Decidability in continuous-time dynamical systems

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Why study dynamical systems

- Describe physical phenomena
- Describe biological phenomena
- Simulate computation models

The natural questions on those systems are motivated by the applications:

- Population extinction $\longrightarrow$ the system reaches 0.
- A cloud goes over a region $\longrightarrow$ the trajectory intersects a region.
- A program loops infinitely $\longrightarrow$ the system is ultimately periodic.

Discrete time dynamical systems

Definition

A discrete-time dynamical system is given by (X, f) where X is a space, and f a function from X to X.
Given a point x_n, the successor is $x_{n+1} = f(x_n)$.
An initial point describes uniquely the whole trajectory.

Example : second order linear recurrent sequence

$$X = \mathbb{Z}^2$$

$$f : (x, y) \mapsto (x + y, x)$$

A trajectory: $(1, 1) \rightarrow (2, 1) \rightarrow (3, 2) \rightarrow (5, 3) \rightarrow (8, 5) \rightarrow ...$

Can be represented by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Example: Turing machine

Given a one tape Turing machine with
$\delta : \mathbb{N} \times \{0, .., 9\} \rightarrow \mathbb{N} \times \{0, .., 9\} \times \{l, s, r\}$

$$X = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \text{ (state, left and right parts of tape)}$$

	1	3	2	$5\star$	0	1	

is coded by $a = 1325$ et $b = 10$

$$\begin{aligned} f : (n, a, b) \mapsto \quad & (n', a', b') \text{ with } ab \text{ coding for the tape,} \\ & \sigma = a \bmod 10 \ ; \ \delta(n, \sigma) = (n', \sigma', \tau) \\ & a' = a - \sigma + \sigma' \ ; \ b' = b \text{ if } \tau = r \\ & a' = a/10 \ ; \ b' = 10 \times b + \sigma' \text{ if } \tau = d \\ & b' = b/10 \ ; \ a' = 10 \times (a - \sigma + \sigma') + (b \bmod 10) \text{ if } \tau = \end{aligned}$$

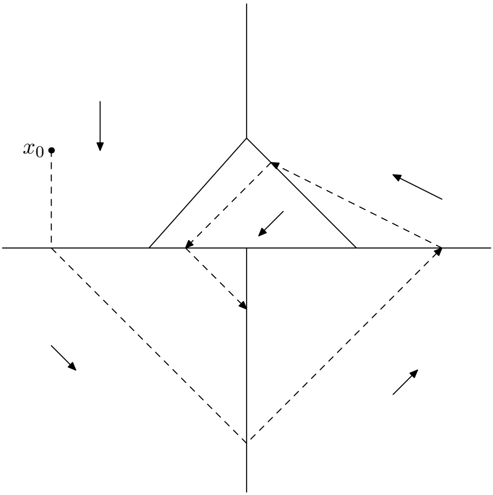
Continuous time dynamical systems

Definition

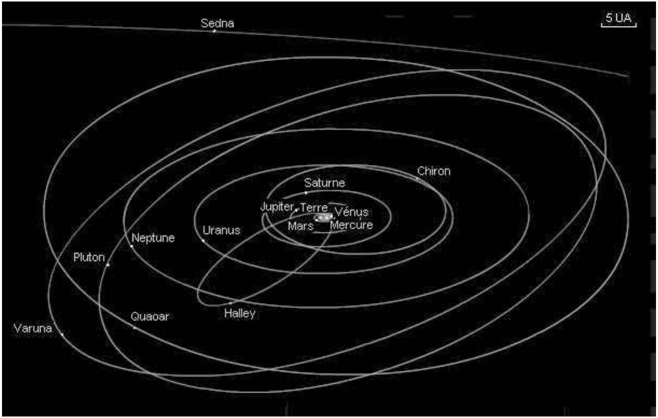
A continuous-time dynamical system is defined by (X, f) where X is the configuration space $(\mathbb{R}^n)$ and $f : X \to X$.
A trajectory of the system is a solution of the Cauchy problem:

$$\begin{cases} y' & = & f(y) \\ y(0) & = & y_0 \end{cases}$$

Example : Piecewise Constant Derivative



Example: n-body problem



n-body problem

Proposition

The n-body problem (with Newton's laws) can be written as a polynomial dynamical system (with n^2 components)

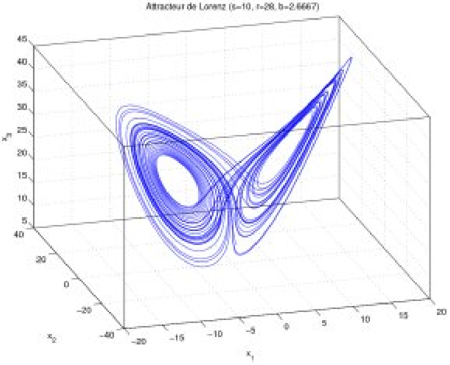
Theorem [Warren D. Smith]

The n-body problem can “solve” the halting problem for Turing machines in constant time.

- This is a polynomial dynamical system.
- The collapsing of the n-body problem is undecidable.

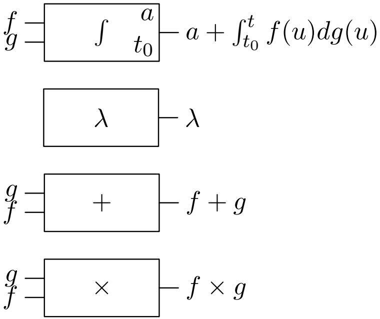
Example: Lorenz' attractor

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 10(y-x) \\ 28x - y - xz \\ xy - \frac{8}{3}z \end{pmatrix}$$

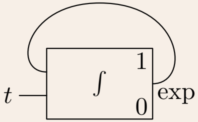


General Purpose Analog Computer

gpac [Shannon 41] consists in circuits interconnecting the following components:

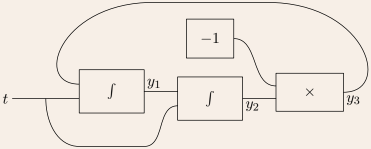


Computing exp with a GPAC



$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

Computing cos with a GPAC



Features of the GPAC

Theorem [Graça Costa 03]

A scalar function $f : \mathbb{R} \to \mathbb{R}$ is generated by a GPAC iff it is a component of the solution of a system

$$y' = p(t, y), \tag{1}$$

where p is a vector of polynomials.

- gpac is a polynomial dynamical system.

Continuous time linear dynamical systems

Definition

A continuous-time linear dynamical system is described by a dimension n, a sqaure matrix A of size n^2 with rationnal coefficients.

$$X = \mathbb{R}^n$$
$$f : Y \mapsto AY$$

A trajectory issued from $Y_0 \in \mathbb{Q}^n$ is a solution of the Cauchy problem: $\left\{ \begin{array}{l} Y' = AY \\ Y(0) = Y_0 \end{array} \right.$. *Id est* $Y(t) = \exp(tA)Y_0$.

Reachability

Definition

Given a dynamical system (X, f), and two points A and B, does the trajectory issued from A reach B?

- Safety problem.

ω-limit set

Definition

Given a dynamical system with solution y, the ω-limit set is the set of $A \in X$ such there $(t_n) \to +\infty$ such that $\lim y(t_n) = A$.

- Périodicity, divergence.

Skolem-Pisot problem

Definition

Given a recurrent linear sequence, is it sometimes 0?

This problem is equivalent to "given a matrix $A \in \mathbb{N}^n$, does the dynamical system $(\mathbb{N}^n, Y \mapsto AY)$ reach a $(0, \_, \dots, \_)$ point?"

- Reachability of a hyperplane

Undecidability

Theorem

Reachability is undecidable

The halting problem can be written as a reachability question.

Undecidability

Proposition

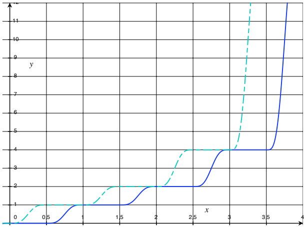
Reachability is undecidable in Polynomial DS.
Hyperplane reachability is undecidable in Polynomial DS.

Proof: From [Graça, Campagnolo, Buescu 2005], Turing machines can be simulated by continuous time polynomial dynamical systems.

Simulating Turing machine with a continuous DS

Dynamical System on $\mathbb{R}^3$ (state, left and right parts of the tape)
$f : \mathbb{N}^3 \rightarrow \mathbb{N}^3$ describing the Turing machine.
Duplicate the state space to simulate the transition:

$$\frac{\partial y_1}{\partial t} = \lambda(f(int(y_2)) - y_1)^3 \theta(\sin(2\pi t))$$
$$\frac{\partial y_2}{\partial t} = \lambda(int(y_1) - y_2)^3 \theta(-\sin(2\pi t))$$
$$y_1(x, 0) = x$$
$$y_2(x, 0) = x$$



with θ Heaviside's function.

Decidability

Proposition [Halava, Harju, Hirvensalo, Karhumäki 2005]

For small dimensions (≤ 5), Skolem-Pisot's problem is decidable.

Proposition [Blondel, Portier 2003]

Pisot's problem is NP-hard.

It is unknown whether it is decidable or not for dimension higher than 5.

Facts in discrete DS

For discrete time dynamical systems,

	ω-limit set	reachability	hyperplane reach.
DS	non computable	undecidable	undecidable
polynomial DS	non computable	undecidable	undecidable
linear DS		decidable	Pisot : open

Facts in continuous DS

For continuous-time dynamical systems,

	ω-limit set	Reachability	hyperplane reach.
DS	non computable	undecidable	undecidable
polynomial DS	non computable	undecidable	undecidable
deg.2 poly DS	non computable	undecidable	undecidable
linear DS	computable	decidable	?

Reachability

Theorem [Hai08a]

Reachability is decidable in continuous-time linear dynamical systems.

$$f : \begin{array}{ccl} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ X & \longmapsto & A \cdot X \text{ with } A \in \mathbb{Q}^{n \times n} \end{array}$$

X_0 initial point

Y target

Omega-limit set

Theorem [Hai08b]

The ω-limit set is computable for continuous-time linear dynamical systems.

Theorem

The ω-limit set for a continuous-time linear dynamical system is semi-algebraical.

$$f : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ X & \longmapsto & A \cdot X \text{ with } A \in \mathbb{Q}^{n \times n} \end{array}$$

X_0: initial point

Ω: ω-limit set

Prerequisite

Theorem [Baker]

Given $\alpha \in \mathbb{C}^\star$, either α or $\exp(\alpha)$ is transcendantal.

Theorem [Gelfond-Schneider]

Let α and β algebraic, if $\alpha \notin \{0, 1\}$ and $\beta \notin \mathbb{Q}$, then α^β is transcendantal.

Definition

An algebraic number x is represented by its minimal polynomial χ, a and ϵ such that x is the only root of χ in $\mathcal{B}(a, \epsilon)$

Proposition

$+$, $-$, $\times$, $/$ are computable for algebraic numbers.
Deciding whether an algebraic number is rational is decidable.

Simple case

If the matrix A is diagonal, everything is simple : the $y_i(t)$ are independent and $y_i(t) = \exp(t\lambda_i)y_{i_0}$.
If $\lambda_i > 0$, the ω-limit set is empty.
Otherwise, the ω-limit set is a singleton $\{y_i^\star\}$ with

$$\begin{array}{ll} y_i^\star = 0 & \text{if } y_{i_0} = 0 \\ y_i^\star = 0 & \text{if } \lambda_i < 0 \\ y_i^\star = y_{i_0} & \text{otherwise} \end{array}$$

Reachability is also a disjonction of simple cases.

General case

In the general case, we transform the matrix to a form close to a diagonal matrix: Jordan form

$$A = \begin{pmatrix} D_1 & 0 & & 0 \\ 0 & D_2 & 0 & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & D_k \end{pmatrix}$$

$$D_i = \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}$$

or $D_i = \begin{pmatrix} B & & & \\ I_2 & B & & \\ & \ddots & \ddots & \\ & & I_2 & B \end{pmatrix}$ avec $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ et $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

General case

Then the difficult cases are those when different components follow circular trajectories.

Using Baker's and Gelfond's theorem, it is possible to rule out many impossible cases.

In the end, a few algebraic expression describe the ω-limit set.

A few algebraic tests allow to answer the reachability question.

See [Hai08a, Hai08b]

Continuous Skolem-Pisot

The continuous Skolem-Pisot problem (reachability of a hyperplane) is still open.
Some cases can be reduced to the existence of a positive root to a polynomial (see [BDJB08]).
But many cases are not yet shown as decidable or undecidable.

Conclusion

As for discrete DS, reachability is decidable for continuous-time linear Dynamical Systems but undecidable for polynomial DS.

Also, the ω-limit set is computable for continuous linear DS.

What about Skolem-Pisot? Reachability of a polyedron?

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