Automata on linear orderings

Olivier Carton

LIAFA
Université Paris Diderot

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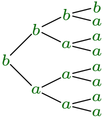
Infinite computations

There are automata on

- infinite words,
- bi-infinite words,
- transfinite words,
- trees 

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Words

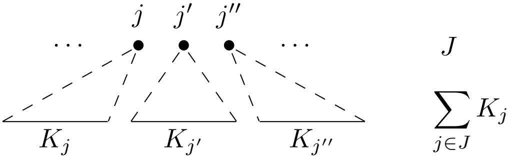
word: function from an ordering J (its length) to an alphabet A

Example $(a_j)_{j \in J}$	length J
$abaab$	5
$(ab)^\omega$	ω
$(ab)^{-\omega}$	$-\omega$
$(ab)^\omega c$	$\omega + 1$
$((ab)^\omega c)^\omega$	$\omega^2 = \omega \cdot \omega$
$(a^{-\omega} b^\omega)^\omega$	$\zeta \cdot \omega$
a^{ω^ω}	ω^ω
$\begin{cases} r \mapsto a & \text{if } r = n/2^m \\ r \mapsto b & \text{if } r \in \mathbb{Q} \setminus \{n/2^m \mid m \geq 0\} \end{cases}$	η
$\begin{cases} x \mapsto a & \text{if } x \in \mathbb{Q} \\ x \mapsto b & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$	λ

Operations on orderings and words

- Mirror image: $-J$ backwards linear ordering
- Sum of orderings and concatenation of words
  - $J + K$ places K to the right of J

  - if x and y are of length J and K, xy is of length $J + K$.
- Generalized sum and concatenation
  - $\sum_{j \in J} K_j$ generalized sum



  - If each word x_j is of length K_j, $\prod_{j \in J} x_j$ is of length $\sum_{j \in J} K_j$

Scattered orderings

- dense ordering: $\forall i < j \; \exists k \quad i < k < j$.
- scattered ordering: no dense subordering.

Theorem (Hausdorff)

The class of countable scattered orderings is $\bigcup_{\alpha \in \mathcal{O}} V_\alpha$ *where*

- $V_0 = \{\mathbf{0}, \mathbf{1}\}$,
- $V_\alpha = \{\sum_{j \in J} K_j \mid J \in \{\omega, -\omega\}$ *and* $K_j \in \bigcup_{\beta < \alpha} V_\beta\}$.

Example

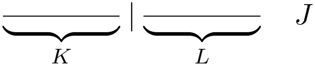
$$(\omega + (-\omega)) \cdot -\omega = \cdots \quad (\bullet \bullet \bullet \cdots \quad \cdots \bullet \bullet\bullet) \quad (\bullet \bullet \bullet \cdots \quad \cdots \bullet \bullet\bullet)$$

$$= \sum_{i \in -\omega} \bullet \bullet \cdots \quad \cdots \bullet \bullet = \sum_{i \in -\omega} \left(\sum_{j \in \omega} \mathbf{1} + \sum_{j \in -\omega} \mathbf{1} \right)$$

Cuts of a linear ordering

A cut of a linear ordering J is a pair (K, L) of intervals such that

- $K \cup L = J$
- $\forall k \in K, \forall l \in L \quad k < l.$



$\hat{J}$ linear ordering of the cuts

Examples

$$J = 3 \qquad J \cup \hat{J} = \;|\bullet|\bullet|\bullet|$$

$$J = \omega \qquad J \cup \hat{J} = \;|\bullet|\bullet|\bullet|\cdots|$$

$$J = \zeta + \zeta \qquad J \cup \hat{J} = |\cdots|\bullet|\bullet|\bullet|\cdots|\cdots|\bullet|\bullet|\bullet|\cdots|$$

Consecutive cuts

$$\begin{array}{c} \qquad\qquad\qquad j \\ \cdots \bullet \mid \bullet \mid \bullet \mid \bullet \mid \bullet \mid \bullet \mid \bullet \cdots \qquad J \cup \hat{J} \\ \qquad\quad c_j^- \quad c_j^+ \end{array}$$

Fact

Each pair of consecutive cuts is of the form (c_j^-, c_j^+).

Automata on linear orderings

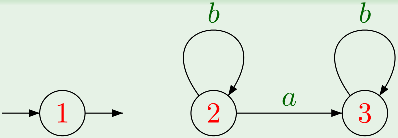
states, initial states

$$\mathcal{A} = (Q, A, E, I, F)$$

transitions, final states

$$
\begin{aligned}
E \subseteq\ & Q \times A \times Q && \text{successor transitions} \\
\cup\ & \mathcal{P}(Q) \times Q && \text{left limit transitions} \\
\cup\ & Q \times \mathcal{P}(Q) && \text{right limit transitions}
\end{aligned}
$$

Example



$1 \to \{2\}$

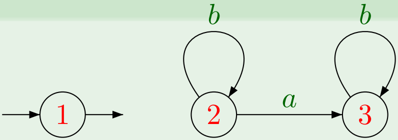
$\{3\} \to 1$

Paths

A path labeled by a word $x = (a_j)_{j \in J}$ is a sequence of states $\gamma = (q_c)_{c \in \hat{J}}$ such that

- If $\underset{p}{|}\ a\ \underset{q}{|}$, then $p \xrightarrow{a} q \in E$,
- If $\cdots_{P}\ \underset{q}{|}$, then $P \to q \in E$,
- If $\underset{q}{|}\ {}_{P}\cdots$, then $q \to P \in E$.

Example



$1 \to \{2\}$

$\{3\} \to 1$

$$(b^{-\omega}ab^{\omega})^2 \in L(\mathcal{A}) \quad \text{but} \quad (b^{-\omega}ab^{\omega})^{\omega} \notin L(\mathcal{A})$$

$$\underset{1}{|}\ \underset{\{2\}}{\cdots} b \underset{2}{|} b \underset{2}{|} a \underset{3}{|} b \underset{3}{|} b \underset{\{3\}}{\cdots} \underset{1}{|}\ \underset{\{2\}}{\cdots} b \underset{2}{|} b \underset{2}{|} a \underset{3}{|} b \underset{3}{|} b \underset{\{3\}}{\cdots} \underset{1}{|}$$

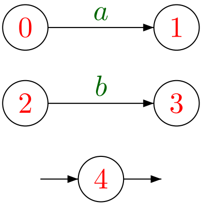
Precise definitions

- For any cut c of J, define the sets

$$\lim_{c^-} \gamma = \{q \mid \forall c' < c \; \exists c' < k < c \quad q = q_k\},$$
$$\lim_{c^+} \gamma = \{q \mid \forall c < c' \; \exists c < k < c' \quad q = q_k\}.$$

-
  - For any consecutive cuts c_j^- and c_j^+, $q_{c_j^-} \xrightarrow{a_j} q_{c_j^+}$ is a successor transition,
  - For any cut c with no predecessor, $\lim_{c^-} \gamma \to q_c$ is a left limit transition
  - For any cut c with no successor, $q_c \to \lim_{c^+} \gamma$ is a right limit transition

Another example



$\{0,1,2,3,4\} \to 0$

$\{0,1,2,3,4\} \to 2$

$\{0,1,2,3,4\} \to 4$

$0 \to \{0,1,2,3,4\}$

$1 \to \{0,1,2,3,4\}$

$3 \to \{0,1,2,3,4\}$

$$\text{Word} \quad \begin{cases} r \mapsto a & \text{if } r = n/2^m \\ r \mapsto b & \text{if } r \in \mathbb{Q} \setminus \{n/2^m \mid m \geq 0\} \end{cases}$$

$$\text{Path} \quad \begin{cases} c \mapsto 0 & \text{if } c = c_r^- \text{ and } r = n/2^m \\ c \mapsto 1 & \text{if } c = c_r^+ \text{ and } r = n/2^m \\ c \mapsto 2 & \text{if } c = c_r^- \text{ and } r \neq n/2^m \\ c \mapsto 3 & \text{if } c = c_r^+ \text{ and } r \neq n/2^m \\ c \mapsto 4 & \text{if } c \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Kleene theorem

Rational expressions

- $\varnothing$ empty set and $a \in A$ letter
- Rational operations
  - union, product, and star iteration
  - ω and $-\omega$ iterations
  - ordinal and backwards ordinal iteration
  - iteration on all linear orderings
  - shuffle iteration for non-scattered orderings

Theorem (C.-Bruyère, C.-Bès)

A set of words on linear orderings is rational iff it is accepted by some finite automaton.

Complementation

Theorem

For scattered and countable orderings, the complement of a rational set is also rational

- Büchi has already pointed out that complementation does not hold for non-countable ordinals (greater than ω_1).

  $\rightarrow$ Only countable orderings are considered.

- The set of all scattered orderings is not rational while its complement is rational.
- Determinization is not possible. Some rational sets like $(A^{-\omega})^{-\omega}$ cannot be accepted by a deterministic automaton.

Logic

Theorem

For scattered and countable orderings, MSO is equivalent to automata.

Theorem

For all orderings, MSO is more expressive than automata.