *We shall write* l_i *to indicate the length of a word* w_i, $l_{\mathbf{max}}$ *and* $l_{\mathbf{min}}$ *denote the length of the lengthiest word* w_i *rsp. the length of the shortest word* w_j *of* T, $0 \leq i, j < \mu$.

b. $\#a_i$ *denotes the total sum of the number of* a_i*'s in* $w_0, \ldots w_{\mu-1}$.

c. $\dot{x}$ *rsp. x indicate an odd rsp. an even number.*

d. *Given a string* $A = a_1 a_2 \ldots a_{l_A}$, *we will say that A is entered with shift x, when the tag system erases its first x symbols, the first symbol scanned in A being* a_{x+1}.

Some basic results

- **Post 1921** Proof that halting and reachability problem for tag systems with $v = 1$ or $\mu = 1$ or $v = \mu = 2$ are solvable. Never published, but the proof for the case $v = \mu = 2$ involved "*considerable labor*"
- **Minsky 1961** Any Turing machine can be represented in a tag system with $v = 6$, and thus tag systems are recursively unsolvable.
- **Minsky and Cocke, 1961** Any Turing machine can be represented in a tag system with $v = 2$
- **Wang 1963**

  a. Proof solvability halting and reachability problem for tag systems with $v = 1$

  b. For any tag system T, if $l_{max} \leq v$ or $l_{min} \geq v$, then its halting and reachability problem are recursively solvable.

⇒ **Both μ and v can be regarded as decidability criteria for tag systems.**

Three classes of Behaviour

- **Example of Periodicity**: $v = 3, 1 \rightarrow 1101, 0 \rightarrow 00, S_0 = \mathbf{0}01\mathbf{1}01$
  $\mathbf{0}01\mathbf{1}01$
  $\mathbf{1}01\mathbf{0}0$
  $\mathbf{0}01\mathbf{1}01$
- **Example of Halt**: $v = 3, 1 \rightarrow 1101, 0 \rightarrow 00, S_0 = \mathbf{0}01\mathbf{0}01$
  $\mathbf{0}01\mathbf{0}01$
  $\mathbf{0}01 \rightarrow \epsilon$
- **Example of Unbounded growth**: $v = 2, 1 \rightarrow 101, 0 \rightarrow 11, S_0 = \mathbf{0}0\mathbf{1}1\mathbf{0}1$
  $\mathbf{1}1\mathbf{0}1\mathbf{1}1$
  $\mathbf{0}1\mathbf{1}1\mathbf{0}1$
  $\mathbf{1}1\mathbf{1}0\mathbf{1}1\mathbf{1}$
  $\mathbf{1}0\mathbf{1}1\mathbf{1}1\mathbf{0}1$
  $\mathbf{1}1\mathbf{1}1\mathbf{0}1\mathbf{1}0\mathbf{1}$
  $\mathbf{1}1\mathbf{0}1\mathbf{1}0\mathbf{1}1\mathbf{0}1$

$\Rightarrow$ **Proving that any tag system with $v = \mu = 2$ will halt, become periodic or show unbounded growth for arbitrary initial conditions in a finite number of steps, results in proof solvability halting and reachability problem for this class.**

How to prove the solvability of a class of tag systems TS(μ, ν)? Two problems.

1. Two times infinity:
   - For each tag system, an infinite number of initial conditions
   - An infinite number of tag systems
2. The words can have arbitrary lengths

Three basic cases, more subcases (and subsubcases)...

- Wang 1963 ⇒ Only consider cases with $l_0 < 2, l_1 > 2$ (symmetrical case is equivalent) ⇒ Three basic cases: $w_0 = \epsilon, w_0 = 1, w_0 = 0$
- Further differentiation through parameters: l_1, parity of l_1, #1, parity of number of 0's separating consecutive 1's in w_1 ⇒ parameters allow for the determination of certain threshold values which divide each case in a finite class of TS that always halt or become periodic and an infinite class that always shows either unbounded growth, halt or periodicity.

Examples explaining the parameters

Parameter 1: l_1

- $w_1 = 000, w_0 = 1 \Rightarrow$ periodicity or halt
- $w_1 = 0000000000, w_0 = 1 \Rightarrow$ unbounded growth

Parameter 2: Parity of l_1

- $w_1 = 1010, w_0 = \epsilon \Rightarrow$ unbounded growth or halt depending on parity length initial condition
- $w_1 = 10100, w_0 = \epsilon \Rightarrow$ periodicity

Parameter 3: $\#1$

- $w_1 = 101, w_0 = 0 \Rightarrow$ periodicity
- $w_1 = 10101, w_0 = 0 \Rightarrow$ unbounded growth

Parameter 4: Parity of $\#0$ **separating 1's in** w_1

- $w_1 = 1001, w_0 = \epsilon \Rightarrow$ periodicity
- $w_1 = 100010, w_0 = \epsilon \Rightarrow$ unbounded growth

The table method

Given a tag system T with deletion number v, words $w_0, \ldots, w_{\mu-1}$ and alphabet $\Sigma = \{a_0, \ldots, a_{\mu-1}$ then:

Step 1 For each of the words, write down all the strings that can be produced by entering it with different shifts 0, 1,..., v - 1. If any of the strings produced in this way has already been written down or is equal to the empty string ϵ, it is marked.

Step 2 For each of the strings left unmarked, write down all the strings that can be produced by entering it with different shifts 0, 1,..., v - 1. If any of the strings produced in this way has already been written down or is equal to the empty string ϵ, it is marked.

Step 3 If all strings produced in the previous step have been marked, stop, if not, goto step 2.

⇒ **Basic tool to prove solvability of halting and reachability problem for a given tag system. If it halts, the solution immediately follows, if not, it is still possible to deduce certain structural properties that lead to the result.**

Examples of some cases proven through the table method.

Case I.2. $w_0 = \epsilon$, $\#1 = 1$, $l_1 \equiv 0 \bmod 2, w_1 = 0^{x_1} 10^{y_1}$

	w_0	w_1
S_0	ϵ	ϵ
S_1	ϵ	w_1 ✓

Case II.2. $w_0 = 1, \#1 = 2, l_1 = 3$. There are three different tag systems to be taken into account here.

Table 2: Case $0 \to 1, 1 \to 100$

	w_0	w_1	$w_1 w_0$	$w_0 w_1$
S_0	w_1	$w_1 w_0$	$w_1 w_0$ ✓	$w_1 w_0$ ✓
S_1	HALT	w_0 ✓	$w_0 w_1$	$w_1 w_0$ ✓

Table 3: Case $w_0 = 1, w_1 = 010$

	w_0	w_1	$w_0\,w_0$
S_0	w_1	$w_0\,w_0$	w_1 ✓
S_1	w_1	w_1 ✓	w_1 ✓

Table 4: Case $w_0 = 1, w_1 = 001$

	w_0	w_1	$w_0\,w_1$	$w_1\,w_0$
S_0	w_1	$w_0\,w_1$	$w_1\,w_0$	$w_0\,w_1$ ✓
S_1	w_1	w_0 ✓	$w_0\,w_1$ ✓	$w_0\,w_1$ ✓

More difficult subcases for case III ($w_0 = 0$)

SubSubcase 3.3.2.1. $\#1 = 2$, $l_1 \equiv 0 \bmod 2$, $w_1 = t_1 1 \dot{x}_1 1 \dot{s}_1$

From w_1:

Shift 1 : A sequence of 0's ✓

Shift 0 :

$$A_1 = t_2\, w_1\, \lfloor \dot{x}_1/2 \rfloor\, w_1\, s_2 \tag{1}$$

From (1) we get:

- If $\dot{s}_1 + \lfloor \dot{x}_1/2 \rfloor + t_1$ even then:

  Shift a :

$$t_3\, A_1\, 0^{n_1} \;\checkmark \tag{2}$$

  Shift b :

$$t_3 0^{n_1}\, A_1 \;\checkmark \tag{3}$$

- If $\dot{x}_1 + \lfloor \dot{x}_1/2 \rfloor + t_1$ odd then:

  Shift a :

$$A_2 = t_4\, A_1\, \lfloor \dot{x}_1/4 \rfloor\, A_1\, s_3 \tag{4}$$

  Shift b : A sequence of 0's ✓

From (4):

- $\dot{x}_1 + s_2 + \lfloor \dot{x}_1/2 \rfloor + t_2 + t_1$ is even

  Shift a:

$$t_5 A_2 0^{n_2} \checkmark \tag{5}$$

  Shift b:

$$t_5 0^{n_2} A_2 \checkmark \tag{6}$$

- $\dot{x}_1 + s_2 + \lfloor \dot{x}_1/2 \rfloor + t_2 + t_1$ is odd

  Shift a :

$$A_3 = t_6 A_2 \lfloor (x_1 - 1)/8 \rfloor A_2 s_4 \tag{7}$$

  Shift b :

A sequence of 0's ✓

Two Possibilities

- $\exists n$: length $\dot{s_1} + s_2 + s_3 + \ldots + s_n + \lfloor (x_1 - 1)/2^n \rfloor + t_n + \ldots + t_2 + t_1$, separating two consecutive A_{n-1} in A_n ($n \in \mathbb{N}, A_0 = w_1$) is even ⇒ Periodicity
- $\exists n$: length $\dot{s_1} + s_2 + s_3 + \ldots + s_n + \lfloor (x_1 - 1)/2^n \rfloor + t_n + \ldots + t_2 + t_1$, separating two consecutive A_{n-1} in A_n ($n \in \mathbb{N}, A_0 = w_1$) is odd ⇒ Unbounded growth or halt

⇒ It can be determined in a finite number of steps for any tag systems from this class whether there exists an n such that $\dot{s_1} + s_2 + s_3 + \ldots + s_n + \lfloor (x_1 - 1)/2^n \rfloor + t_n + \ldots + t_2 + t_1$ will ever become even:

Lemma 1 *For any tag system from the class 3.3.2.1. it can be proven that there is always an n, $n \in \mathbb{N}$ such that for any $i \geq n$ the sequence of 0's $\dot{s_1} + s_2 + s_3 + \ldots + s_i + \lfloor (x_1 - 1)/2^i \rfloor + t_i + \ldots + t_2 + t_1$ between a pair of A_{i-1} in A_i is of the same length as $\dot{s_1} + s_2 + s_3 + \ldots + s_n + \lfloor (x_1 - 1)/2^n \rfloor + t_n + \ldots + t_2 + t_1$.*

Proof. To prove the lemma, consider again the sequence:

$$A_2 = t_4 1 s_1 + s_2 + \lfloor \frac{x}{4} \rfloor + t_2 + t_1 1 s_3 \tag{8}$$

Since for any tag system from this class, any sequence of 0's ultimately converges to ϵ, while for every iteration, each s_i rsp. t_i is converted to s_{i+1} rsp. t_{i+1}, the tag system will ultimately produce a sequence:

$$A_n = X_{n-1} s_1 + s_2 + s_3 + \ldots + s_n + \lfloor \frac{x}{2^n} \rfloor + t_n + \ldots t_3 + t_2 + t_1 Y_{n-1} \tag{9}$$

from (8) such that $s_n = \lfloor \frac{x}{2^n} \rfloor = t_n = \epsilon$, with X_{n-1} rsp. Y_{n-1} equal to A_{n-1} minus its rightmost rsp. leftmost sequence of 0's. This string can be rewritten as:

$$A_n = X_{n-1} s_1 + s_2 + s_3 + \ldots + s_{n-1} + t_{n-1} \ldots t_3 + t_2 + t_1 Y_{n-1} \tag{10}$$

If the tag system now scans A_n it produces:

$$A_{n+1} = X_n s_1 + s_2 + s_3 + \ldots + s_{n-1} + s_n + + \lfloor \frac{x}{2^n} \rfloor + t_n + t_{n-1} \ldots t_3 + t_2 + t_1 Y_n \tag{11}$$

However, since $t_n = s_n = \epsilon$, (10) = (11) and we have thus proven the lemma. $\square$

Discussion: Some open questions.

- Possibilities for finding a shorter more elegant proof?
- Applicability of the methods of the proof for other classes of tag systems, e.g. TS(3,2) or TS(2, 3)?
- What about other decision problem? Does the result exclude universality for this class?