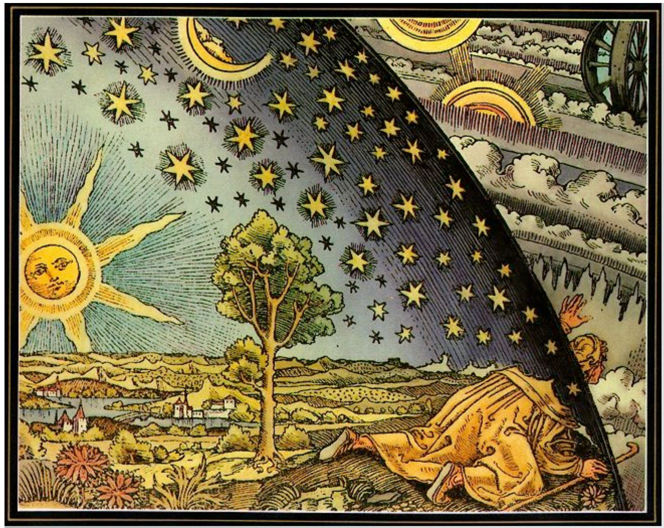
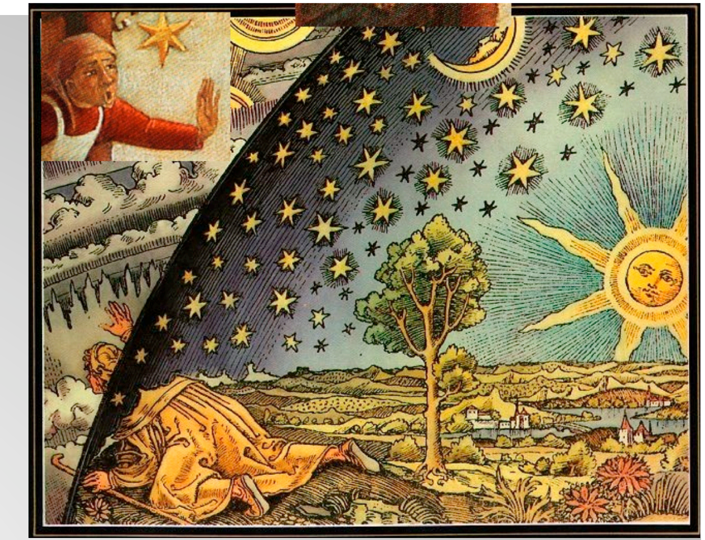


Halfway Up to the Mathematical Infinity



Non-Discreteness, Non-Sequentiality, and Non-Locality of Post-Cantorian Transfinite Designs and the Prospectives of Quantum Formal Reasoning and Computing

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The Early Dawns of the Mathematical Infinity

« Heraclitus is the one who first declared the nature of the infinite and first grasped nature as in itself infinite, that is, its essence as process. »

Georg Wilhelm Friedrich Hegel

- Two Basic Mathematical Habitats, N and R

More than two thousand years before Georg Cantor has entered the scene of Mathematical infinity, two types of infinite totalities, the denumerable N and the continuum R, were already well known « experimentally » to the mainstream Western mathematical and communities, starting apparently with those of Ancient Greece.

- Dismissal of the Actual Infinite

Still, accepting with the characteristic realism *the utility of the idea of a potential infinity*, Greeks were dismissing *the actual infinity*. The following statement from Aristotle's Physics (Phys. III, 7, 207b27-34), anticipates the « ultraintuionistic criticism » of the late XXth century and explicitly — quite possibly, for the first time — raises the problem of the philosophical causality principles universally but mostly implicitly underlying all mathematical queries :

- « *Our account does not rob the mathematicians of their study, by disproving the actual existence of the infinite ... In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. It is possible to have divided in the same ration as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes.* »

- It is instructive to compare Aristotle's courageous and open to compromises and amendments realism to the panicky and eschewing any ontological responsibility policy of a XXth century formalist:

  « *My position concerning the foundations of Mathematics is based on the following two main points or principles.*

  (i) Infinite totalities do not exist in any sense of the word (i.e., either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, meaningless.

  (ii) Nevertheless, we should continue the business of Mathematics "as usual", i. e., we should act as if infinite totalities really existed. » (Abraham Robinson, 1965)

- Ontological and Logical Incompatibility of Discreet and Continuous

  The Greeks were also fully conscious of the ontological differences, if not conceptual incompatibility between two infinities, N and R, reflecting two different types of human experiences: the external and omnipresent (i.e., present everywhere at the same time and continuous), and the intellectual, inner, local, sequential, and discreet, — as it is abundantly clear from the famous paradoxes advanced by Zeno of Elea (ca. 490 BC – ca. 430 BC)

- Zeno, like Heraclitus before him, clearly doubted that the two infinities could be reconciled: one can run, but one cannot adequately understand or otherwise fully and exhaustively capture in a pure intellectual way, and in particular formally, this phenomenon, because our reasoning proceeds by finite, sequential steps, by discreet, local markers or buoys occasionally dropped in the mowing waters, whereas our and the river movement – a mystery in itself – is, as an ever-changing face of a river, continuous:
- « *On those stepping into the same river, other and other waters flow.* » (Heraclitus of Ephesus, ca. 535 – ca 475 BC)

- Local Causality Worries

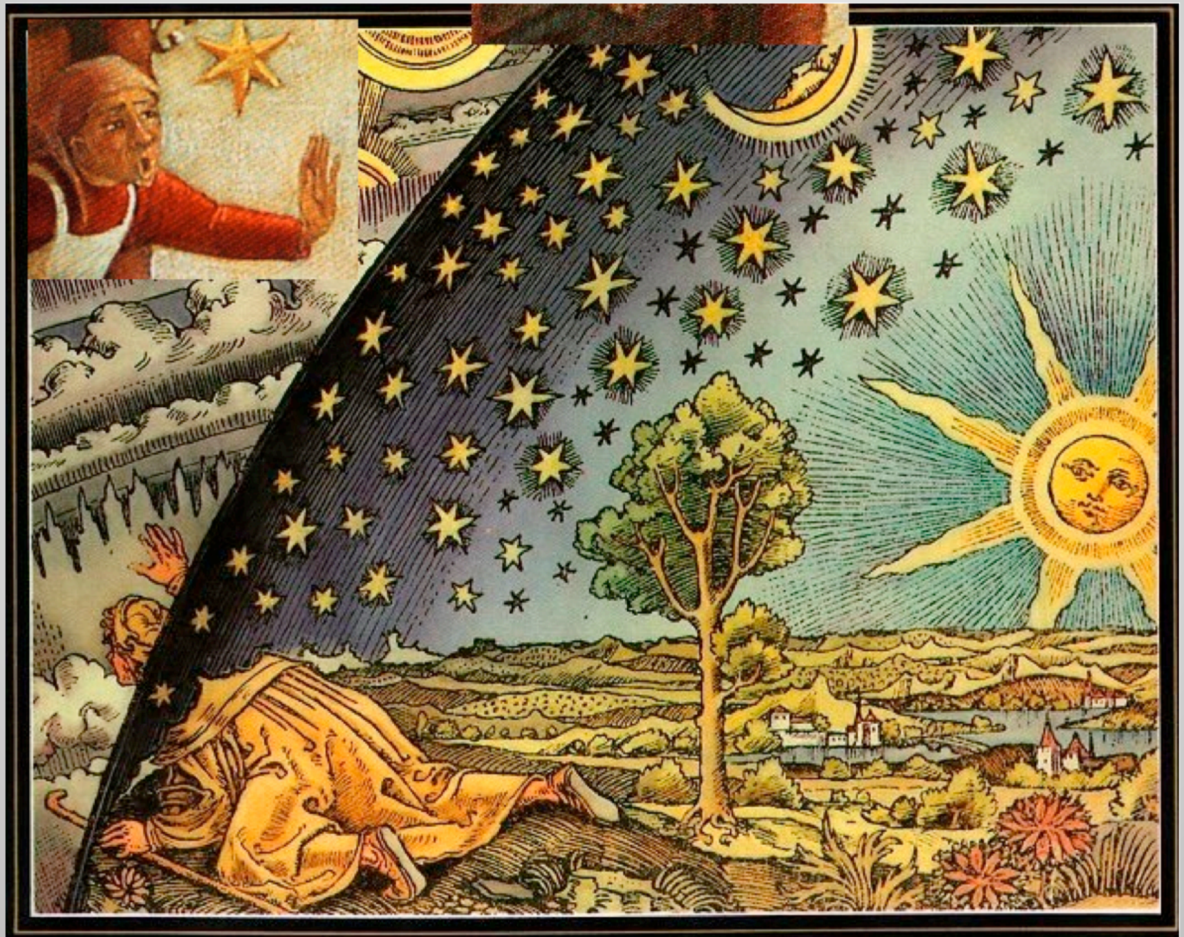
Zeno's paradox "Achilles and Tortoise" displays the thorough understanding of the underlying Aristotle's pragmatism *Local Causality Principle* — the local, finite, and sequential propagation of all action in the Universe:

Consider the setting of Zeno's paradox "Achilles and Tortoise" consisting of two independent, simultaneously unfolding processes: Tortoise's steady advancement along the real line R and Achilles ever shortened strides along the same line. Achilles strategy and its « man's » realization are familiar to us: they are discrete, sequential, recursive, abiding by local causation. But what about Tortoise's ? Why is Zeno suggesting that hers is a more enigmatic logical and mathematical enterprise?

- It is not by chance that Zeno has chosen a man, Achilles, who has to catch up with a fabulously slow and yet mysteriously unattainable Tortoise.

  The answer, we believe, is that Zeno allots to the Tortoise the « divine ability » to calmly swim along the flow of the Continuum, whereas Achilles, the man, has no choice but to dangerously hang over the bottomless depths of the primeval continuum stream, being able to advance only by jumping from a one point-size location of this stream — the location carefully chosen and temporally frozen solid — to the next one

« Our point of view is to describe the mathematical operations that can be carried out by finite beings, man's mathematics for short. In contrast, classical mathematics concerns itself with operations that can be carried out by God. » Errett Bishop, 1985



- In other words, Zeno's paradox suggests that the real Continuum, in all its unfathomable richness, somehow « divinely » exists (taking Errett Bishop at his word !), both at any given moment and at any point of our probing into it — the scenario clearly implying some non-local causation — whereas, according to our experience, any precise mathematical or numerical question concerning the continuum cannot be humanly resolved but within an appropriate discrete, sequential mathematical framework abiding by local causation.

Here Comes Georg Cantor, with His Actual Infinities: Called Ordinals

« Following the finite there is a transfinite, that is an unbounded ascending ladder of definite modes, which by their nature are not ‡finite but infinite, but which just like the finite can be determined by definite well-defined and distinguishable numbers. »
Georg Cantor, 1983

Became acquainted with Cantor's transfinite numbers, David Hilbert didn't mince the words to praise it:

« This appears to me to be the most admirable flower of the mathematical intellect and in general one of the highest achievements of pure rational human activity. »

Aristotle and Robinson Get It Wrong: Evidences of « Necessary Uses » of the Transfinite

- Alan Turing: Ordinals as Iterative Transfinite Codes

  In 1949, Alan Turing has given a remarkable general interpretation of explicitly defined countable ordinals as succinct symbolic notations for iterative algorithmic structures with multiple loops: the interpretation which inspired a series of remarkable results on the program verification.

- Elementary Tokens of Incompleteness

  In 1944, Reuben L. Goodman designed an elementary problem in Peano arithmetic whose proof necessary uses the ordinal $\varepsilon 0$.

- Ordinal Analysis

  In 1936, Gerhard Gentzen proved the ordinal «strength» of Peano arithmetic being $\varepsilon 0$.

Georg Cantor's Powerset Construction:

The Comprehensive Ever-Emerging System of Disparate Discrete Ascending Ladders Leading to Discernable Individual Points of the Continuum

- Georg Cantor has resolved (1874) one particular aspect of « Achilles and Tortoise » paradox, formally confirming Zeno's intuition that the Continuum is infinitely richer than the Discreet.
- Using his method of one-to-one correspondence to both identify infinite totalities of the same power, or equipotent totalities, and to establish which of two totalities is bigger, Cantor gave his proofs, topological and diagonal, that the set of reals is uncountable.

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Georg's Cantor's Preconceived, Peremptory Transfinite Design

Encouraged by his third, Powerset proof of the uncountability of the Continuum, Cantor proceeded to definitely « dislodge » the non-sequential and not abiding by the principle of local causality specificity of the Continuum, clearly discerned already by the Greeks, — by subjecting it to invented by him universal transfinite counting procedure, well-ordering:

- (1) Countable Ordinals Form a Well-Ordered Set, the Smallest Uncountable Ordinal, Thus Extending the Ordinal Hierarchy beyond the Countable.
- (2) The Powerset Device is Iteratively Applicable, Extending the Hierarchy of Sets beyond the Continuum.
- (3) Hence Cantor's « Fundamental Law of Thought »: Every Set Could Be Actually Well-Ordered

Both Cantor's Assumptions Fail, Voiding His Fundamental Law of the Universal Applicability of the Well-Ordering :

- **Ordinal Deadlock:** In the light of the Ordinal Analysis results, the collection ω1 of countable ordinals is the authentic, universal, ever emerging and never completed formal ordinal measure scale of the power and sophistication of iterative logical arguments and methods. In particular, ω1 is a proper class.
- **Powerset Debacle:** « This does not prove the legitimacy of the [universal] Powerset principle. For the argument is not: we have a perfectly clear intuitive picture of the continuum, and the Powerset principle enables us to capture this set-theoretically. Rather, the argument is: the Powerset principle was revealed in our attempts to make our intuitive picture of the continuum analytically clearer; in so far as these attempts are successful, then the Powerset principle gains some confirmatory support. » (Michael Hallett, 1984)

Salvaging Cantor's Transfinite Design and the Soundness of Mathematical Logic

- To save Cantor's transfinite design, a vast axiomatic machinery has been invented, developed, and exploited during the last century, starting with Ernest Zermelo's Axiom of Choice and his axiomatic, known today as Zermelo-Fraenkel's, ZF or ZFC, and
- Simultaneously, Bertrand Russel and his followers, sympathisers, and demurrers confronted the logical paradoxes and the intrinsic to them *self-referentiality*, or as it is known today *circularity* of formal arguments — such as Liar paradoxes, «*This sentence is false*» or «*This sentence is not true*», or circular and overreaching sets definitions, like: «*The set of all sets*», «*The set of all sets which are not an element of itself*» , «*The ordinal of all ordinals*», etc.

Constative versus Performative Axiomatic Paradigms

- Our main foundational argument concerns the nature of axioms of Set Theory, from ZF to ZFC to Axioms of Determinacy to Axioms of Large Cardinals, which are typically performative inductive principles and procedures of emergence of set-theoretical entities acting on swaths of transfinite ordinals and cardinals:

  Turing-like Halting Barrier for Performative Set-theoretical Axioms of Iterative Nature. There exists no general logical criterion or procedure to establish the eventual truth or ontological relevancy of a performative set-theoretical axiom of iterative nature postulating the existence of transfinite objects outside the already existing transfinite scale – otherwise than either to prove its inconsistency or « to run » the theory completed with the new axiom until it would be discovered an independent « necessary use » of the object in question.

ZF-Based Axiomatic as High-Level Programming Language

- There is no other modern domain of formal studies where this post-modern paradigm is so pronounced as in Set Theory and its ZF-based axiomatic. The trouble is hidden exactly where our ZF pride resides: in the powerful built-in iterative mechanisms of set generation.

- In other words, ZF has gained in its creative and performative power on the expense of its descriptive power, becoming a sophisticated programming language, which is successfully mimicking some aspects of the Mathematical Infinity but whose main thrust lies with the providing to advanced « users » sophisticated options of generation of, and manipulation with artificial transfinite totalities, similarly to computer graphic imagery of video games – with the Axiom of Determinacy opening the advent of Infinite Games.

Axiomatic Consistency and Independence Are Not Substitutes for Proof-theoretic Efficiency

- Thanks to this interpretation, finds its proper place, in particular, the puzzling and disconcerting predominance in modern set theory of results on Z F consistency and independency:

- « *When modern set theory is applied to conventional mathematical problems, it has a disconcerting tendency to produce independence results rather than theorems in the usual sense. The resulting preoccupation with 'consistency' rather than 'truth' may be felt to give the subject an air of unreality.* » (Saharon Shelah, 1992

Consistency and Independence as Program Verification Criteria

- Moreover, we claim that all such consistency results are just the instances of successful program verification. In other words, the totalities in question are, in fact, pure mathematical notations not related to any set-theoretical «reality» outside the tight structure of their definitions and relationships. Still, the related results might turn out one day to be relevant to, if not suggestive of our permanently evolving iterative programming ability:
- « *Only the first few levels of the cumulative hierarchy bear any resemblance to external reality. The rest are a huge extrapolation based on a crude model of abstract thought processes. Gödel himself comes close to admitting as much.* » (Stephen G. Simpson, 1999)

Why Non-Locality of Physical Universe ?

- For at least two millennia – starting with the paradoxes of Zeno – the practitioners of logic and mathematics were facing the Discrete–Continuum and closely related to it Sequential–Parallel dualities of formal world descriptions. And since at least two centuries, science has been alerted to the pertinence of the Local–Non-Local duality, to become eventually dominated by Laplace's Local Causality, or Locality Principle — the local propagation of all action in the Universe — and its implications.
- This apparently definitive victory of the Discrete–Sequential–Local over the Continuum–Parallel–Non-Local came at a price; thus, Albert Einstein, the genuine discoverer of Quantum Mechanical Non-Locality which he was able to appreciate only as a paradox, has spent the last twenty years of his research activity on Local-Causation theories of everything. Ultimately, the Non-Locality – even if still referred to as counter-intuitive – has won over the physical research and Quantum Computing.

Why Non-Locality of Formal Reasoning ?

- From its explicit, external, real-world imperativeness, the Locality Principle has been implicitly assumed to hold for the intellectual activity of man, too, — starting with the Perfect and Omniscient Intellect invented by Laplace..
- Thus, Georg Cantor has extended into the Transfinite the limits of both Discrete, Sequential, and Local, with the explicitly stated purpose to definitely submit to them their Continuum and, respectively, Parallel and Non-local counterparts.
- Then, in the middle of the first foundational crisis, it was the turn of David Hilbert to enter the fray with his global argumentative reductionism designed to reduce all mathematical reasoning to formalized mechanical proof-theoretic procedure

… Why Non-Locality of Formal Reasoning ?

- Following suit, Emil Post, Alonso Church, and Alan Turing have advanced their computing schemata of pure Sequential and Local Causality nature, claiming that such schemata ultimately cover all imaginable computing activity (Church–Turing Thesis).
- It is this theory which, in the absence of any formal alternative of a comparable importance, is deemed today by many to somehow stand for the theory of human thinking, with Cantor's aforementioned « fundamental law of thought » being a clear tributary to such a reductionist, outdated, but still beguiling philosophical appeal

Consistency and Independence as Program Verification Criteria

- Now, how about Mathematics of a free surfing on the Continuum of real line, starting with the logically unimpeded Classical Analysis and passing by the famous, ZF-independent, and still open Suslin's conjecture?
- **Suslin's Conjecture.** Let K be a linearly ordered set without the first or last element, connected in order topology, with no uncountable family of pairwise disjoint open intervals. Is K isomorphic to the real line R ?
- « *Only the first few levels of the cumulative hierarchy bear any resemblance to external reality. The rest are a huge extrapolation based on a crude model of abstract thought processes. Gödel himself comes close to admitting as much.* » (Stephen G. Simpson, 1999)

Non-Locality Characterization of the Continuum

- (1) The striking feature of classical analytic machinery conceived to deal with, and perfectly adapted to the Continuum habitat, and which immediately distinguish it from methods and theories subject to the discrete, Ordinal Analysis related treatment, is tits intrinsic, inextinguishable, fundamental, outside of the Continuum not existing and not obtainable non-locality — in the sense this term is understood in Quantum Information Processing, or QIP, for short
- (2) Any phenomenologically and ontologically faithful or at least fully relevant Axiomatisation of the Continuum should include a Non-locality Postulate, or Non-locality Axiomatic Scheme, to formally account for the following property of the Continuum:

  All «points», or «elements» of the Continuum are at any moment and non-locally, i.e., universally, simultaneously, and independandly accessible.

Circularity as a Symptom of Non-Locality

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Conclusion: Quantum Theoretical Insights and Their Suggestive Power

- With the discovery of quantum-theoretical non-locality and quantum entanglement and its applicability to QIP, non-local causation became the source of dramatic improvements in the efficiency of algorithms.
- It's true that, at the present juncture, the prospects of a radical improvement of the strength and versatility of algorithmic methods of QIP are seriously hampered by the extreme narrowness of the only open today window of genuinely quantum-theoretical opportunities, the entanglement phenomenon.
- However, this should not prevent us from fully appreciating an epistemological and explanatory potential of the non-locality, bearing on a mysterious duality between non-discrete, non-locl continuum phenomena in (logically unrestricted) Mathematics and non-discrete and non-local nature of a genuine Mathematical Reasoning.