A simple P-complete problem and its representations by language equations

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Representing hardest sets

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1. Turing machines
2. Linear context-free grammars
3. Trellis automata

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Problem

Find small Boolean grammars for P-complete sets.

Boolean grammars

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$$A \to \alpha$$

"If w is generated by α, then w is generated by A".

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Boolean grammars (Okhotin, 2003) Rules of the form

$$A \to \alpha_1 \& \ldots \& \alpha_m \& \neg\beta_1 \& \ldots \& \neg\beta_n$$

"If w is generated by each α_i and by none of β_j, then w is generated by A".

Formal definition

- Quadruple $G = (\Sigma, N, P, S)$, where $S \in N$ and rules in P are

$$A \rightarrow \alpha_1 \& \ldots \& \alpha_m \& \neg\beta_1 \& \ldots \& \neg\beta_n, \quad \text{with } A \in N,\ \alpha_i, \beta_i \in (\Sigma \cup N)^*$$

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- Language equations for G:

$$A = \bigcup_{A \to \alpha_1 \& \ldots \& \alpha_m \& \neg\beta_1 \& \ldots \& \neg\beta_n \in P} \left[\bigcap_{i=1}^{m} \alpha_i \ \cap \ \bigcap_{j=1}^{n} \overline{\beta_j} \right] \qquad (*)$$

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- Another semantics: Kountouriotis et al. (DLT 2006).

Properties of Boolean grammars

- Generate $\{a^n b^n c^n \mid n \geqslant 0\}$, $\{ww \mid w \in \{a,b\}^*\}$, $\{a^{2^n} \mid n \geqslant 0\}$, etc.

Example

$$\begin{array}{lcl} S & \to & AB \& \neg DC \\ A & \to & aA \mid \varepsilon \\ B & \to & bBc \mid \varepsilon \\ C & \to & cC \mid \varepsilon \\ D & \to & aDb \mid \varepsilon \end{array}$$

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- Practical parsing methods: recursive descent, generalized LR.
- ✓ Completion of the context-free grammars.

Trellis automata

(one-way real-time cellular automata)

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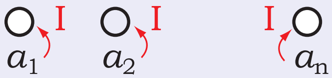

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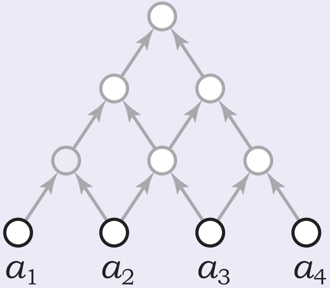

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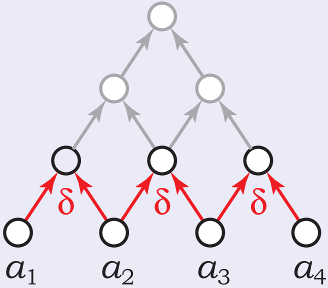

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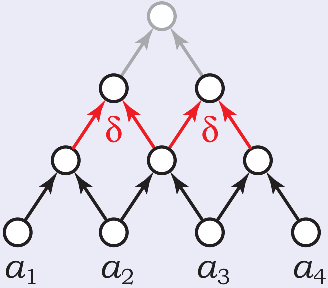

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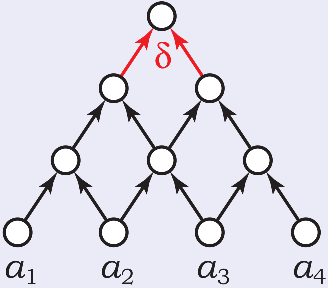

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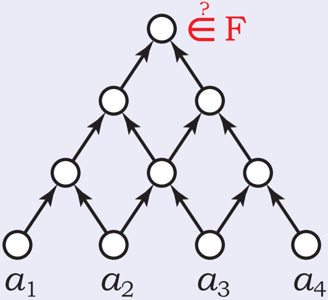

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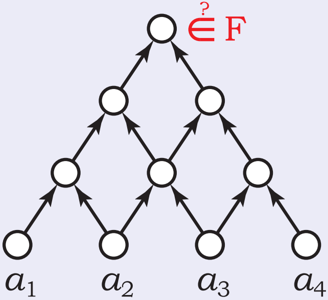

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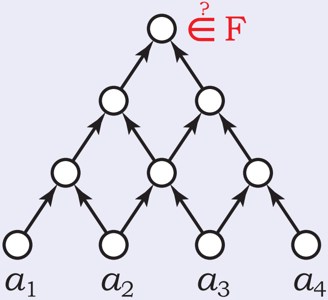

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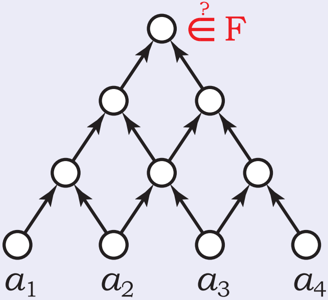

- Equivalent to linear conjunctive grammars.
- Closed under $\cup, \cap, \sim$, not closed under concatenation and star.
- Can recognize $\{wcw\}$, $\{a^n b^n c^n\}$, $\{a^n b^{2^n}\}$, VALC.

P-completeness results

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Problem

Construct smaller descriptions.

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Circuit Value Problem (Ladner, 1975)

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Planar Circuit Value Problem (Goldschlager, 1977)

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✓ A new variant of CVP.

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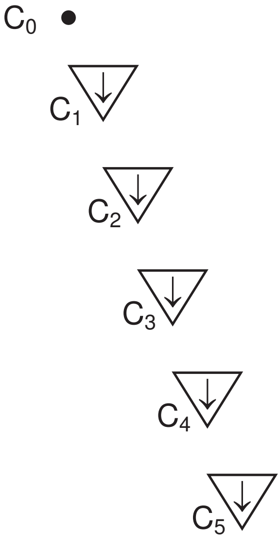

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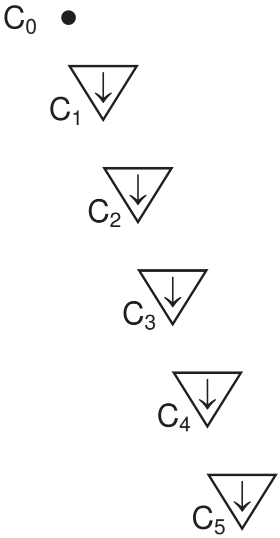

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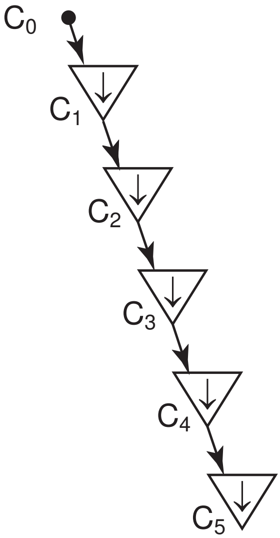

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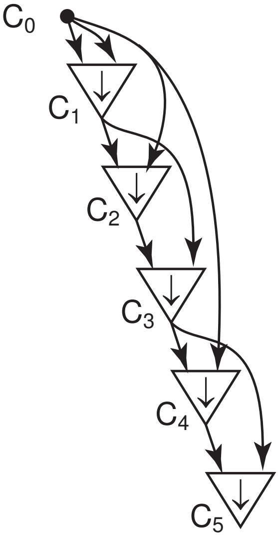

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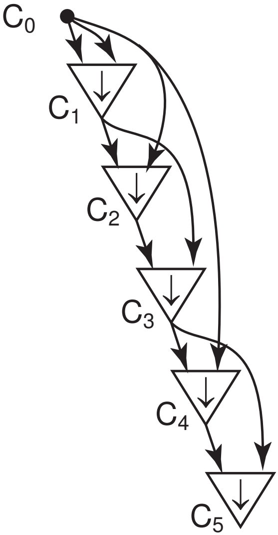
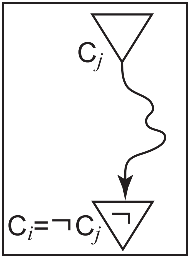


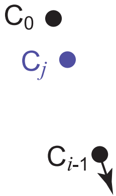
Theorem

Sequential NOR CVP is P-complete.

Reducing CVP to Sequential NOR CVP

- Simulating negation.





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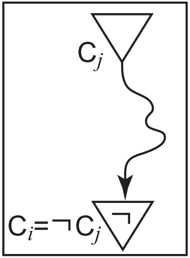

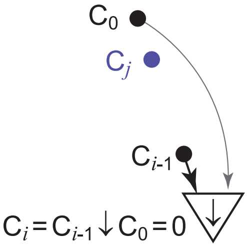

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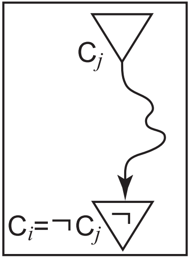

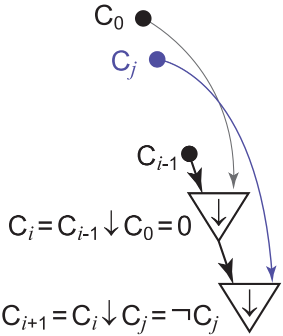

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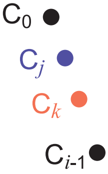

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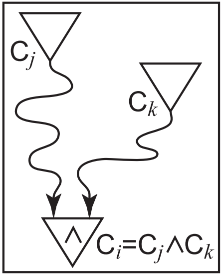

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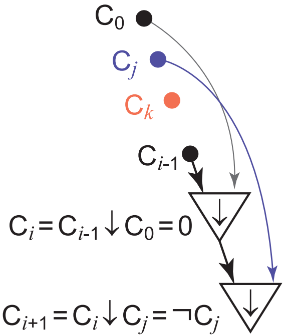

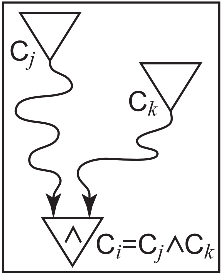

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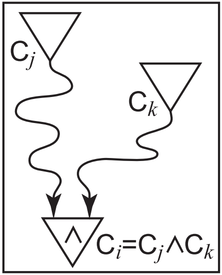

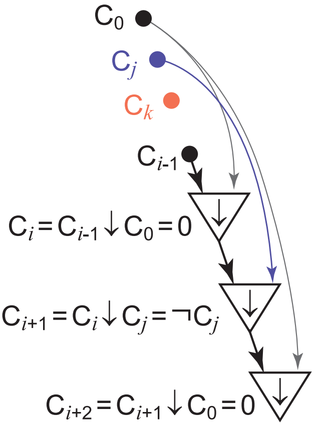

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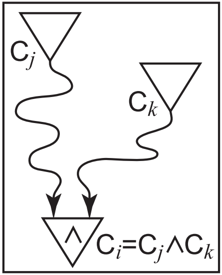

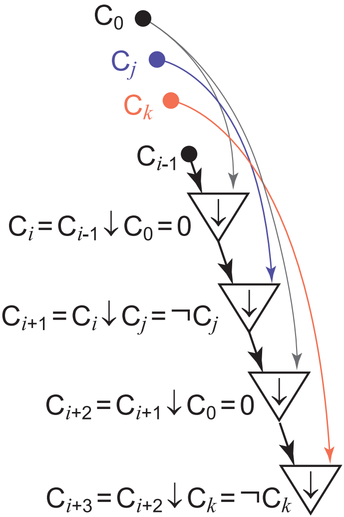

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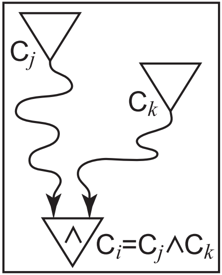

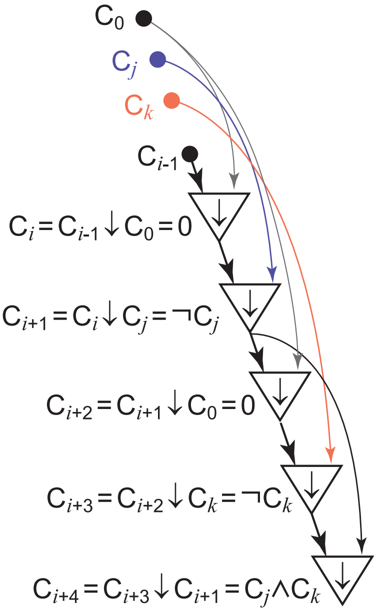

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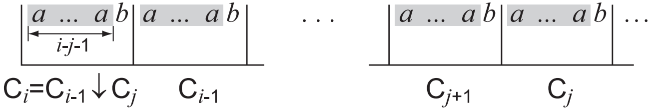

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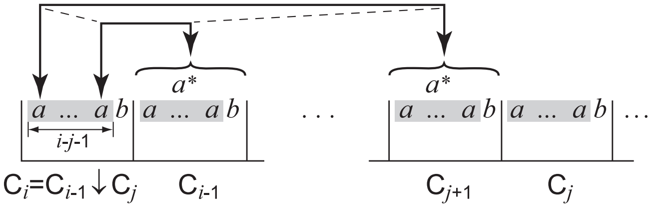

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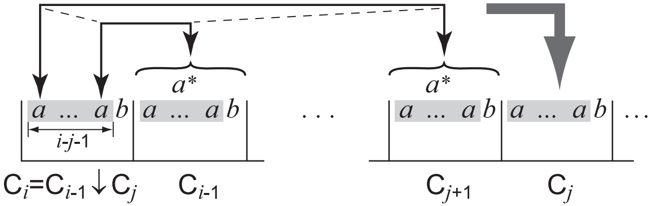

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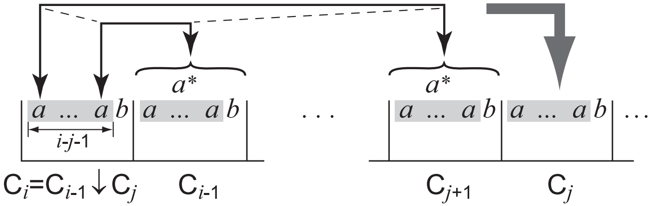

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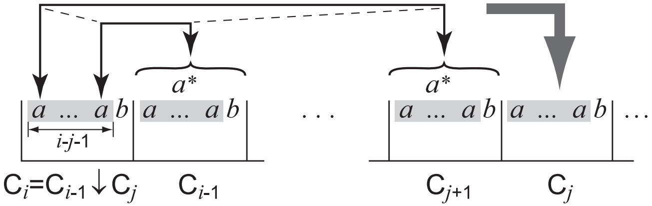

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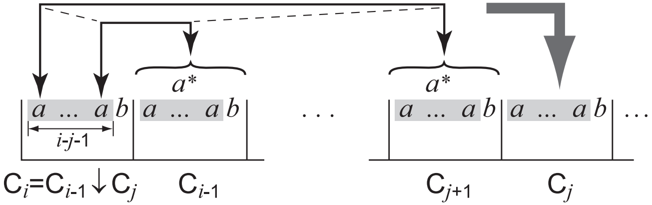

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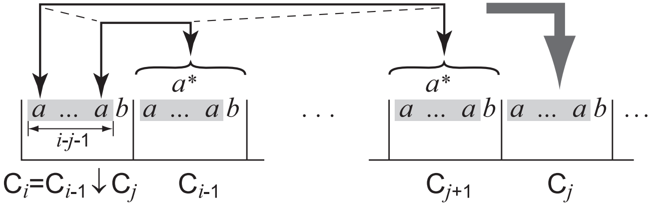

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