Computations in hyperbolic spaces

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in this talk:

1. reminding hyperbolic geometry
2. coordinates for tilings in the hyperbolic plane
3. application to tiling problems
4. application to cellular automata in the hyperbolic plane

1. hyperbolic geometry

hyperbolic geometry

absolute geometry

$+$ new axiom (Lobachevsky-Bolyai):
from a point A not on line ℓ,
at least two parallels to ℓ

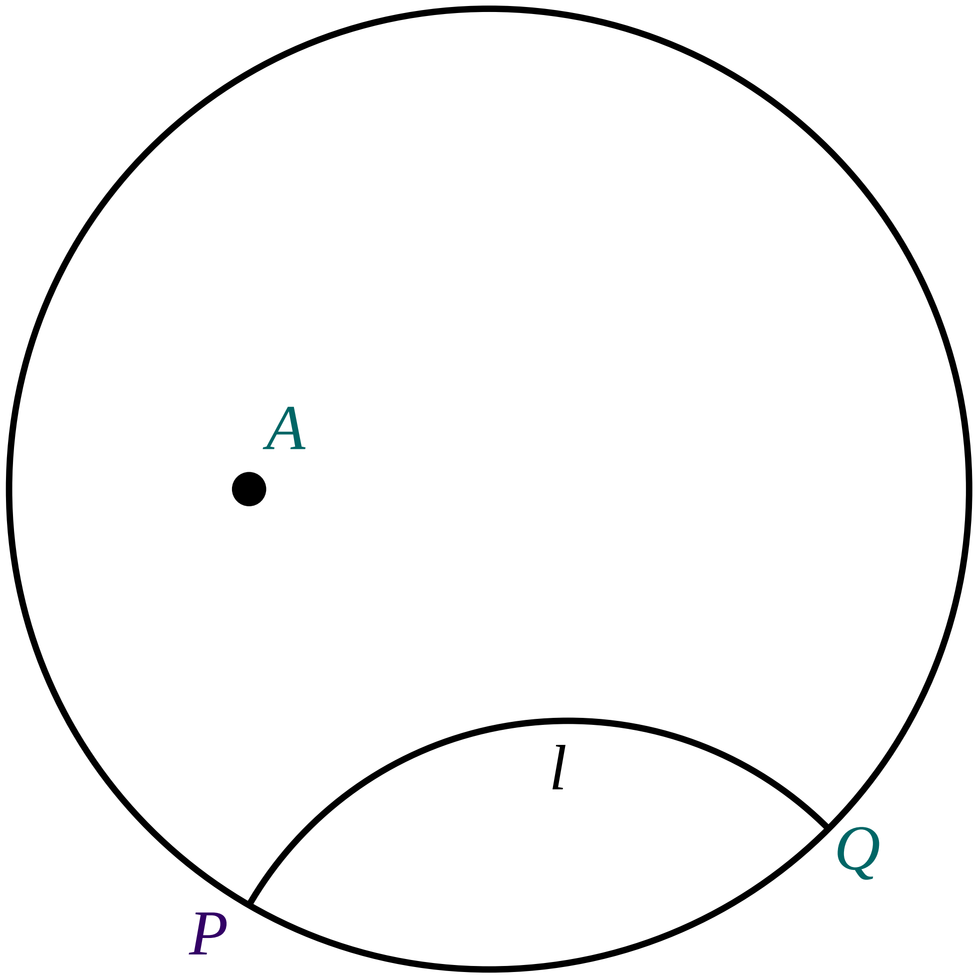
extension to any dimension

many models

Beltrami, Klein, Poincaré,...

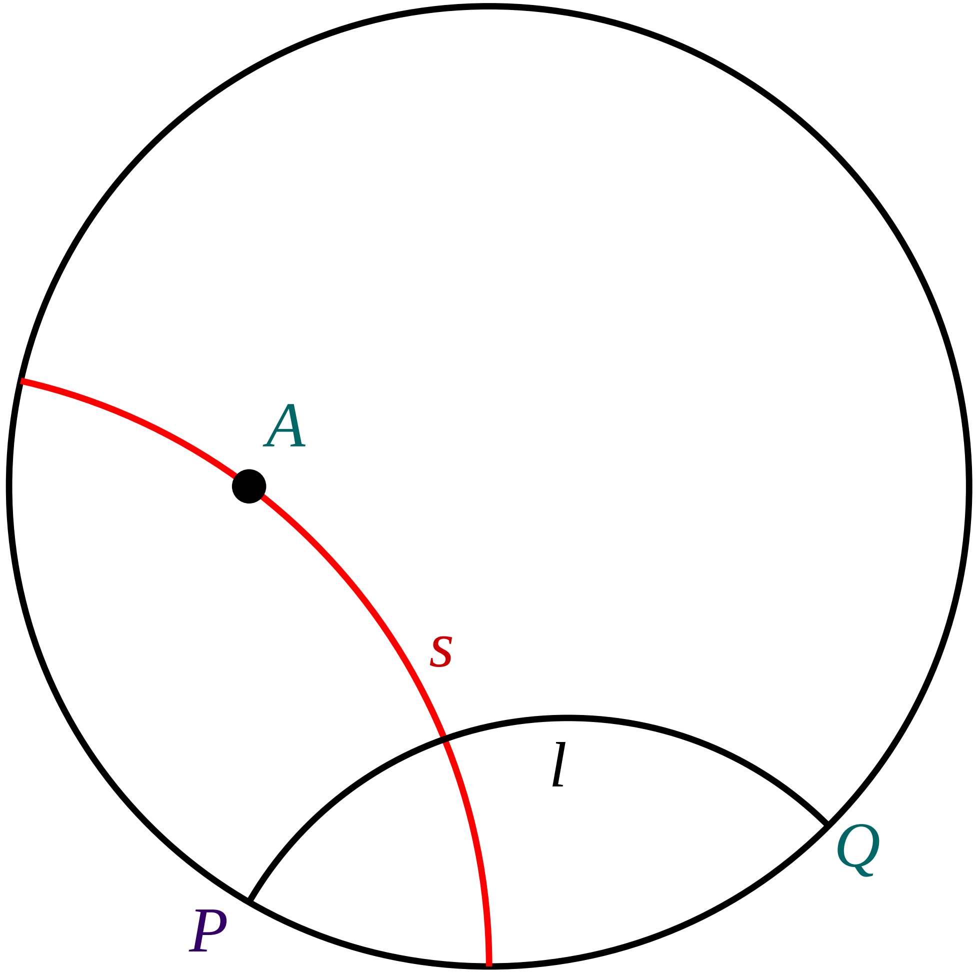
Poincaré's disc model

a point A,
a line ℓ



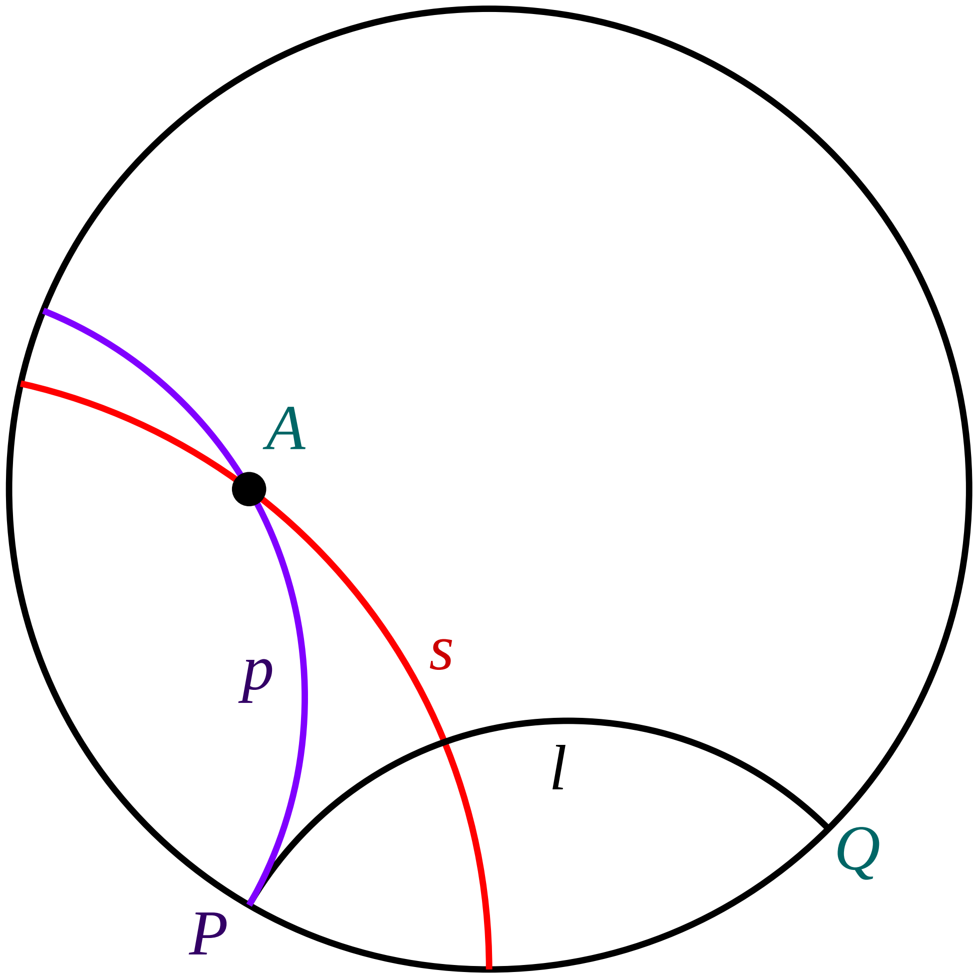
Poincaré's disc model

a **secant** s
to ℓ,
through A



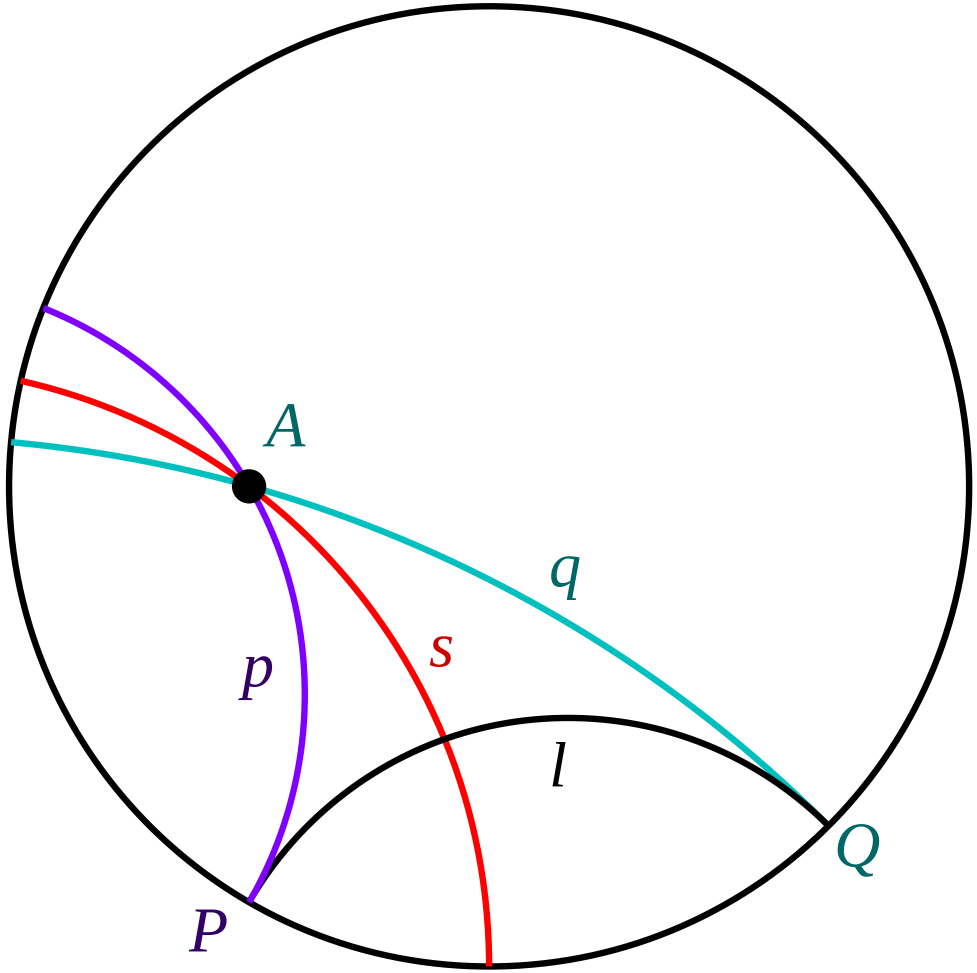
Poincaré's disc model

a **parallel** p
to ℓ
through A



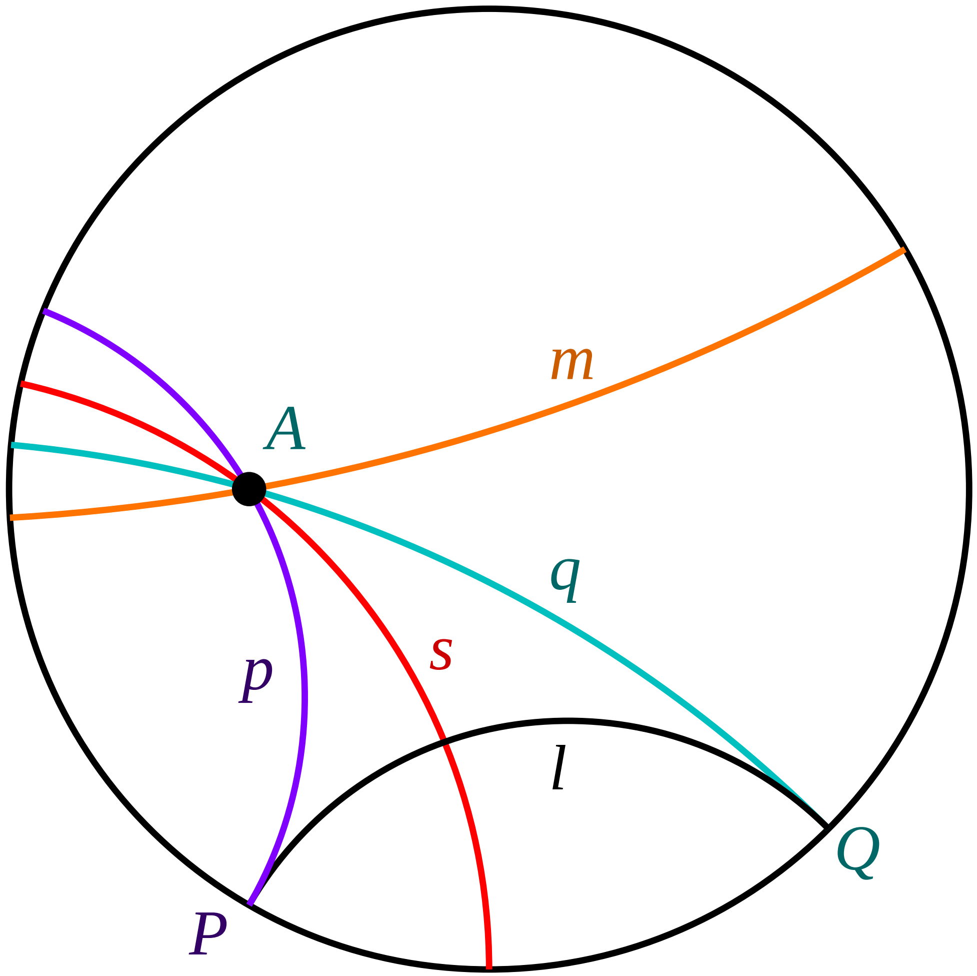
Poincaré's disc model

another
parallel q
to ℓ
through A



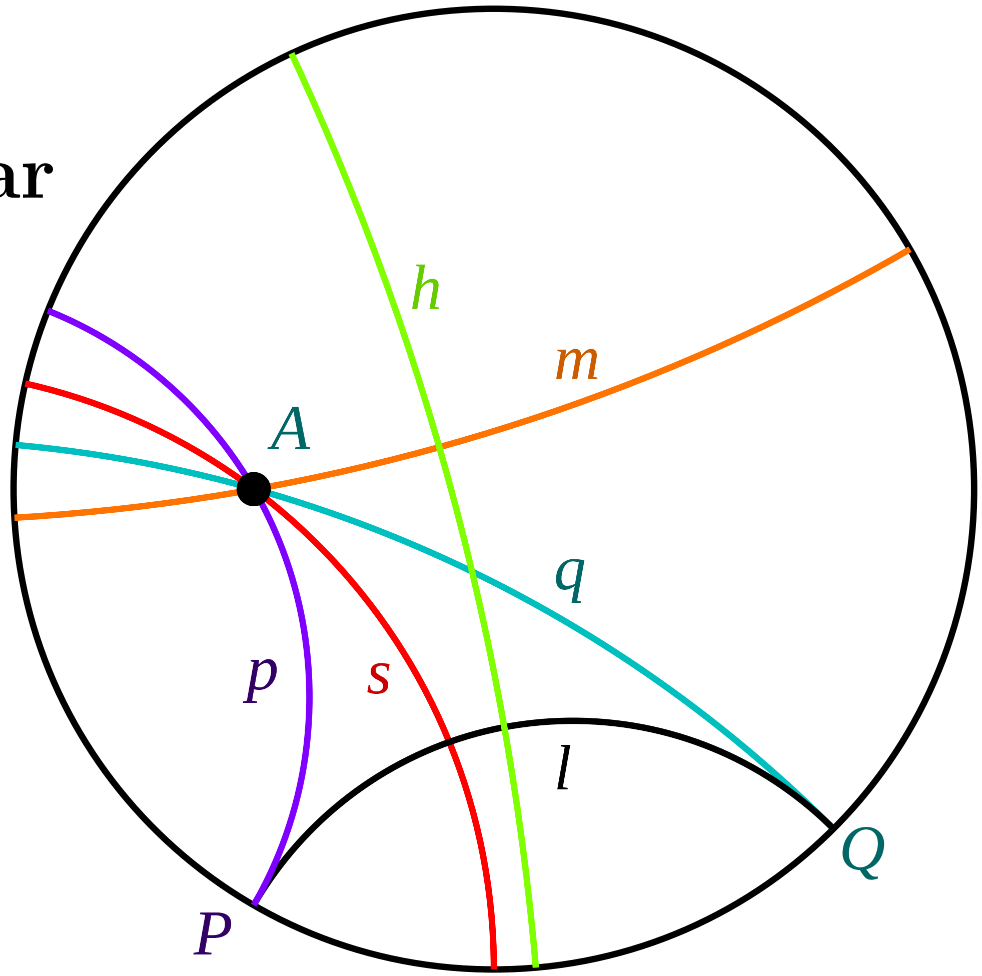
Poincaré's disc model

a **non secant** line m to ℓ through A



Poincaré's disc model

the common perpendicular
to m and ℓ



a few useful properties

sum of angles of triangle:
    always less than π
non-secant lines:
    always a unique common perpendicular

motions in the hyperbolic plane

definition:

finite product of reflections in lines

classification theorem:

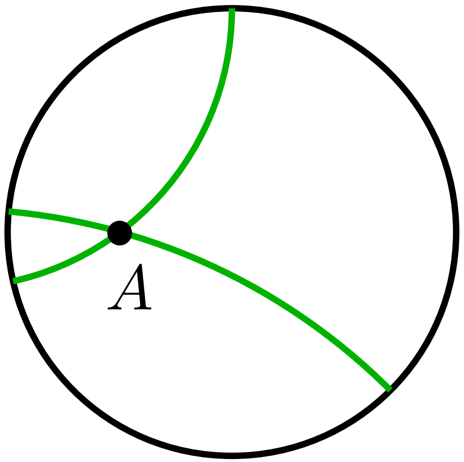
any isometry of the hyperbolic plane is a product of at most three reflections

positive motions:

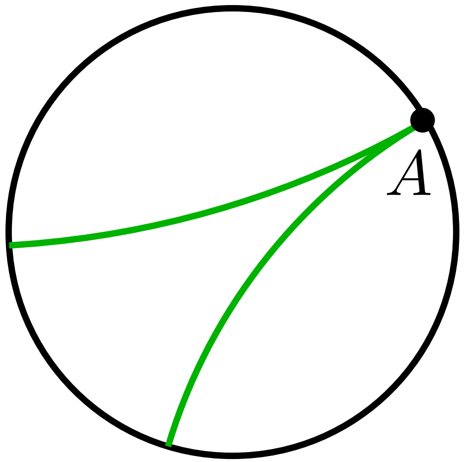
they do not change orientation:
products of two reflections in lines

classification of positive motions

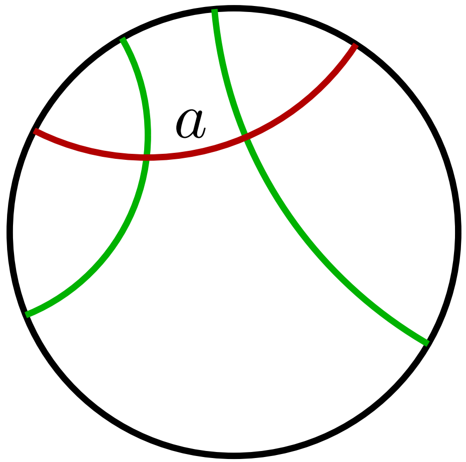
three cases, depending on the intersection of the axes of the reflections:



rotation



ideal rotation



shift along a line

2. coordinates for tilings in the hyperbolic plane

2. coordinates for tilings in the hyperbolic plane

2.1 tilings in the hyperbolic plane

2.2 the splitting method

2.3 application to various location problems

2.1 tilings in the hyperbolic plane

tilings:

sequence $\{T_i\}_{i \in I}$ of **tiles**,

$T_i \subset E$, E geometric space such that:

$i)\ \bigcup\limits_{i \in I} T_i = E$

$ii)\ \forall i, j\ \ (i \neq j \ \Rightarrow\ \mathrm{int}(T_i) \cap \mathrm{int}(T_j) = \emptyset)$

where $\mathrm{int}(T_i)$ is the interior of T_i

here, as usual, only

finitely generated tilings:

there is a **finite** J, $J \subset I$,
such that for all i,
T_i is a copy (isometric image) of some T_j
for $j \in J$

T_j's, for $j \in J$ are called **prototiles**

tessellations

one basic tile P ;

T_i's are obtained by reflections of a convex polygon P in its sides and, recursively, of the images in their sides

classically :

in the Euclidean plane, three tessellations:

square, regular hexagon, equilateral triangle in the hyperbolic plane:

Poincaré's theorem, (1882):

all tessellations based on a triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ *and* $\frac{\pi}{r}$, *with* p, q *and* r *positive integers such that* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ *exist*

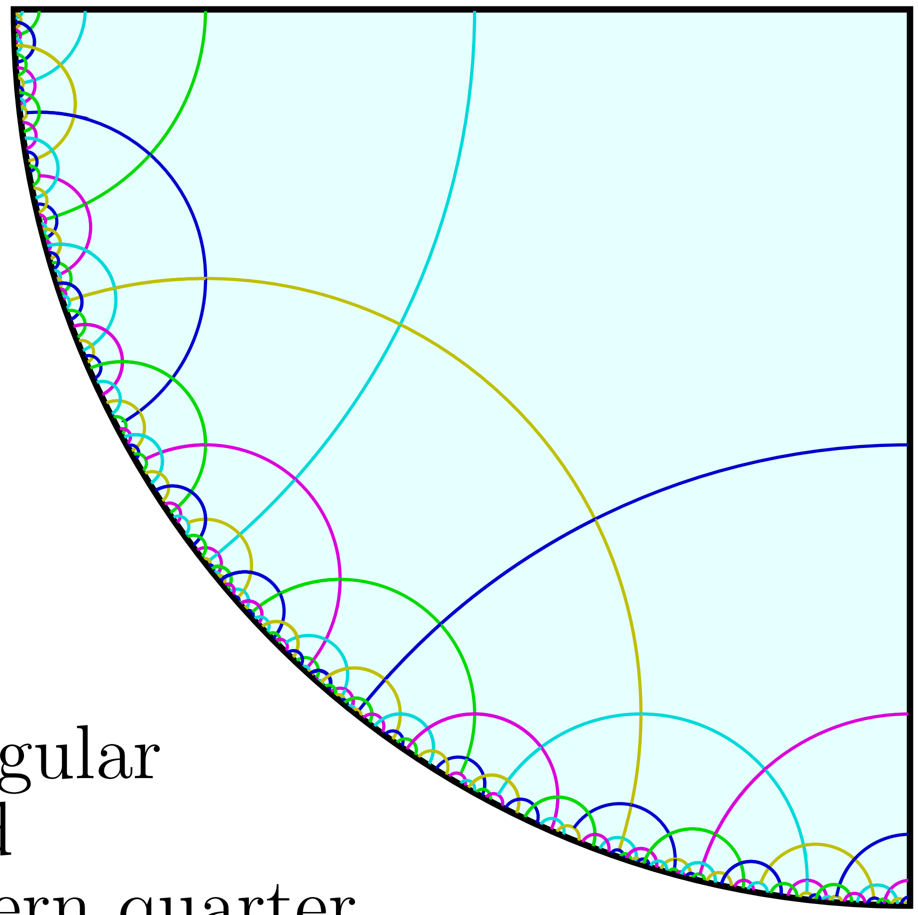
and so, infinitely many solutions

2.2 the splitting method

2.2.a the classical case of the pentagrid

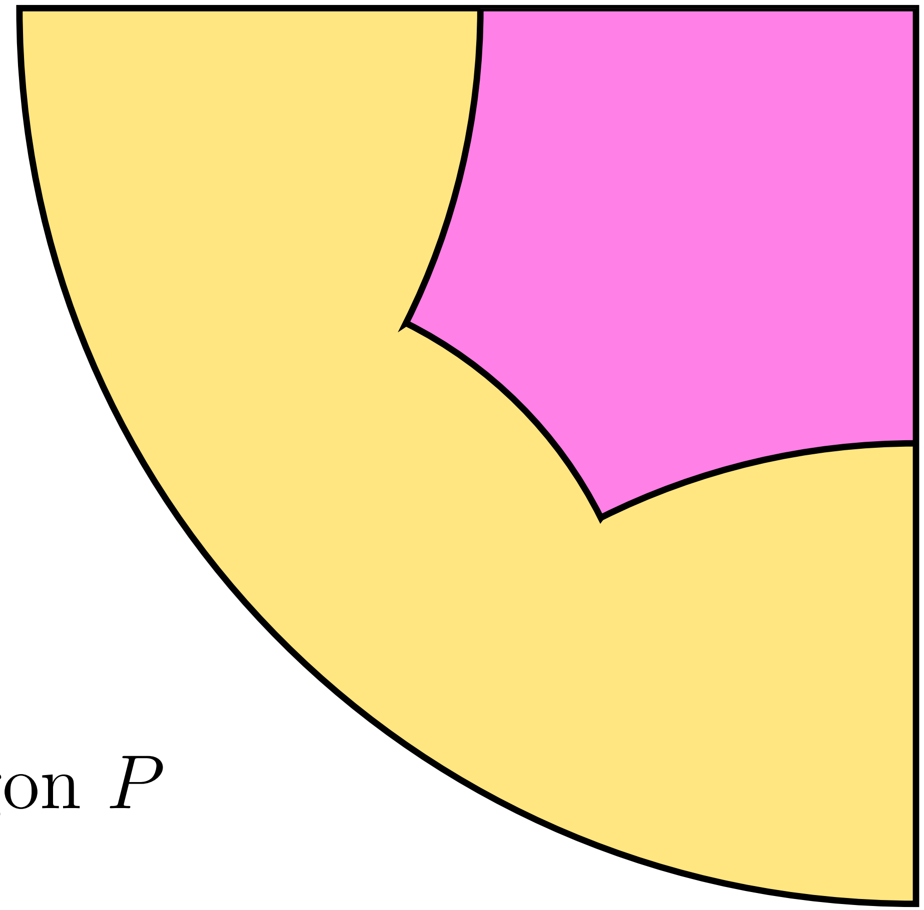
2.2.a combinatoric tilings

2.2. a the classical case of the pentagrid:



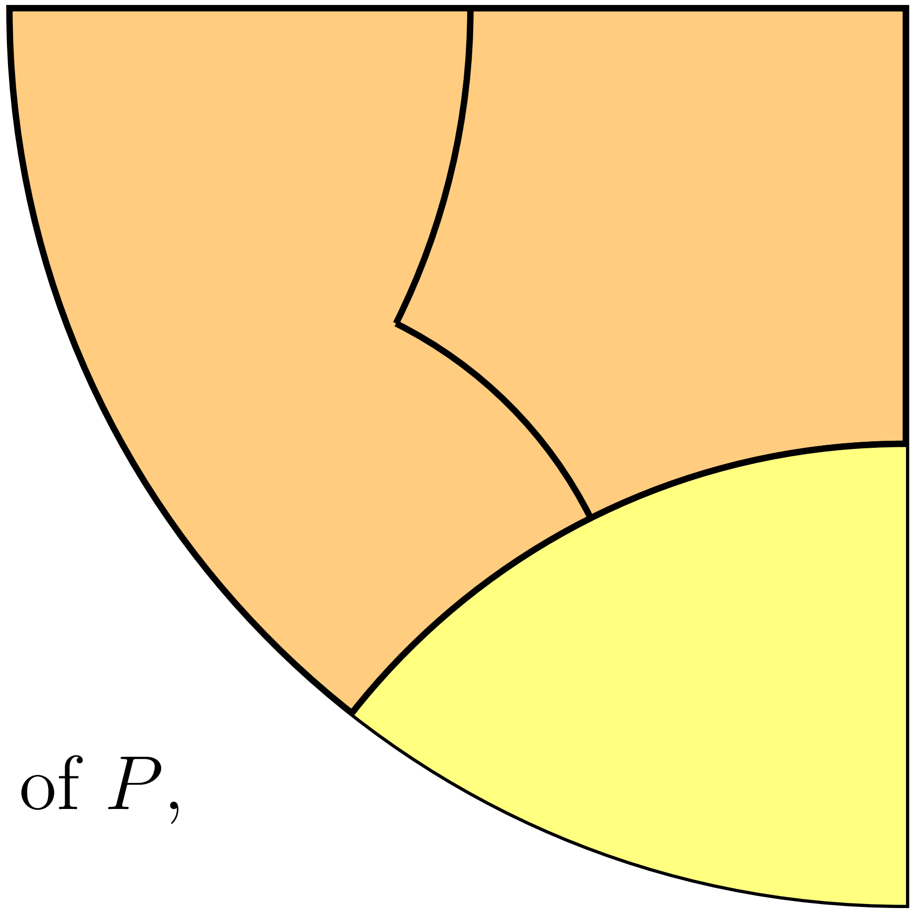
the simpest rectangular
grid here restricted
to the South-Western quarter

$I\!H^2$, the splitting process for the pentagrid:



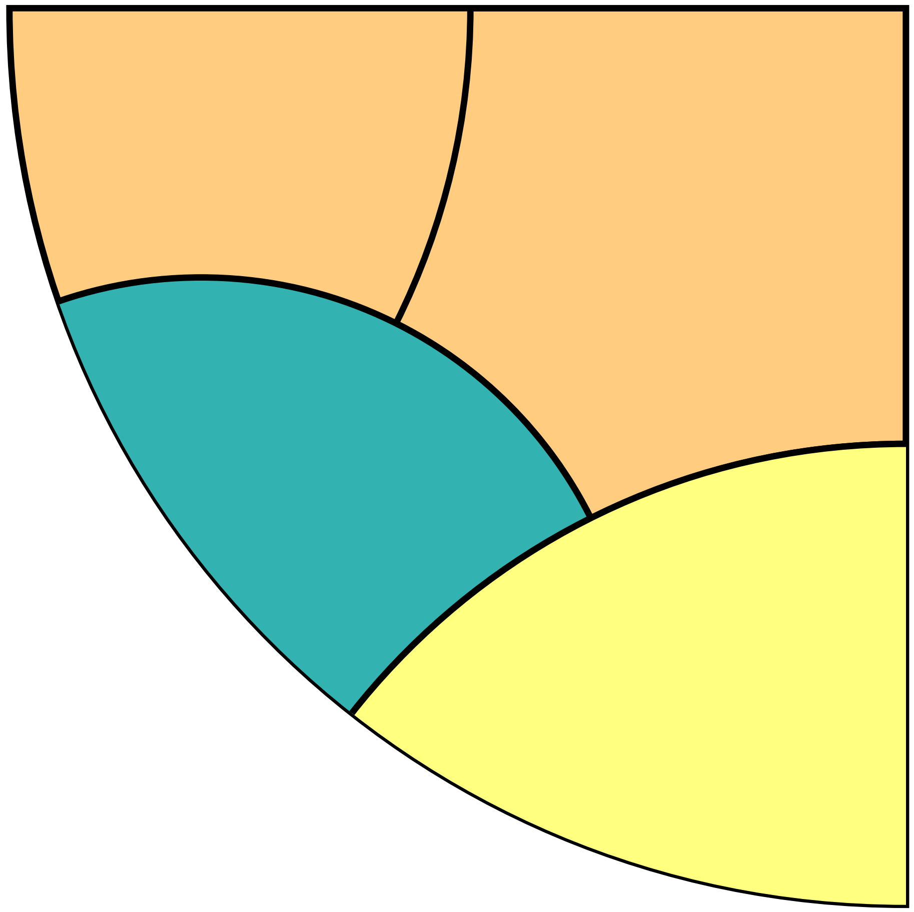
the leading pentagon P
of a quarter

$I\!H^2$, the splitting process for the pentagrid:



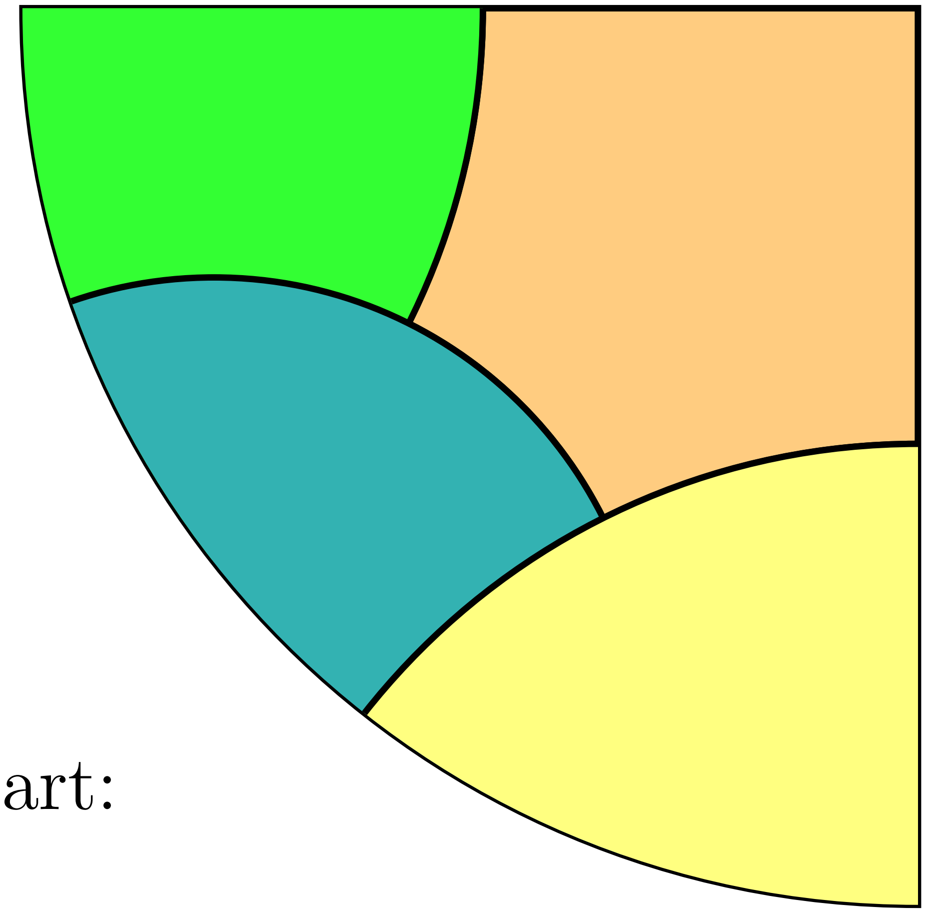
in the complement of P,
a quarter

$\mathbb{H}^2$, the splitting process for the pentagrid:



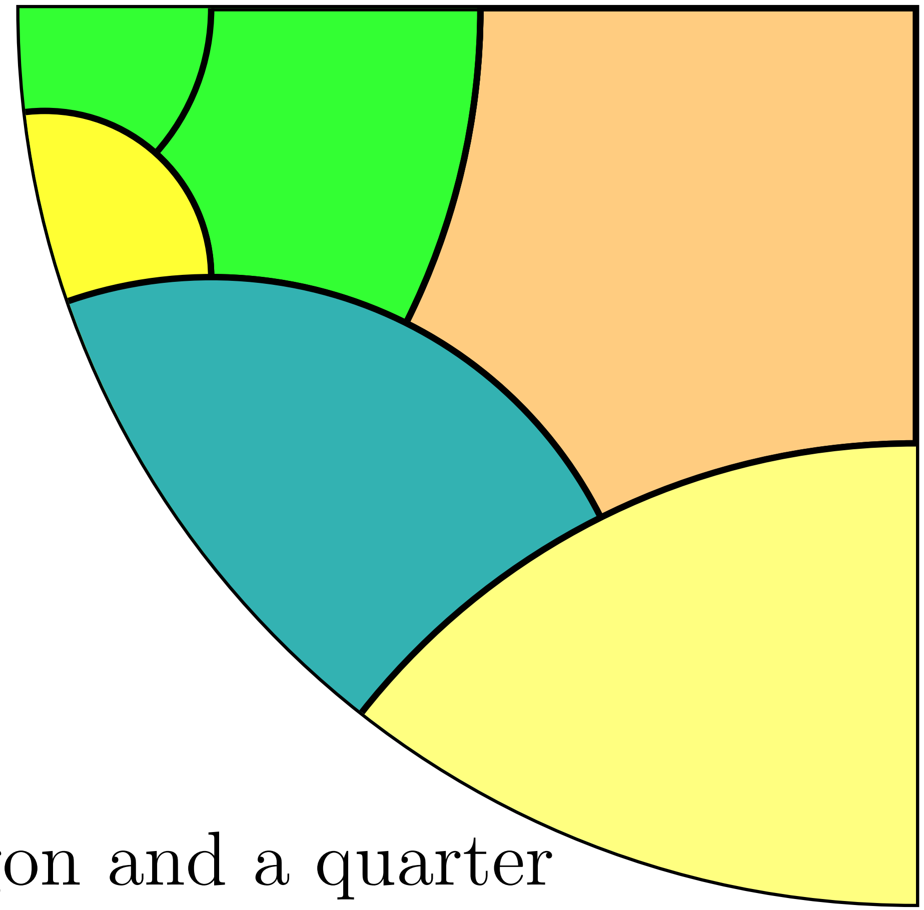
another one

$\mathbb{H}^2$, the splitting process for the pentagrid:



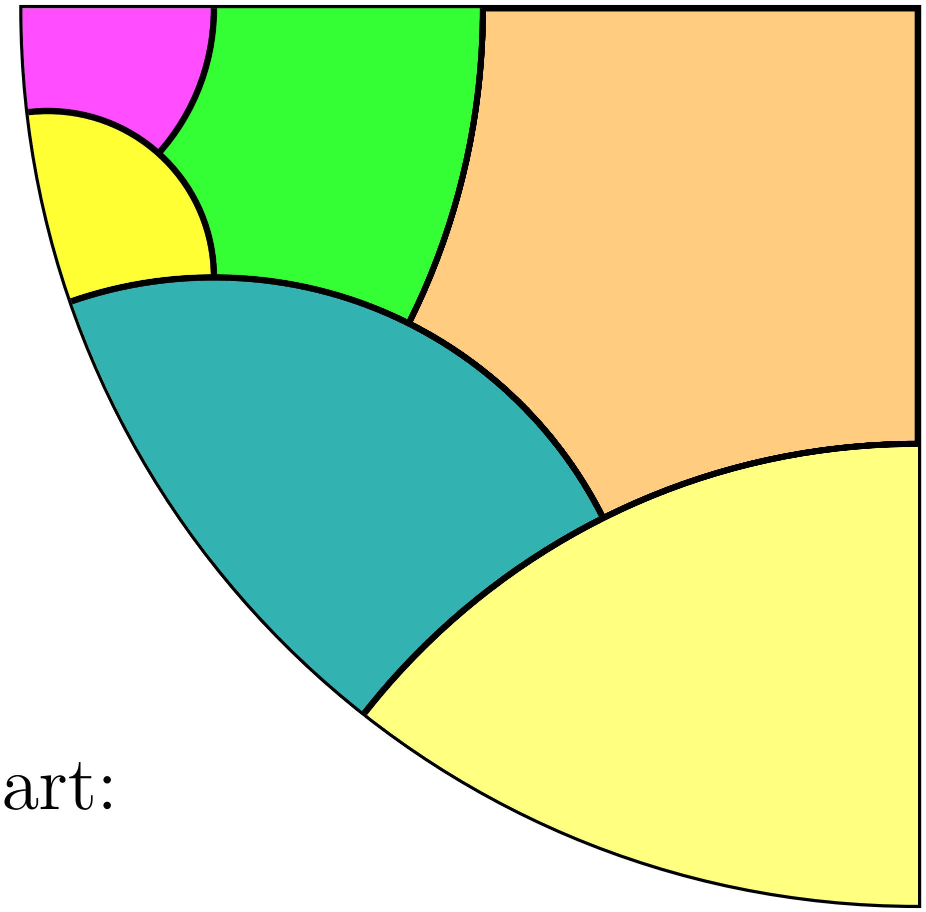
and the remaing part:
a **strip**

$\mathbb{H}^2$, the splitting process for the pentagrid:



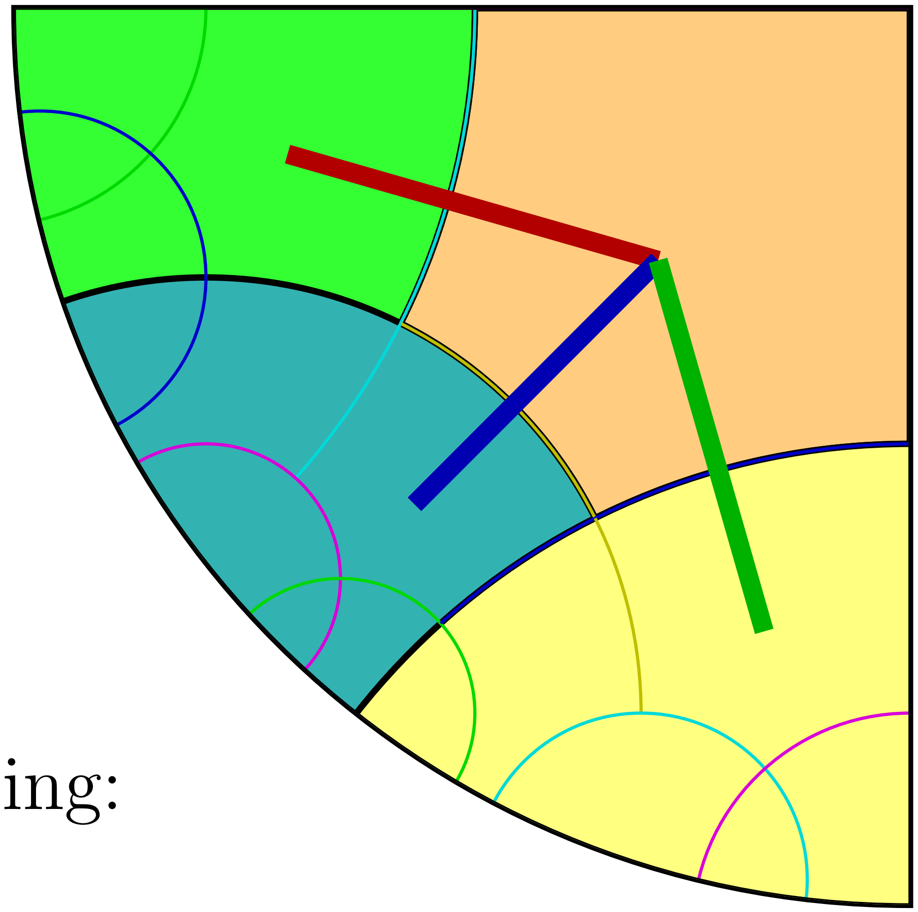
in a strip:
the leading pentagon and a quarter

$\mathbb{H}^2$, the splitting process for the pentagrid:



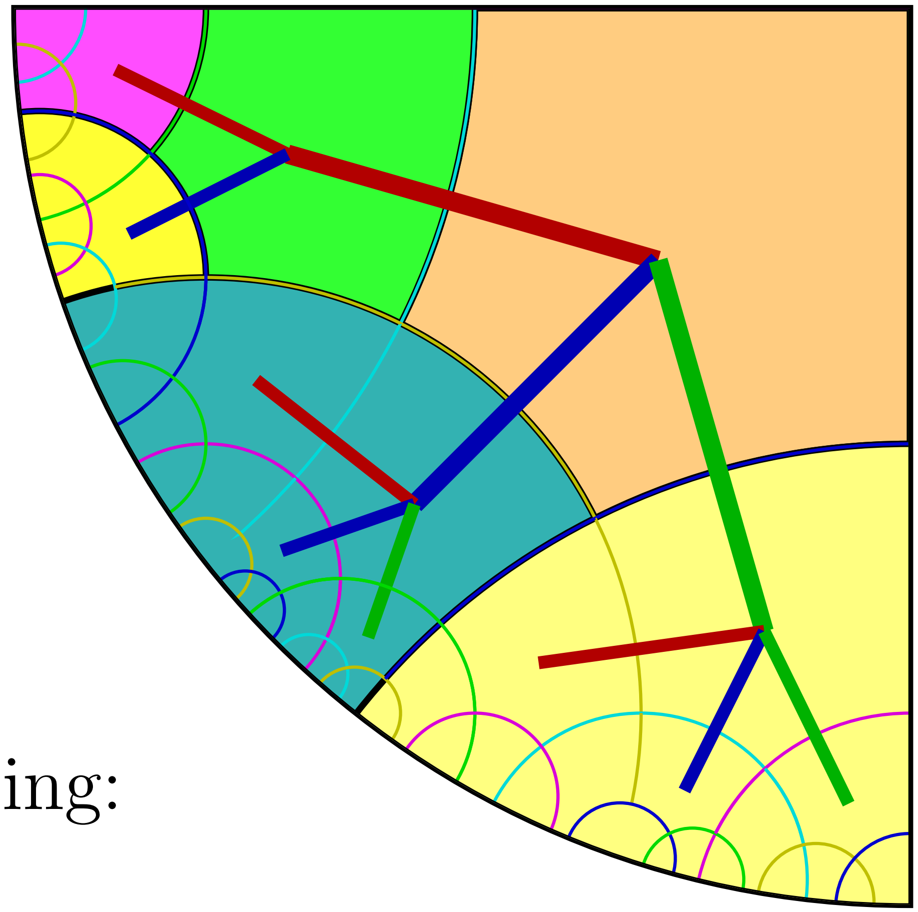
and the remaing part:
a strip again

$I\!\!H^2$, the splitting process for the pentagrid:



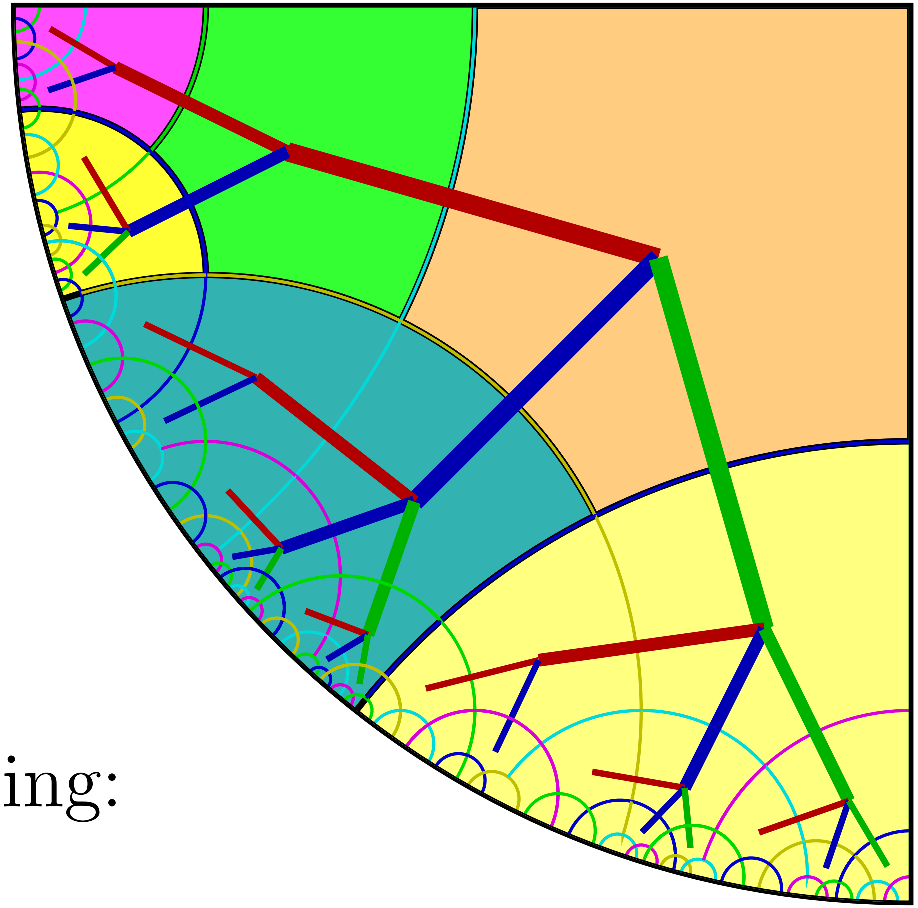
the recursive splitting:
first step

$\mathbb{H}^2$, the splitting process for the pentagrid:



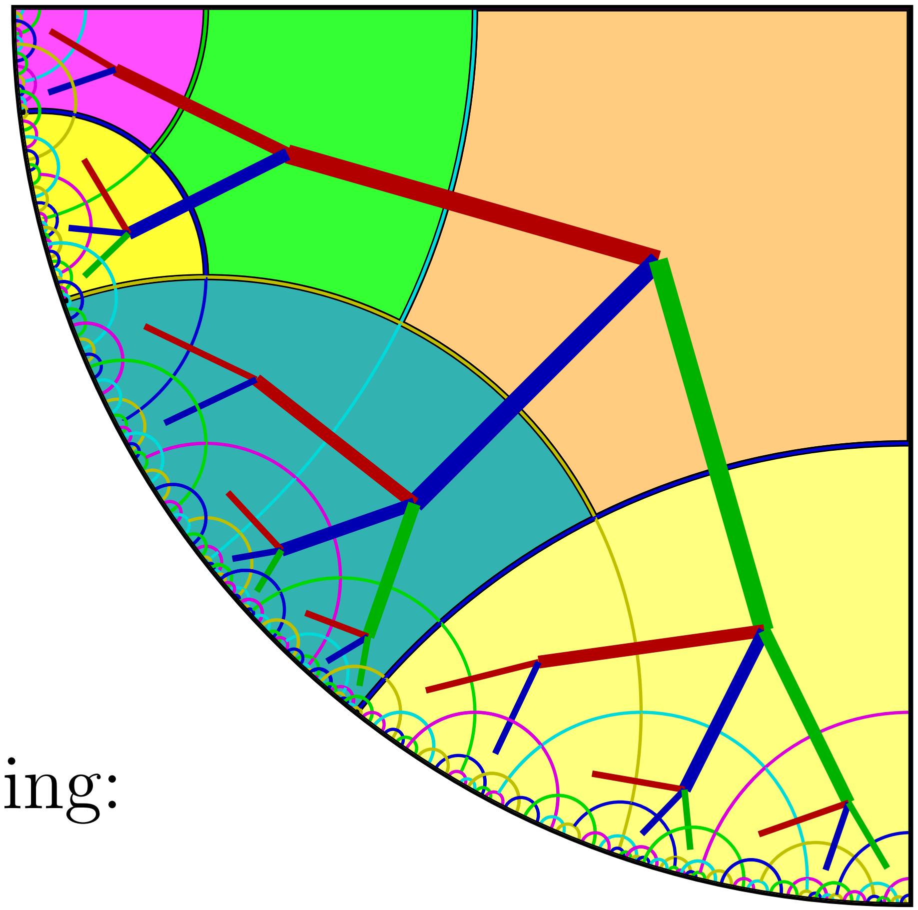
the recursive splitting:
second step

$I\!\!H^2$, the splitting process for the pentagrid:



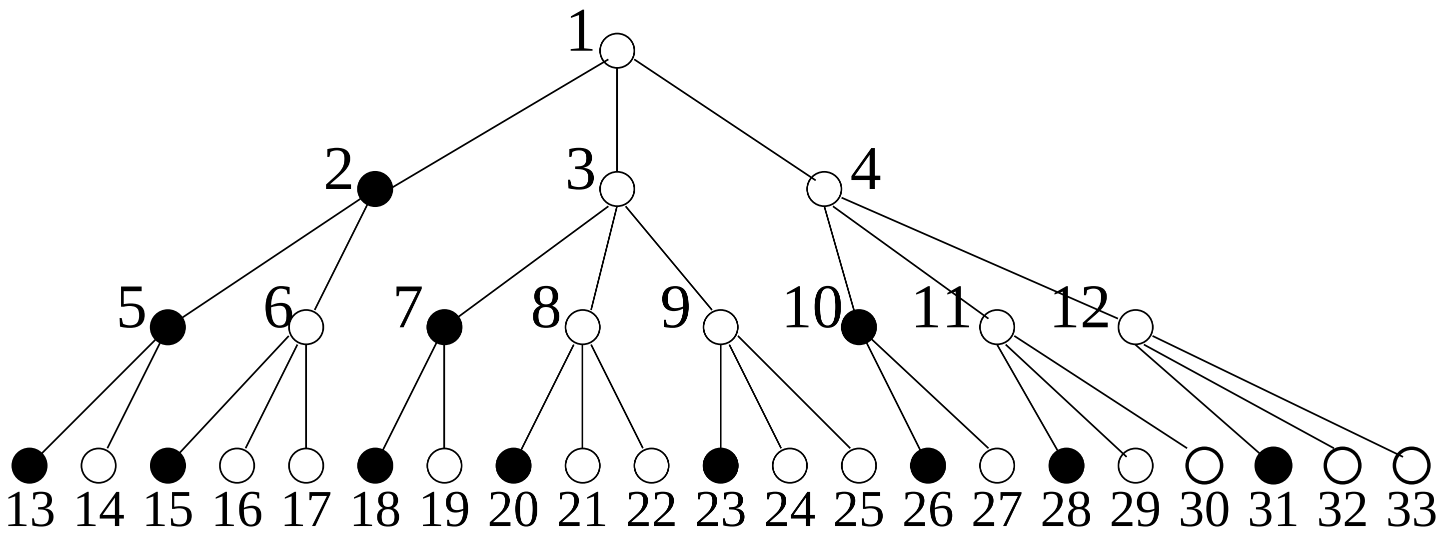
the recursive splitting:
third step

$I\!\!H^2$, the splitting process for the pentagrid:



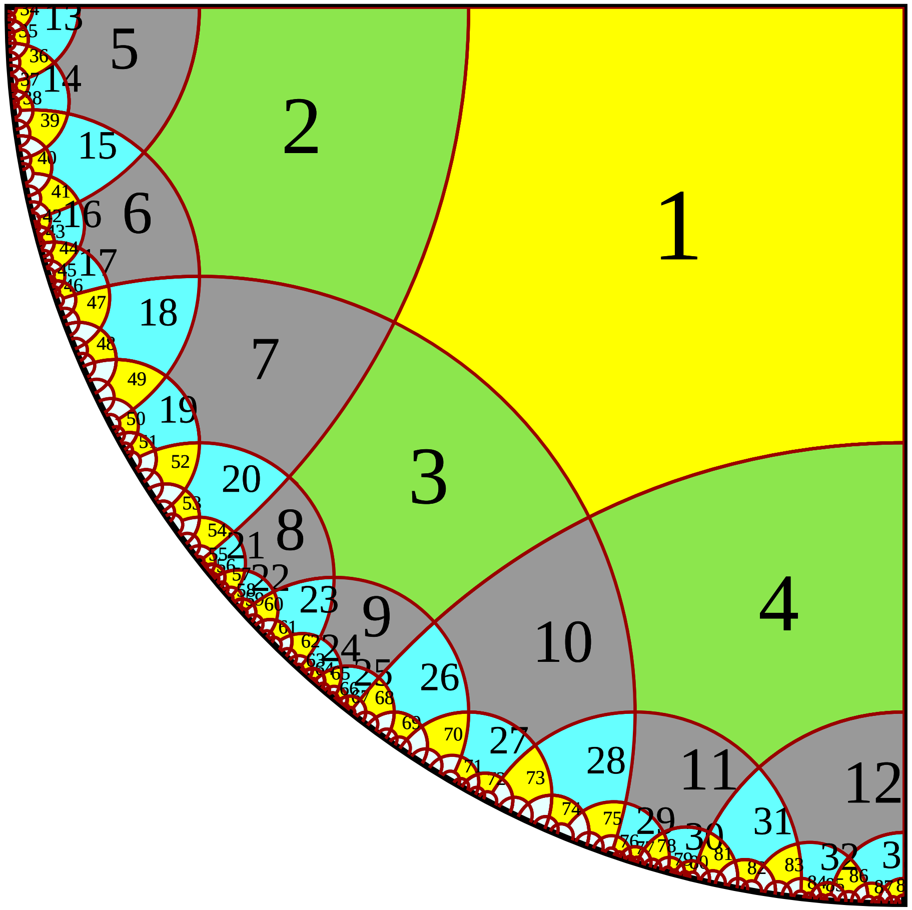
the recursive splitting:
and so on...

the tree being associated to the pentagrid

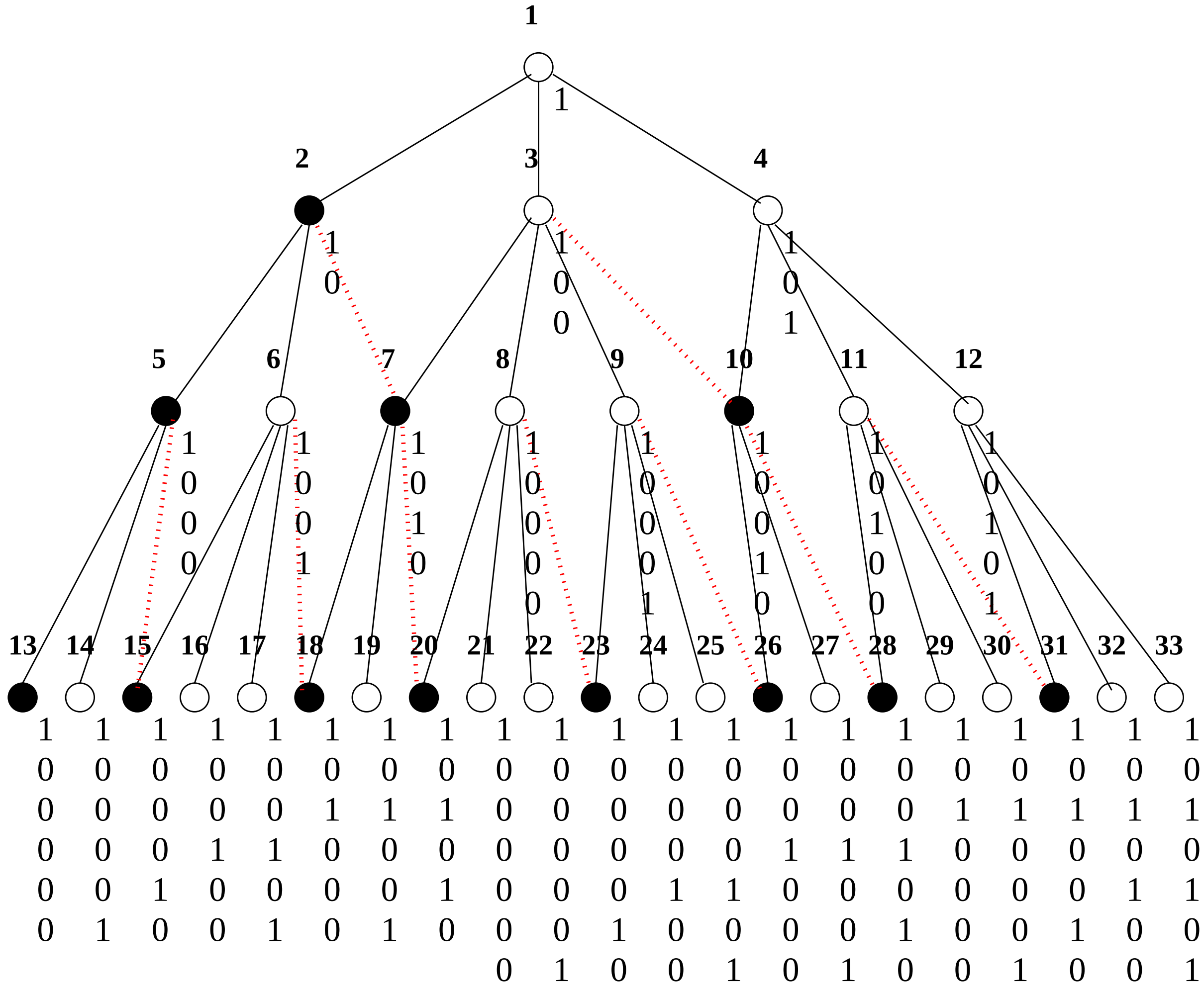


with its numbering

new look on the numbering of the pentagrid



the generating tree:



Fibonacci technology:

recall: for any number n: $n = \sum_{i=0}^{k} a_i f_i$,

where f_i: Fibonacci sequence

the representation is not unique ;

uniqueness obtained by a rule: forbid 11

this representation called **coordinate** of n

the language of coordinates is regular

the property of the preferred son:

let $\alpha_k \ldots \alpha_0$ be the coordinate of ν
and let $\beta_h \ldots \beta_0$ represent $\nu{-}1$

coordinates of the sons of ν:

if ν 2-node: $\boxed{\alpha_k \ldots \alpha_0 00}$, $\boxed{\alpha_k \ldots \alpha_0 01}$

if ν 3-node: $\boxed{\beta_h \ldots \beta_0 10}$, $\boxed{\alpha_k \ldots \alpha_0 00}$, $\boxed{\alpha_k \ldots \alpha_0 01}$

call the node with coordinate $\boxed{\alpha_k \ldots \alpha_0 00}$
the **preferred son** of ν

every node has precisely one preferred node

rule to determine the preferred son:

in 2-nodes, it is the leftmost son

in 3-nodes, it is the middle son

2.2.b **combinatoric tilings:** (2002)

basis of splitting:

k unbounded simply connected parts of $I\!H^n$ $S_0, \ldots, S_k$, and h bounded simply connected parts of $I\!H^n$ $P_0, \ldots, P_h$, $h \leq k$, with:
(i) $I\!H^n$ split into finitely many copies of S_0 (copy = isometric image)
(ii) each S_i split into one copy of some P_ℓ and finitely many copies of S_j

distinguished P_ℓ: **leading tile** of S_i

the spanning tree of the splitting

root : the leading tile of S_0

let level n defined and each node associated to the leading tile of a copy C_j of some S_j
then, sons of leading tile of C_j:
leading tiles of copies of those S_k's occurring in the splitting of C_j

by induction: infinite tree

combinatoric tilings

say that the tiling $\mathcal{T}$ is **combinatoric**

if there is a basis of splitting such that:

the associated spanning tree is in bijection with the restriction of $\mathcal{T}$ to S_0, all tiles of $\mathcal{T}$ being copies of P_ℓ's

later, in most cases, a single generating tile $P = P_0$

matrix and polynomial of the splitting

when combinatoric tiling,

its spanning tree:

$k+1$ types of nodes: type i means S_i

moreover:

let $M_{i,j}$ be the number of S_j in splitting S_i ;

$\Rightarrow$ the number of nodes of level n for a root of type i $=$ sum of row $i+1$ of M^n

M : **matrix of the splitting** ; its characteristic polynomial: **polynomial of the splitting**

language of the splitting

let $u_n = \#\{\text{nodes at level } n \text{ of } \quad\}$

where spanning tree

$\{u\}_n$ satisfies the induction equation defined by the polynomial of the splitting

number the nodes of starting from 1, from the root and level by level:

coordinate of node ν = maximal greedy representation of ν

language of the splitting = language of the coordinates

greedy representation in a basis

let $\{u_n\}_{n\in\mathbb{N}}$ be positive numbers with $u_0 = 1$, $u_n < u_{n+1}$ and $\limsup \frac{u_n}{u_{n+1}} < \infty$

$$\text{let } b = \lfloor \limsup \frac{u_n}{u_{n+1}} \rfloor$$

$$\text{then } n = \sum_{i=0}^{k} \alpha_i u_i \text{ with } \alpha_i \in [0..b]$$

maximal greedy representation:

if k maximal, then unique

results: tilings proved to be **combinatoric**:

$I\!H^2$: $\{5,4\}$: pentagrid (MM-KM, MM)
$\{s,4\}$: s sides and right angle (MM-GS)
$\{p,q\}$: p sides and angle of $\dfrac{2\pi}{q}$ (MM)
most cases of Poincaré's theorem
(MM 2002)
$\{\infty,q\}$: $*$ (MM 2003)

$I\!H^3$: $\{5,3,4\}$: rectangular dodecahedron
(MM-GS)

$I\!H^4$: $\{5,3,3,4\}$: 120-cell (MM, 2004)

2.3 application to various location problems

2.3.a **the shortest path from a tile to another one**

2.3.b **change of coordinates**

2.3.c **locating points in the penta- and the heptagrid**

2.3.d **other connected results**

2.3.a **the shortest path from a tile to another one**

given 2 tiles T_1 and T_2 by their coordinates, find a shortest path from T_1 to T_2:

i.e. find a sequence $\{\tau_i\}_{0\leq i\leq k}$ with $\tau_0 = T_1$, $\tau_k = T_2$ and τ_i, τ_{i+1} sharing a side for $i \in \{0..k-1\}$ and such that k is the smallest as possible

the shortest path from T_1 to T_2

first result:

theorem (MM 2003)
there is an algorithm which gives the path from a tile T in a Fibonacci tree F to the root of F in a time which is linear in the size of the coordinate of T in F

the shortest path from T_1 to T_2

define coordinates for the tiles of the penta- or the -heptagrid as follows:

fix a central tile T_0

define α sectors around T_0, each one spanned by a Fibonacci tree, $\alpha \in \{5, 7\}$

the shortest path from T_1 to T_2

number the sides of a tile:

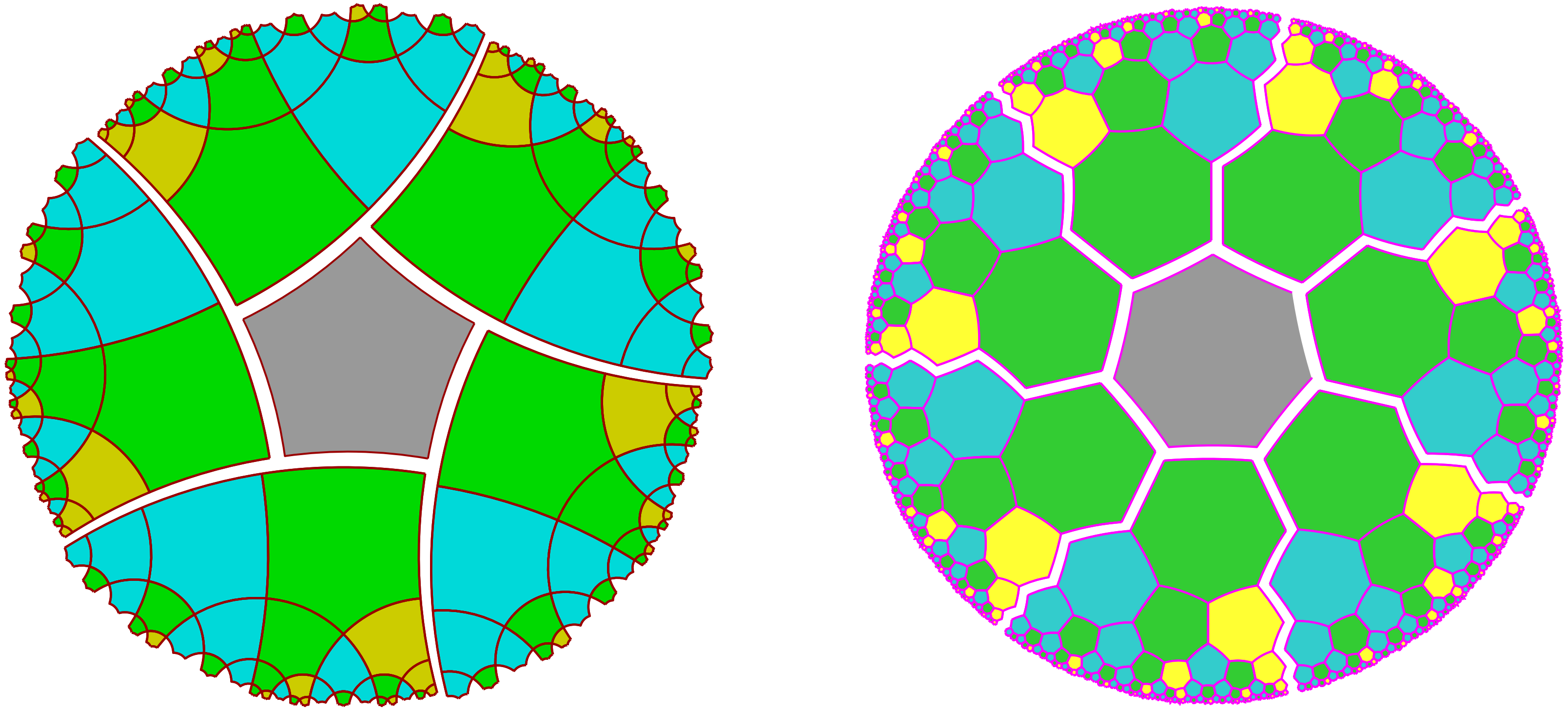
for the central tile T_0

side i defines sector i

for another tile T

side 1 is shared with the father of T

other sides numberd by counterclockwise turning around T



the shortest path from T_1 to T_2

number the sectors around T_0 from 1 to α,
counterclockwise turning around T_0,
sector 1 being fixed once for all

the coordinate of T_0 is 0

the coordinate of $T \neq T_0$ is $\nu(i)$

with i the number of the sector of T
and ν the coordinate of T in the tree
spanning the sector

the shortest path from T_1 to T_2

then as a corollary of the theorem we have:

there is an algorithm which gives a path from a tile T_1 to a tile T_2 of the penta- or the heptagrid in a time which is linear in the size of the coordinates of T_1 and T_2

note that the path given by the algorithm may be not a shortest one

the shortest path from T_1 to T_2

recently, a new result:

theorem (MM, 2008):
the coordinates being fixed in the penta- or the heptagrid, there is an algorithm which, for any pair of tiles T_1 and T_2 computes a shortest path between T_1 and T_2 in a time which is linear in the coordinates of T_1 and T_2

the shortest path from T_1 to T_2

the proof relies on the characterization of the shortest paths between T_1 and T_2

in general the shortest path between T_1 and T_2 is not unique

following two paths π_1 and π_2 between T_1 and T_2, we define the **apartness** between π_1 and π_2, denoted by **apart**(π_1, π_2), as the biggest distance bewteen tiles of π_1 and π_2 which are at the same distance from the origin of π_1 and π_2

the shortest path from T_1 to T_2

lemma
let π_1 *and* π_2 *be two shortest paths from* T_1 *to* T_2 *;*
then $\mathbf{apart}(\pi_1, \pi_2) \leq 1$

the apartness is easily computed,

it allows to characterize the **leftmost** and the **rightmost** shortest paths,

the algorithm computes the leftmost shortest path

2.3.b change of coordinates

consider a central tile O and a system of coordinates based on this tile ; consider two tiles A and T, knowing their coordinates with respect to O

theorem (MM, 2008)
in the above setting, there is an algorithm which computes the coordinates of T in a system of coordinates centred at A which is linear in the size of the coordinates of A and T in the system centred at O

change of coordinates

the theorem is a corollary of the theorem of the shortest path:

> from the coordinates of A and T in the system centered at O, we compute a shortest path between A and T ;
>
> from the shortest path, we compute the coordinate of T in a system of coordinates centered at A
>
> both parts of the computations are linear in the sizes of the coordinates of A and T in the system centered at O

2.3.c locating point in the penta- and the heptagrid

consider a central tile O and a system of coordinates based on this tile ; consider a point P of the hyperbolic plane

question:

can we find a tile T such that $P \in T$?

locating point in the penta- and the heptagrid

the answer depends on how P is defined

define P by (x, y), its cartesian coordinates centered at O

then:

we have $x^2 + y^2 < 1$ and,

if $x, y \in I\!\!R$, the problem is undecidable

locating point in the penta- and the heptagrid

if $x, y \in \mathbb{Q}$, the problem is not only decidable, it has a relatively low complexity:

theorem (KC-MM-BM-IP, 2004 ; MM, 2008) *there is an algorithm to find T such that $P \in T$ such that:*

the number of equations of circles involved in the computation is linear in the size of x and y

for any $r \in \mathbb{Q}$, $0 < r < 1$, the computation of T is polynomial in the size of x and y when $x^2 + y^2 \leq r$

locating point in the penta- and the heptagrid

the proof relies on the following:

$X^2 + Y^2 - 2aX - 2bY + 1 = 0$ is the form of the equations of the circles which support the sides of the tiles

now, $a, b \in \mathbb{Q}(\omega, \zeta)$ where ω is an algebraic integer and ζ, a primitive root of 1 of order α

locating point in the penta- and the heptagrid

this proves the decidability part of the theorem

the complexity part relies on an analysis of the operations involved in computing the new a's, b's from the former ones

2.3.d **connected results:**

two results:

> defining coordinates for points of the hyperbolic plane
>
> constructing a Peano curve in the hyperbolic plane

connected results:

coordinates for points of the hyperbolic plane

from the location algorithm, considering P with $x, y \in I\!R$ we can *define* a tile T such that $P \in T$

note that this is not constructive

the number of T in the Fibonacci tree of its sector is the **integral part** of the **coordinate** of P

coordinates for points of the hyperbolic plane

next: tile T is split into seven triangles T_i^0 constructed on its sides and its centre

> one of these triangles contains P: this defines the first digit in $\{0..\alpha{-}1\}$ of the coordinate of P, i being attached to side $i{+}1$

coordinates for points of the hyperbolic plane

then, for each i: the current tile T^i is split into four triangles T_j^{i+1} constructed on the mid-points of the sides of T^i, $j \in \{0..3\}$

> one of these triangles contains P: this defines the $i{+}1^{\text{th}}$ digit in $\{0..3\}$ of the coordinate of P

coordinates for points of the hyperbolic plane

to be more precise for the digits:

number the vertices of T_i^0 as follows:

the centre is numbered with 2, the other vertices are 0 and 1, in the natural order when counterclockwise turning around T

number the vertices of T_j^{i+1} as follows:

the mid-point of ab of T_j^i is c such that $\{a, b, c\} = \{0, 1, 2\}$

coordinates for points of the hyperbolic plane

then, in T^i_j, T^{i+1}_k has the number of its vertex which is also a vertex of T^i_j; T^{i+1}_3 is the triangle whose vertices are the mid-points of the sides of T^i_j

the orientation in the numbering of the vertices is the same for T^i_j and T^{i+1}_3 and opposite for T^i_j and T^{i+1}_k for $k \neq 3$

coordinates for points of the hyperbolic plane

let ζ be the coordinate of P

the digits of ζ are ultimately stationnary if and only if P is a vertex of some T_j^i or the intersection of all T_3^{m+n} for a certain m

let $\alpha_0\alpha_1\ldots$ be the digits of $\{\zeta\}$

define α'_k by the condition

$(*)$ $\{\alpha'_k, \alpha_{k+1}, \alpha'_{k+1}\} = \{0, 1, 2\}$,

assuming simply $\alpha'_0 \neq \alpha_0$

coordinates for points of the hyperbolic plane

we have:

lemma (MM, 2008)

ζ belongs to a side of some $T_j^{i_0}$ if and only if there is a k_0 such that the condition $(*)$ is true for all $k \geq k_0$ for a certain k_0

in particular, we get:

if the digits of $\{\zeta\}$ contains infinitely many 3's, P lies inside all T_j^i's which contain P

coordinates for points of the hyperbolic plane

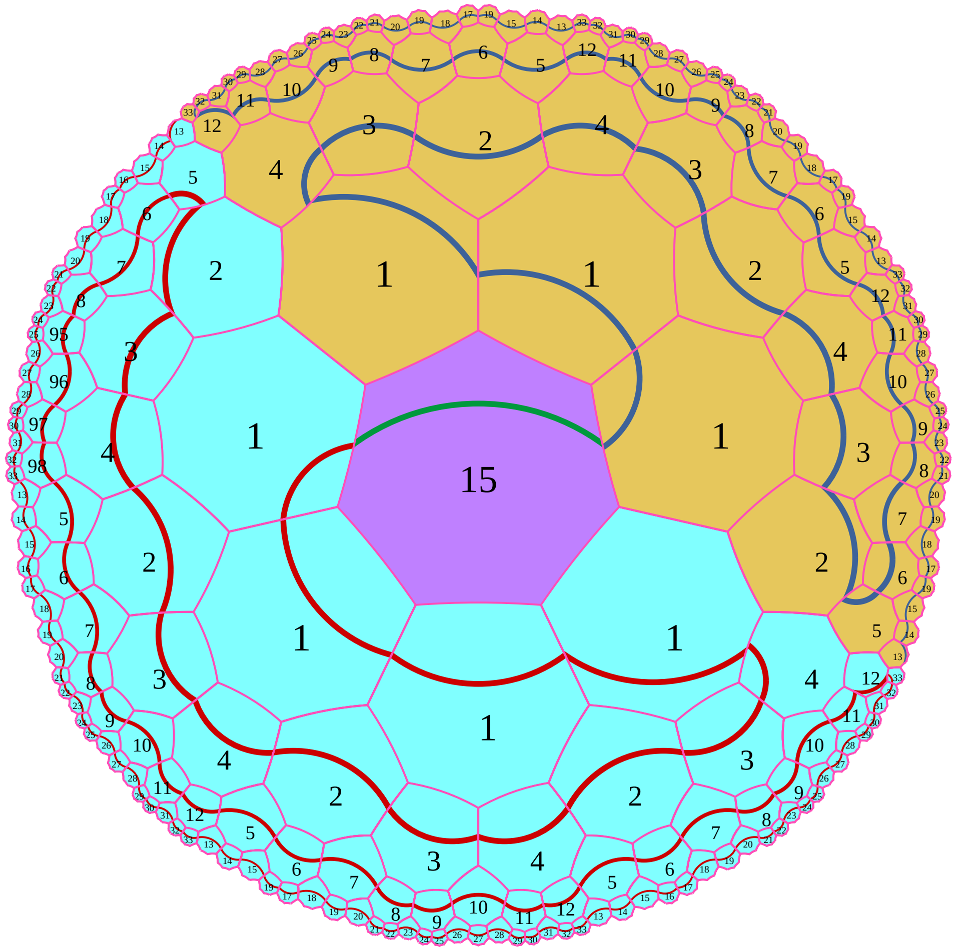
there are examples of P for which the digit of $\{\zeta\}$ are in $\{0,1,2\}$ only such that P is inside all T^i_j's which contain P

here is such an example:

take the sequence of digits defined by $(02)^\infty$

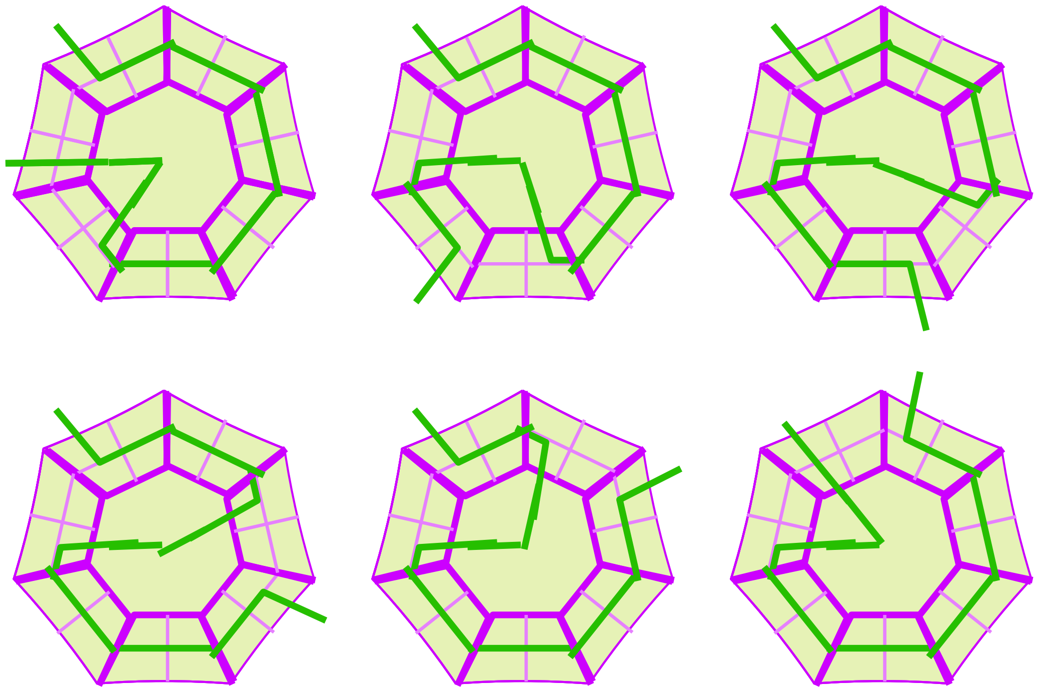
connected results:

a Peano curve in the hyperbolic plane



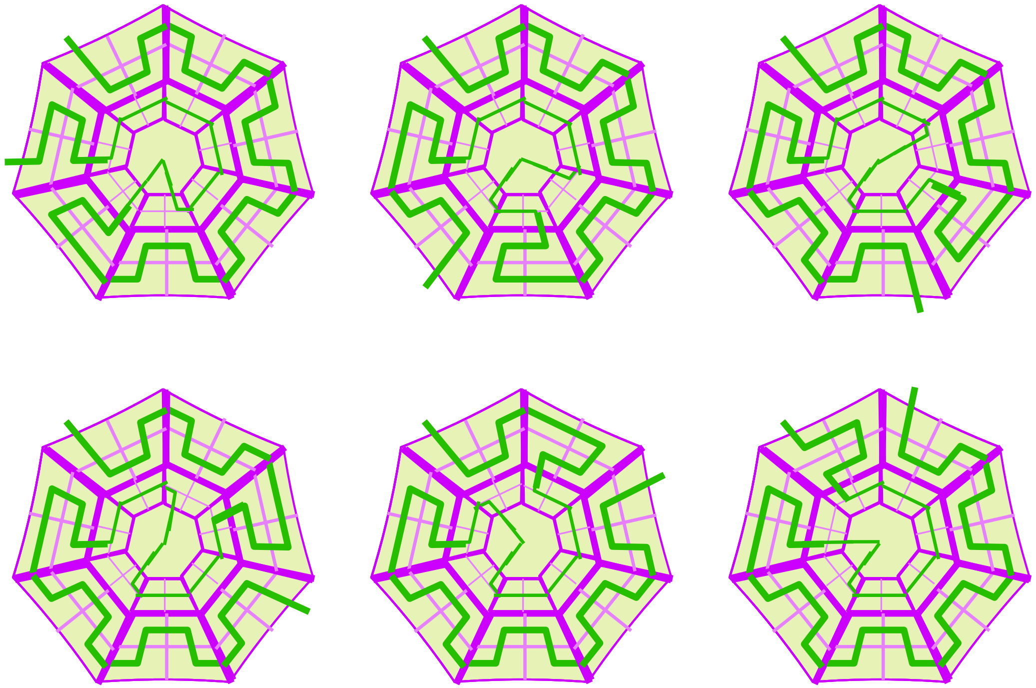
the integral part

a Peano curve in the hyperbolic plane



the first generation

a Peano curve in the hyperbolic plane



the second generation

a Peano curve in the hyperbolic plane

this can be continued for each generation:

the tile is divided into an annulus and a center

then, from the generation n to $n+1$:

the annulus goes from generation n to $n+1$

the center is repaced by a 'scaled' tile of generation n

3. application to tiling problems

3.1 the tiling problem

3.2 construction of a grid

3.3 undecidability of the tiling problem

3.1 the tiling problem

question:

is there an algorithm A such that:

given the description of a finite set $\mathcal{S}$ of tiles, the **prototiles**, A says `yes` if it is possible to tile the plane with copies of the prototiles and `no` if it is not the case in the case of the Euclidean plane:

- problem raise by Hao Wang in 1958
- conjectured decidable by Hao Wang
- proved **undecidable** by Robert Berger in 1966
- complexed proof but a very deep one, involving a bit more than 21,000 prototiles
- in 1971, Raphael Robinson gave a simpler proof of the same result

in the case of the hyperbolic plane:

in the same 1971 paper, Robinson asks:

what can be said for the hyperbolic plane ?

in 1978

Robinson proved that the origin-constrained problem is undecidable in the hyperbolic plane

it is known that he tried to solve the general problem nothing new in the case of the hyperbolic plane until 2006

in march 2006 (published 2008), I proved the undecidability of a problem which is an intermediate step between the origin-constrained problem and the general one:

it is the generalized-origin constrained problem now, in 2007:

the general problem is proved undecidable

first proof, *arXiv:cs/*0701096 (MM2007) presented at the AMS sectional meeting, Davidson, NC, March 2007

at the same meeting, another proof announced by Jarkko Kari, completely different and independent

Oct. 2008,

TCS, MM2008:

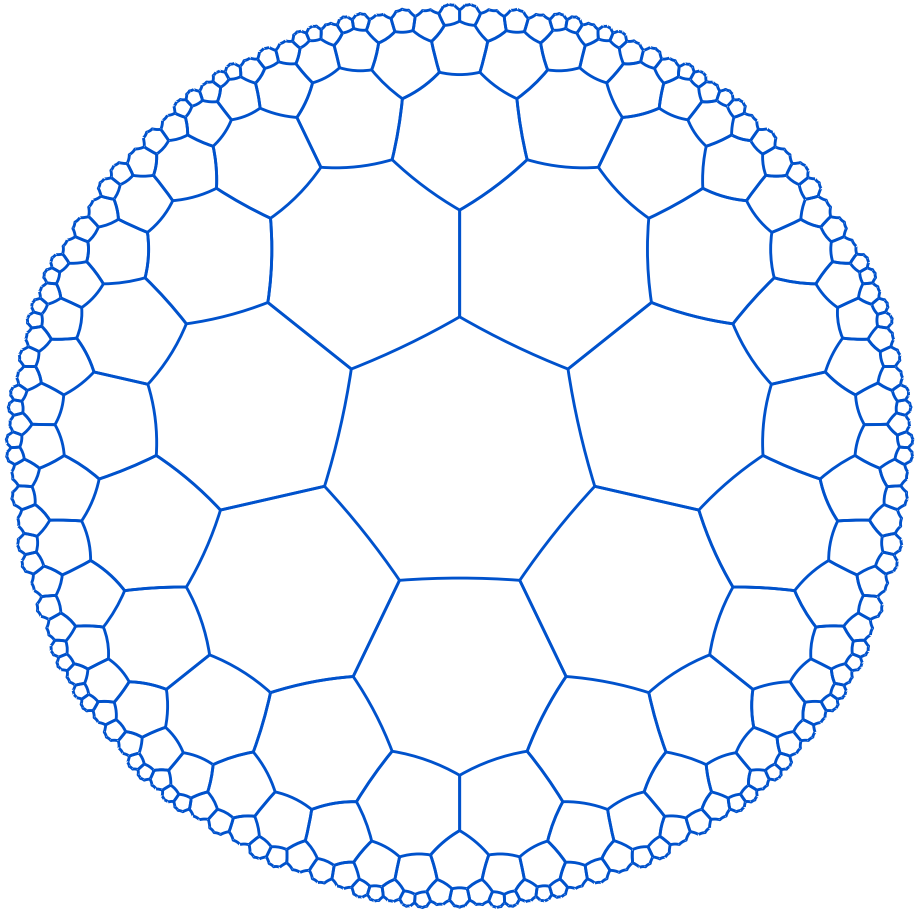
single full proof published up to date

here:

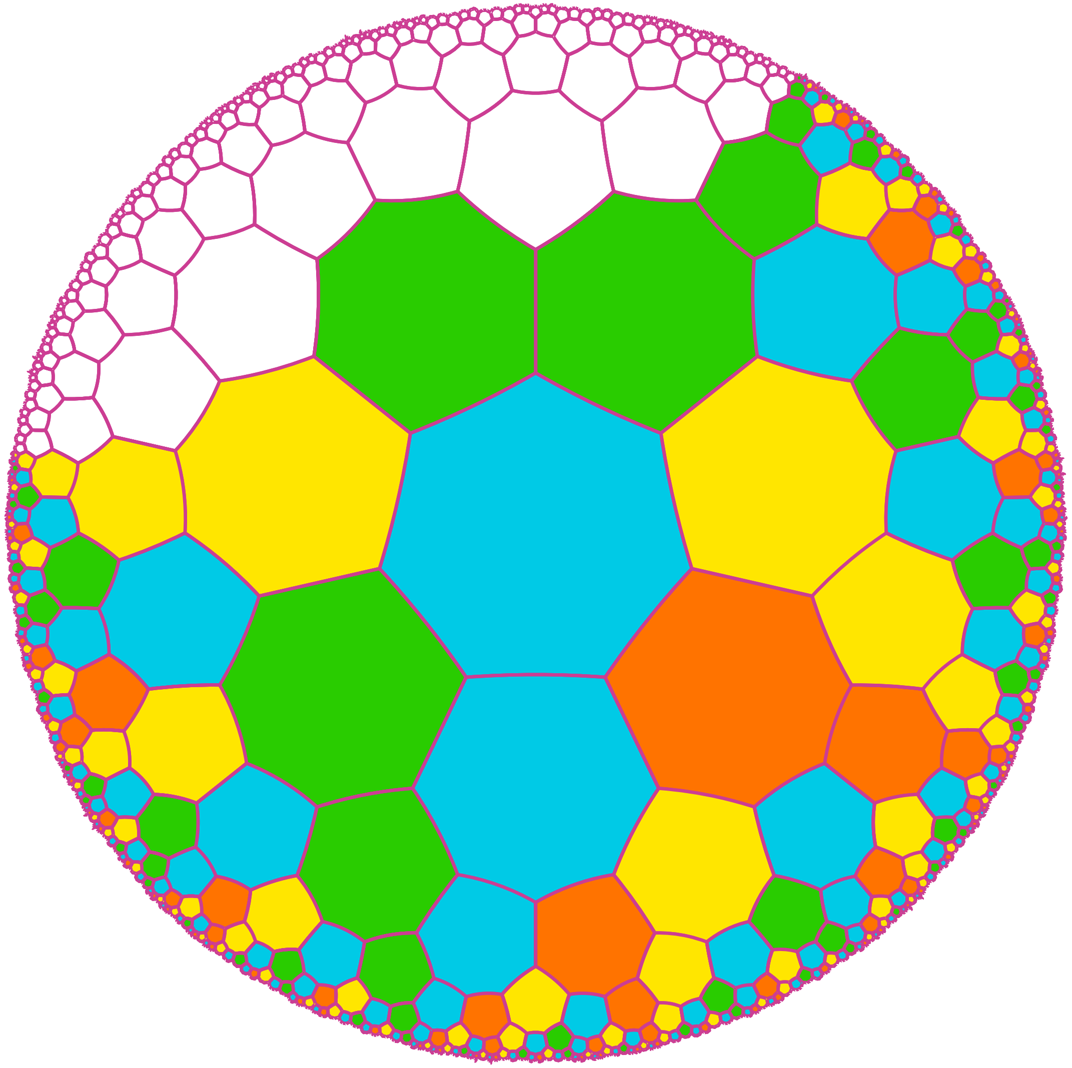
- a variant of the TCS proof
- sketchy outline of the proof:
  - construction of a grid
  - and then:
    - the interwoven triangles
    - their implementation in $I\!H^2$
    - simulating a Turing machine

3.2 construction of a grid

recall **the ternary heptagrid:**



another way to tile the heptagrid:



$$G \longrightarrow YBG$$
$$Y \longrightarrow YBG$$
$$B \longrightarrow BO$$
$$O \longrightarrow YBO$$

two trees with one set of tiles:

central Fibonacci tree:

black: B

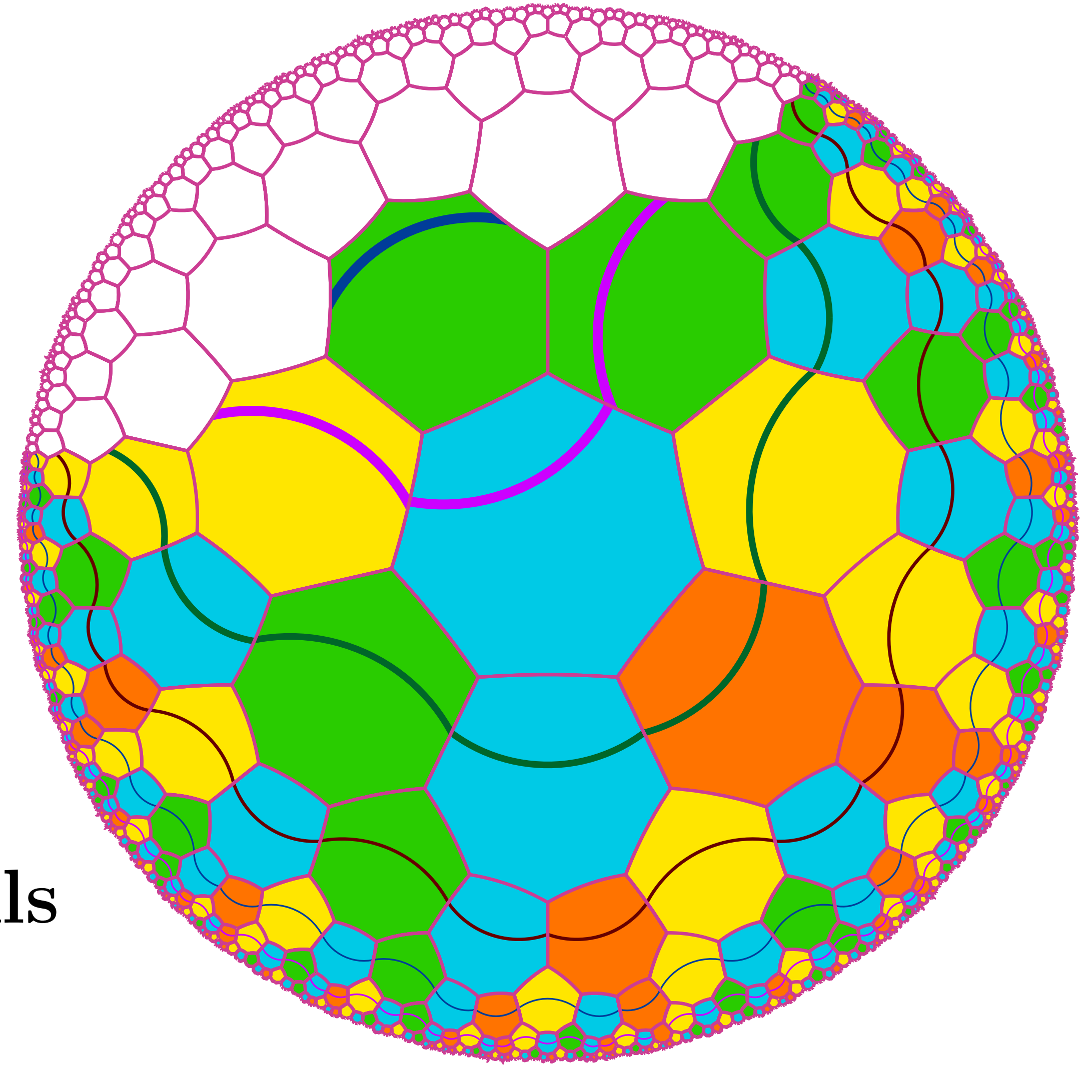
white: G, O and Y

adapted standard Fibonacci tree:

black: G and Y

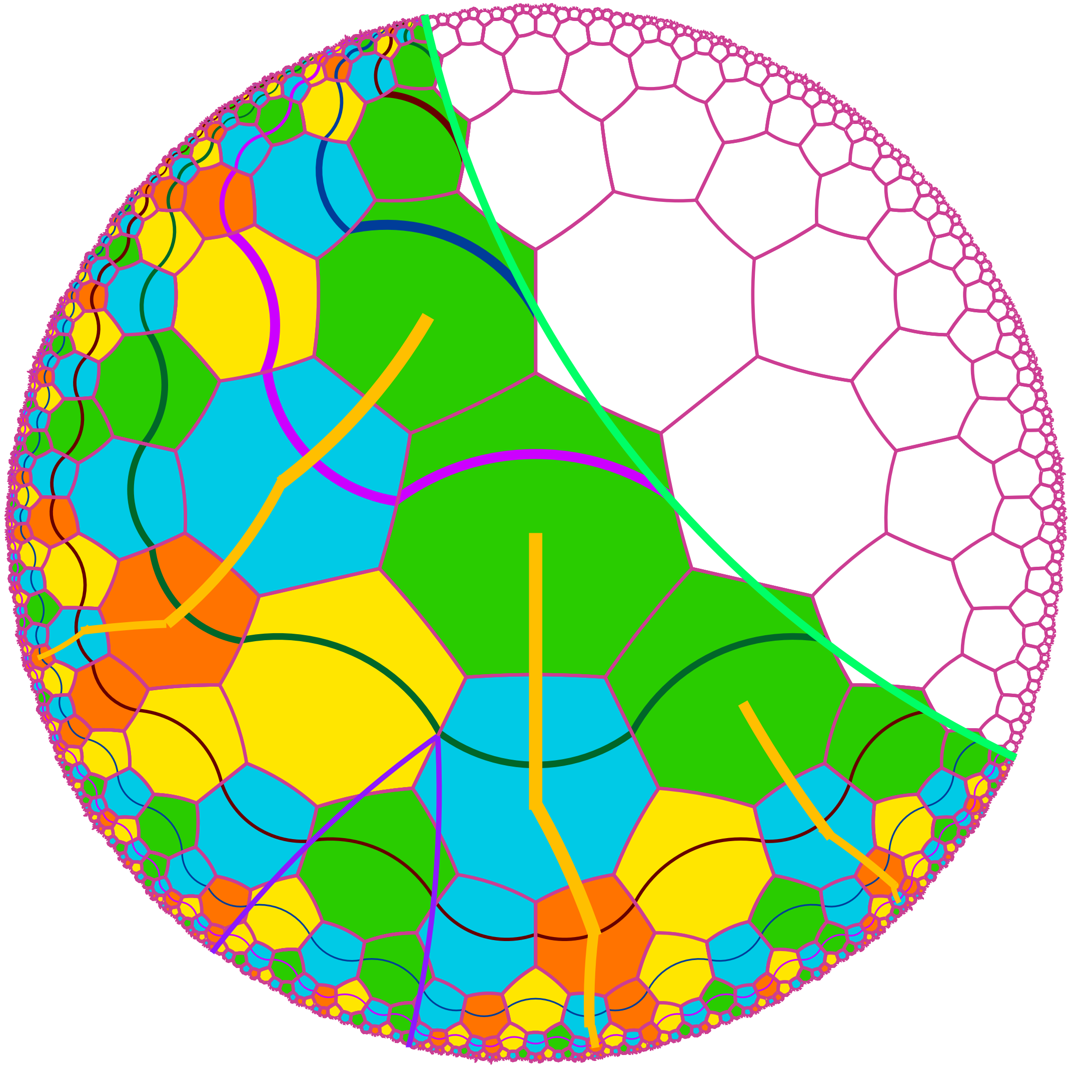
white: B and O

the levels in the adapted standard tree



they are
the **horizontals**

the verticals in the adapted standard tree



they avoid
any standard
subtree
with a G-root

this defines a **grid**

possible to simulate the computation of a Turing machine:

easy construction,
already coming from Hoa Wang

3.3 the undecidability of the tiling problem in the hyperbolic plane

3.3.a the interwoven triangles

3.3.b the trees and the seeds

3.3.c simulations of a Turing machine

three-stepped construction:

first:

a one-dimensional process on brackets

next:

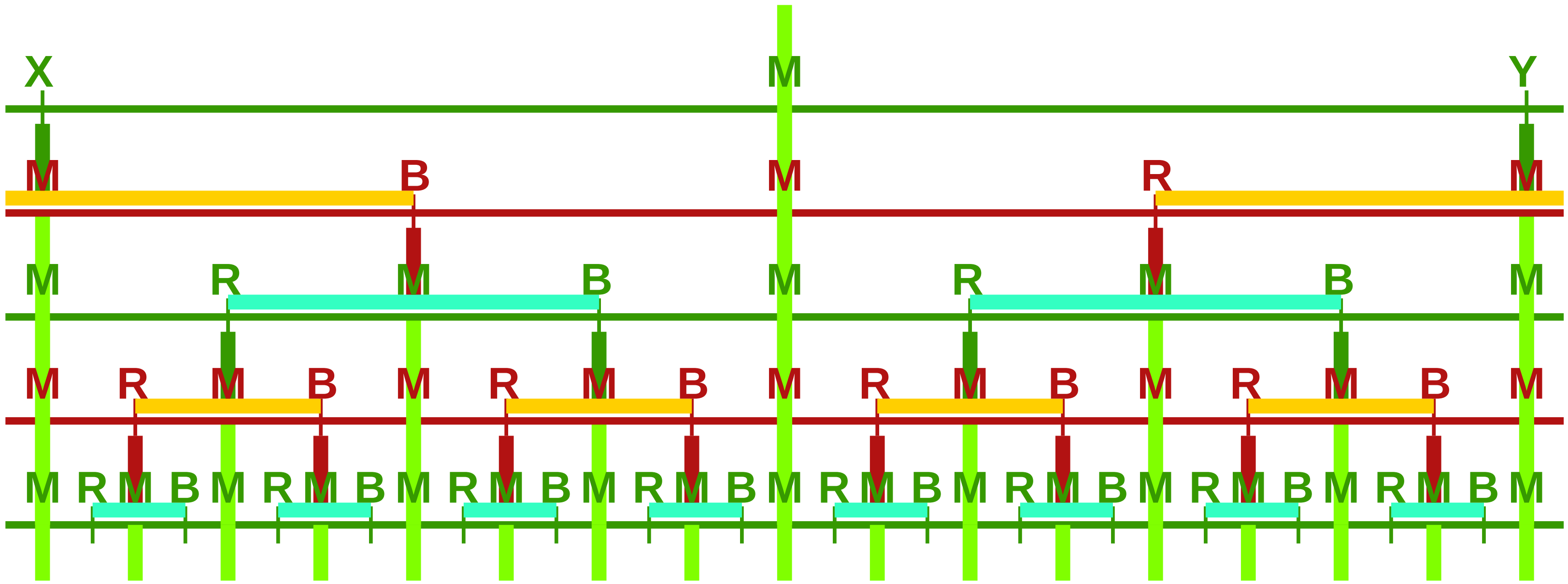
lift up the process in the Euclidean plane

and then,

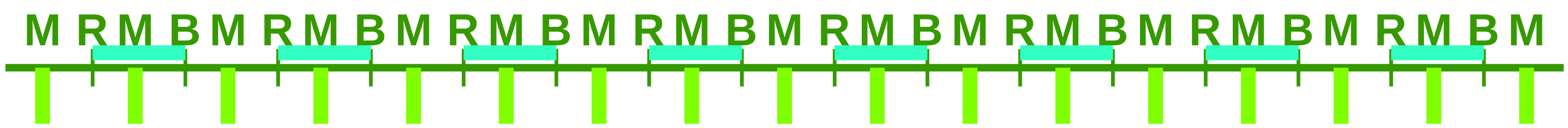
in the hyperbolic plane

a one-dimensional process on brackets

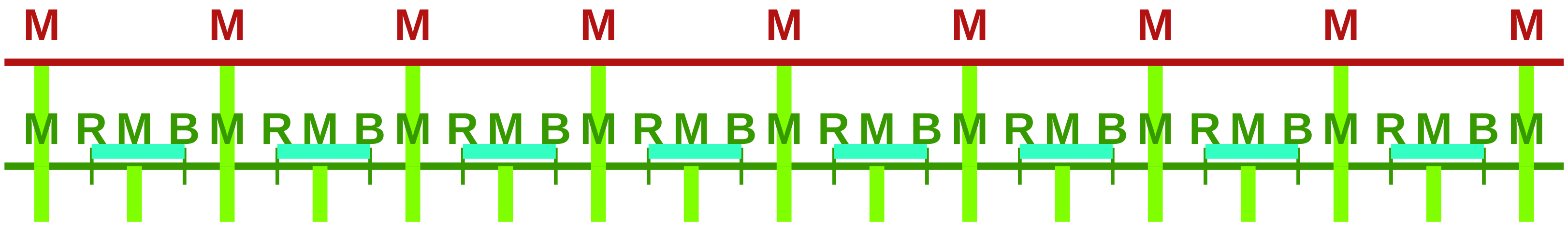
consider the following picture:



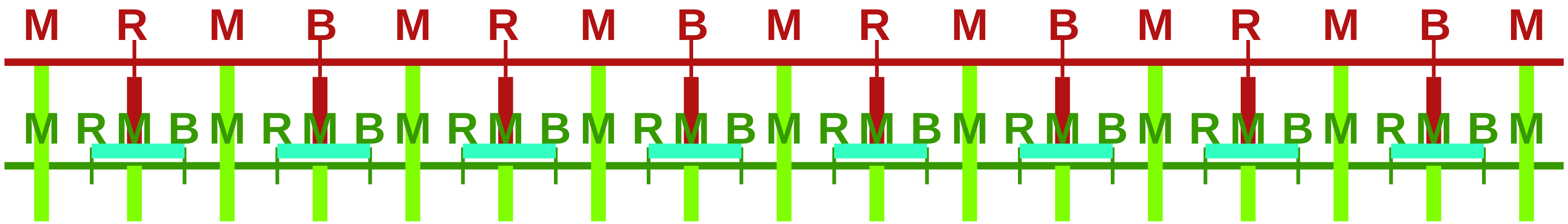
it is a result of the following process:



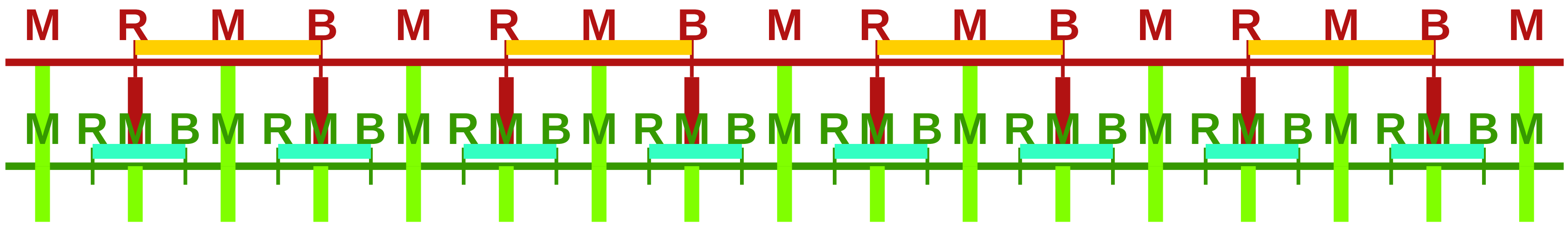
it is a result of the following process:



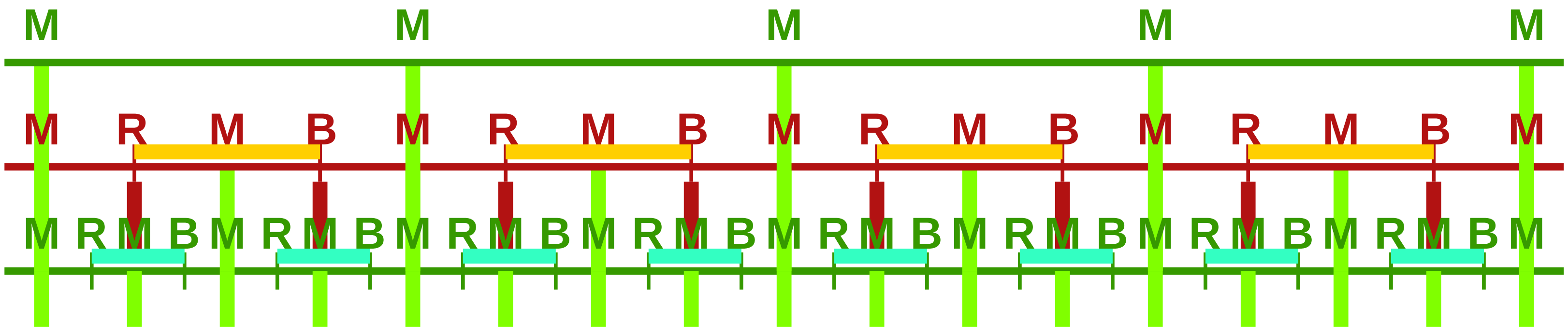
it is a result of the following process:



it is a result of the following process:



it is a result of the following process:



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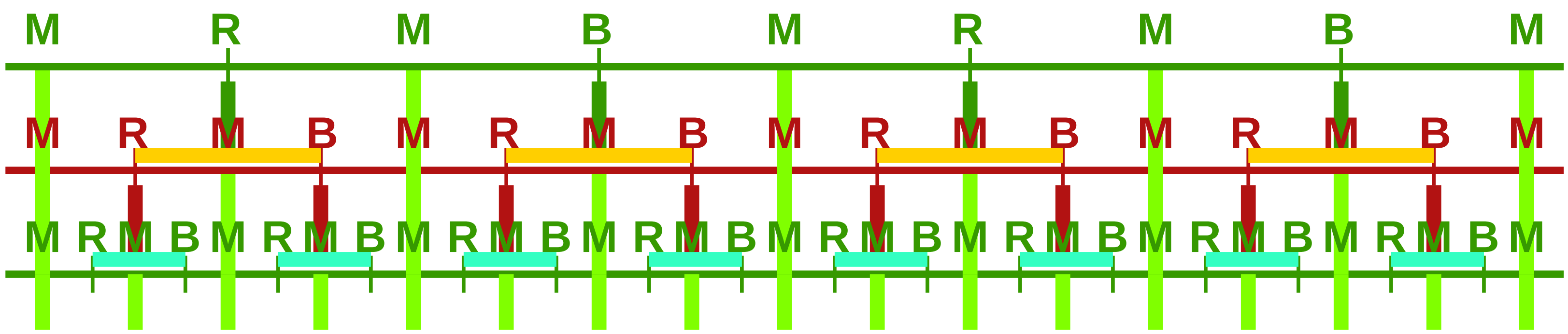
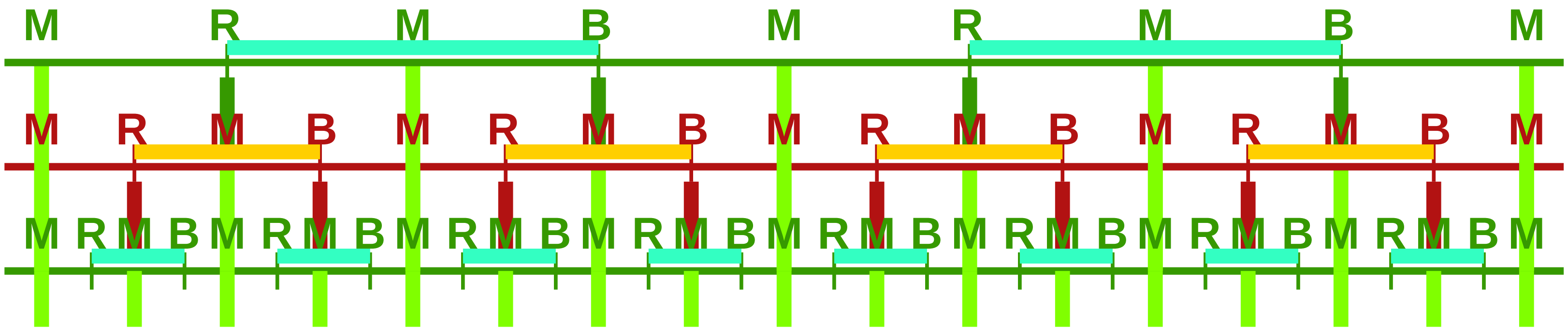
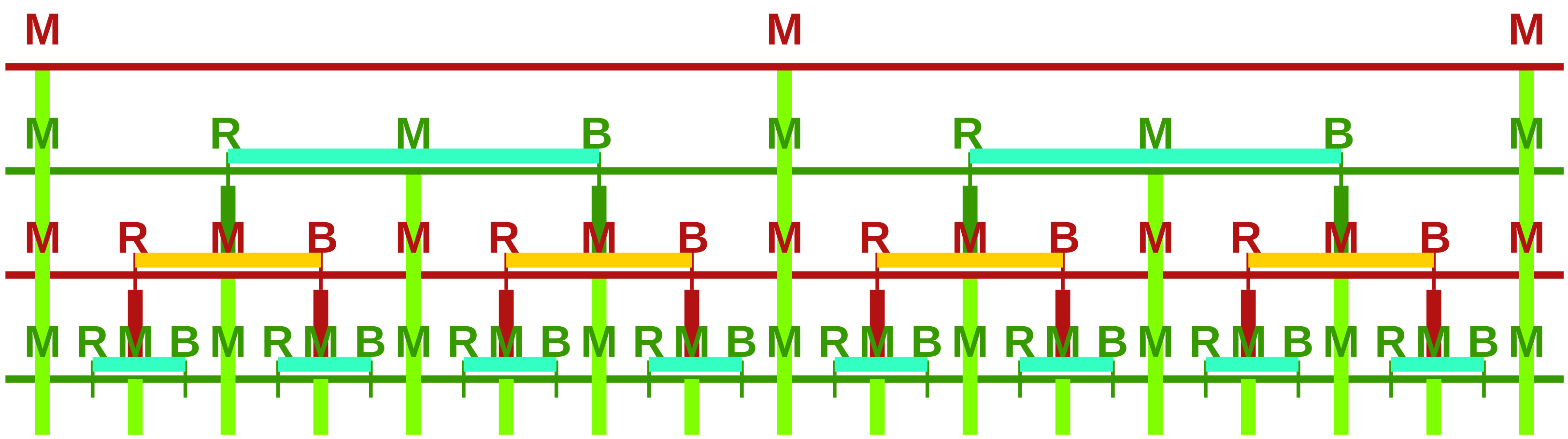


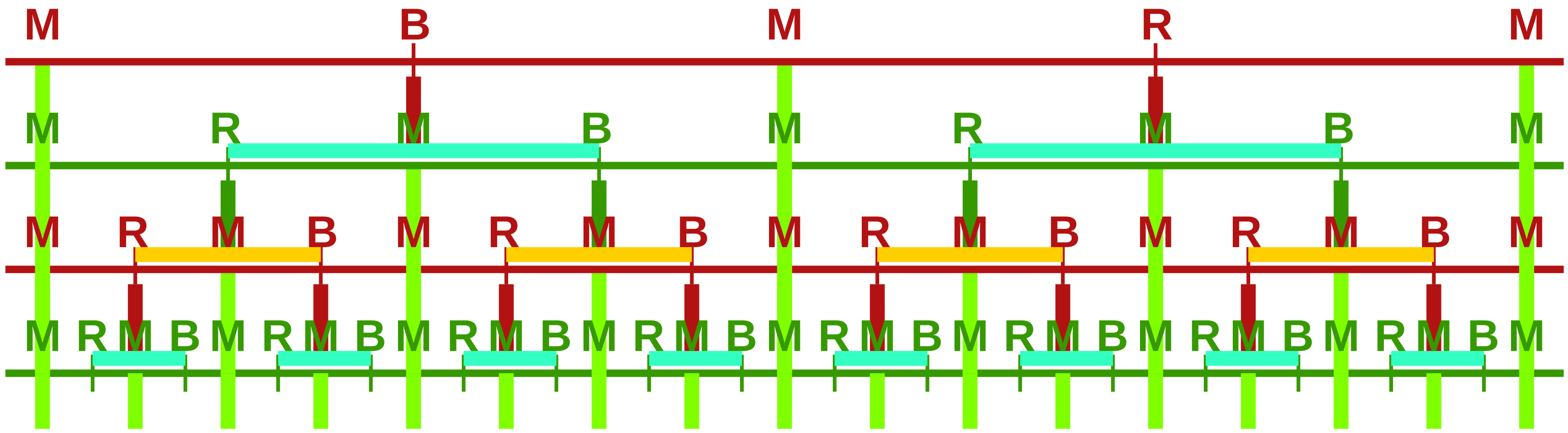
it is a result of the following process:



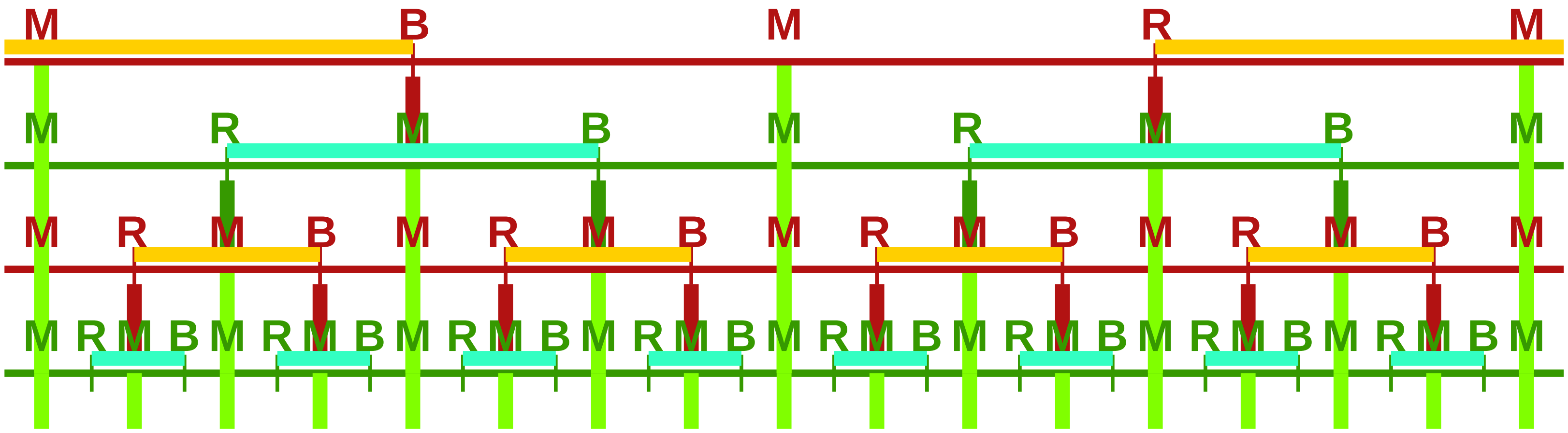
it is a result of the following process:



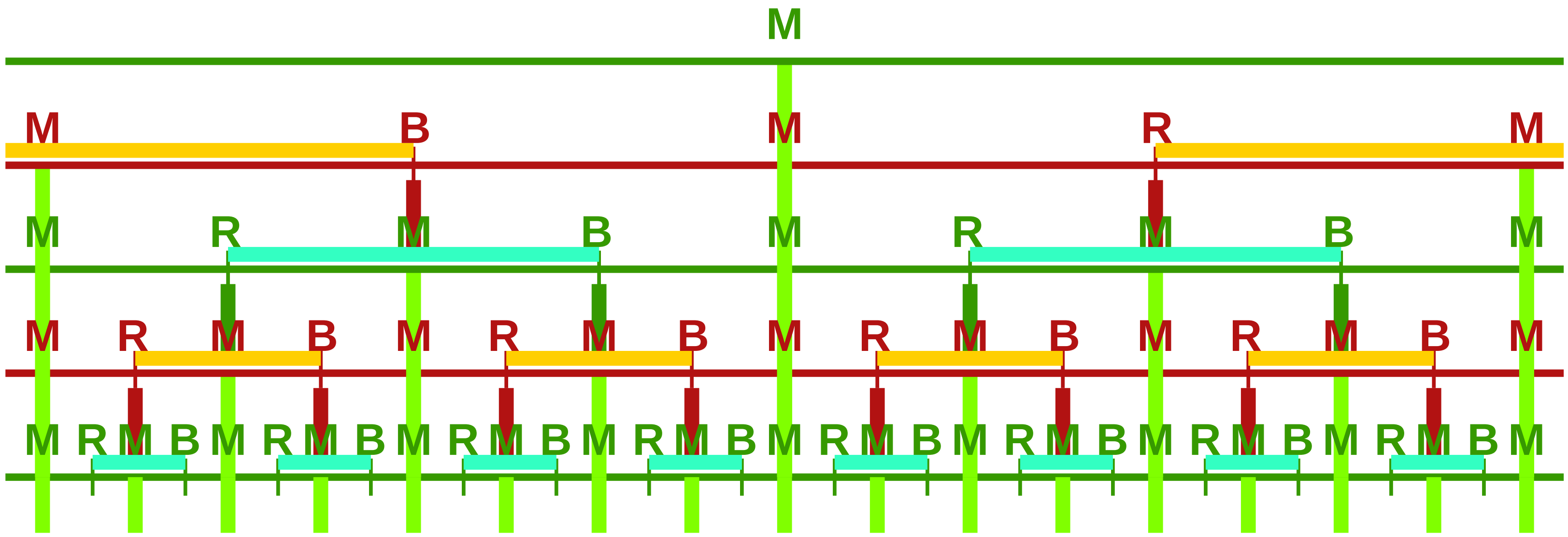
it is a result of the following process:



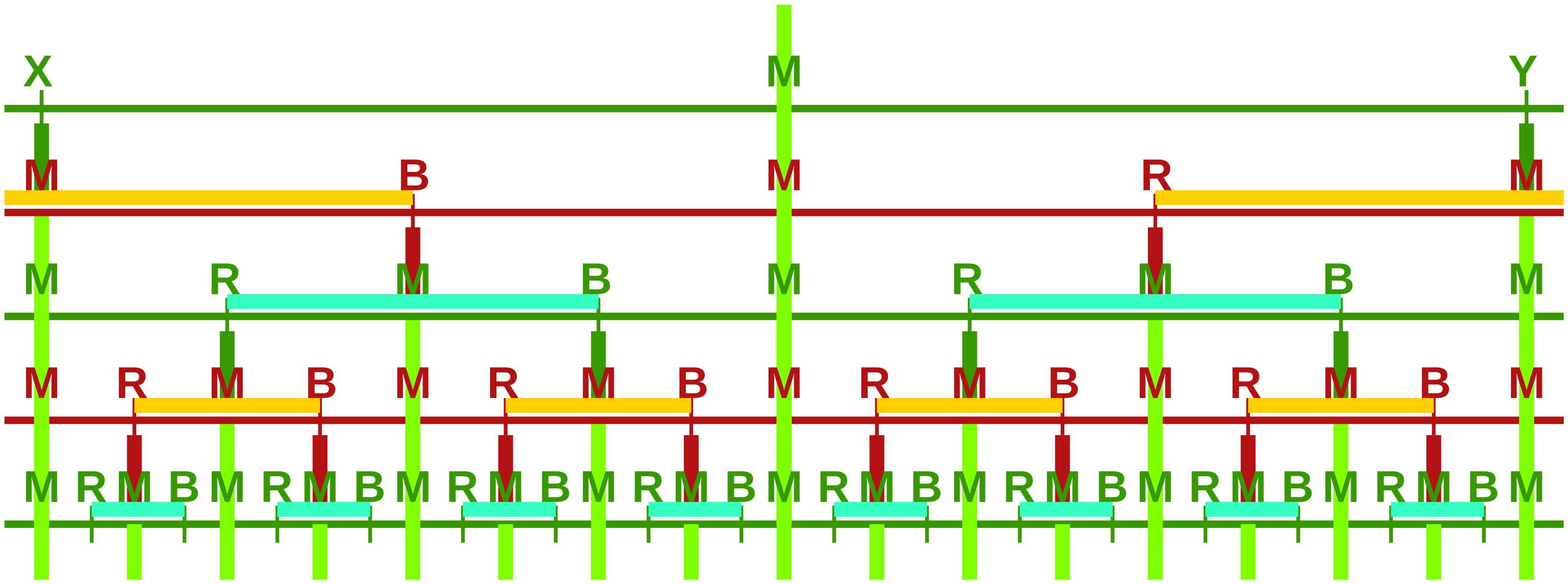
it is a result of the following process:



it is a result of the following process:



it is a result of the following process:



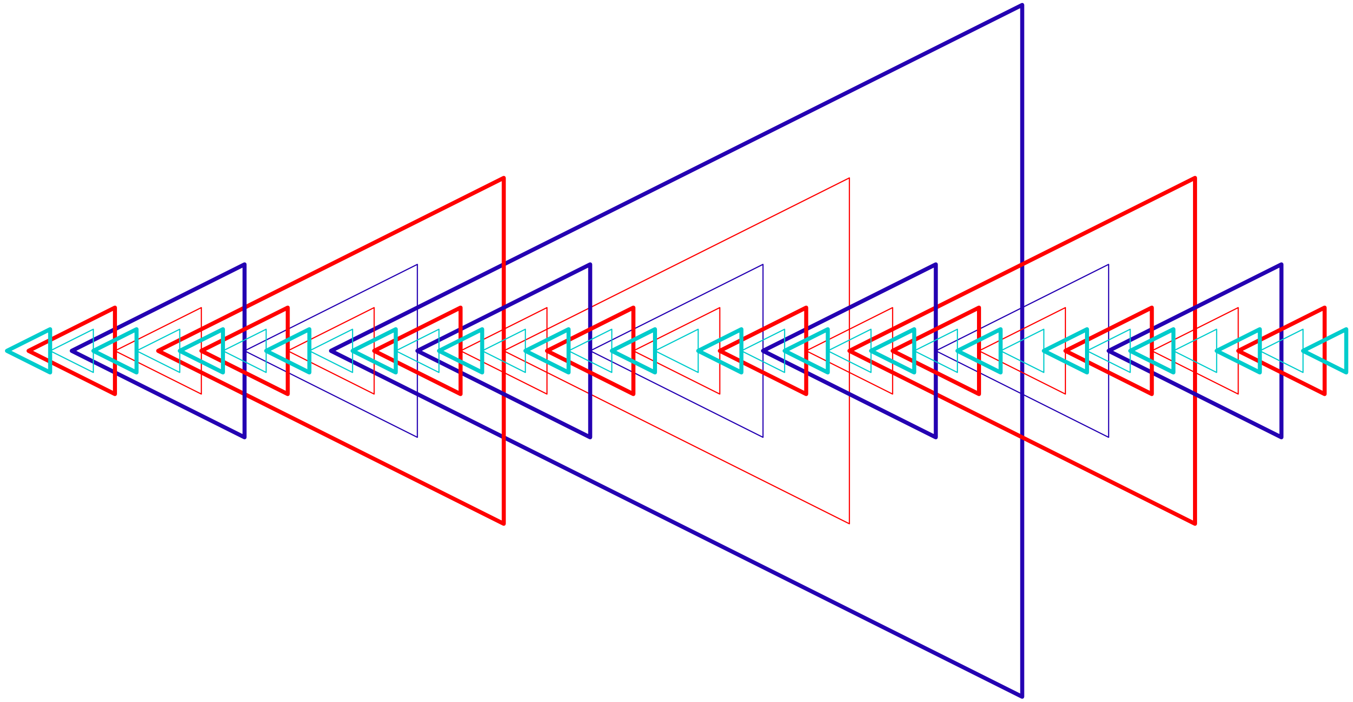
3.3.a the interwoven triangles:

it consists in lifting up the construction:

first into the Euclidean plane

and then into the hyperbolic plane

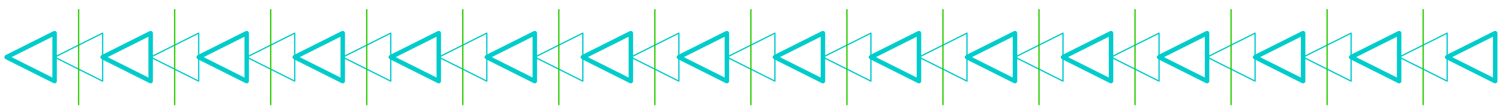
the key pattern:



from this, the Euclidean interwoven triangles:

triangles and **phantoms**,

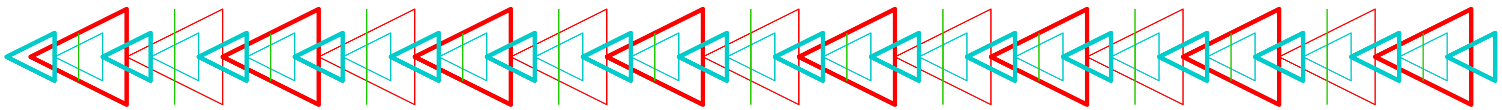
generation 0, blue-0



from this, the Euclidean interwoven triangles:

triangles and **phantoms**,

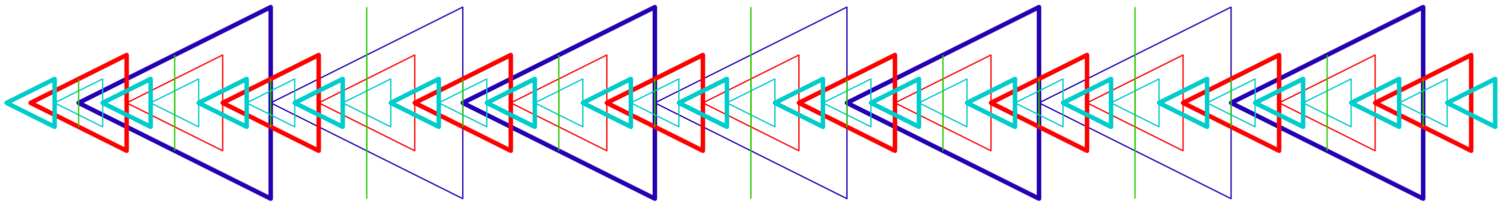
generation 1, red



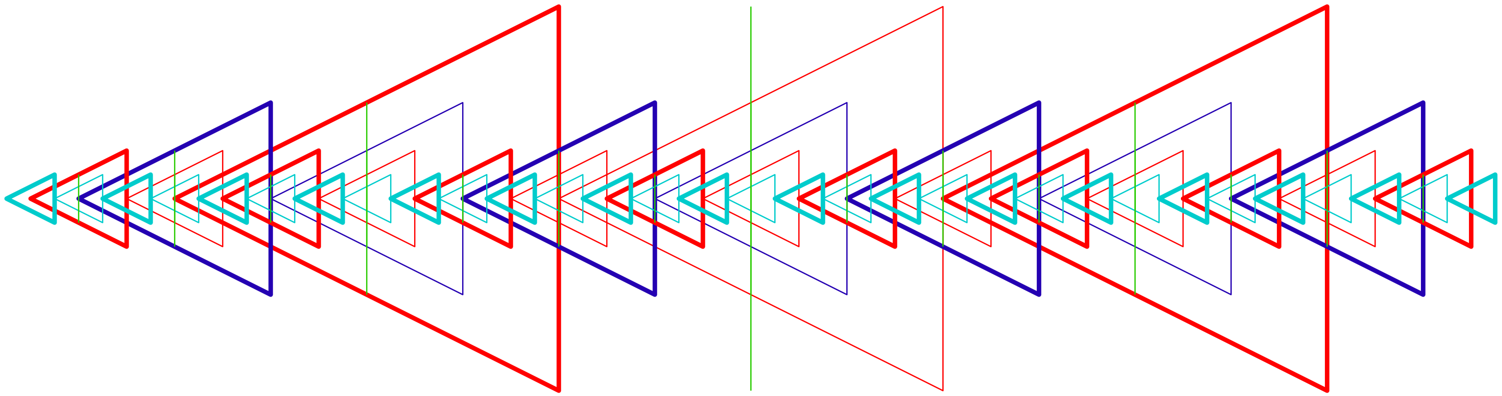
from this, the Euclidean interwoven triangles:

triangles and **phantoms**,

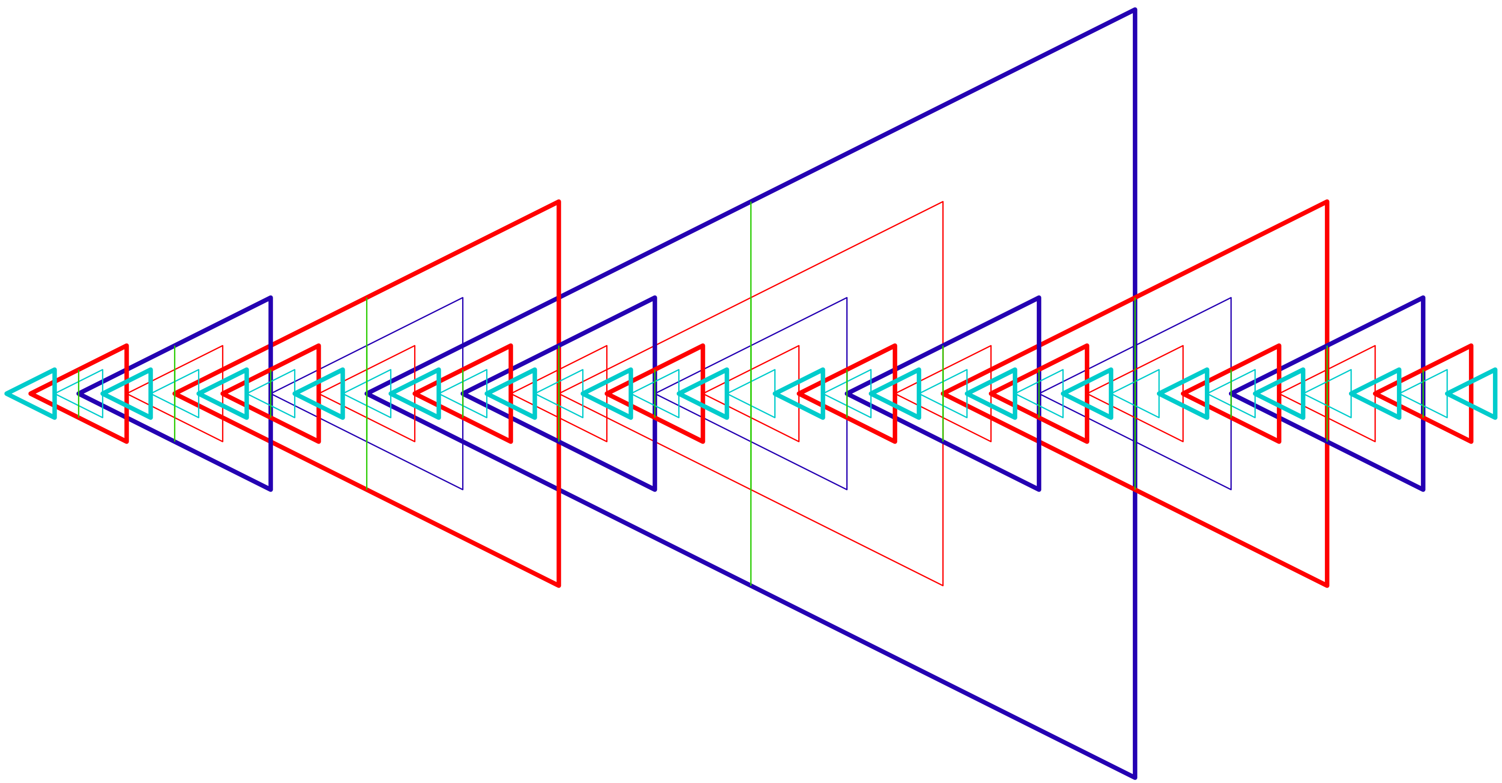
generation 2, blue



from this, the Euclidean interwoven triangles:
triangles and **phantoms**,
generation 3, red



from this, the Euclidean interwoven triangles:
triangles and **phantoms**,
generation 4, blue, and so on...



this figure can be defined by a finitely generated tiling of the Euclidean plane (190 tiles) which forces a relization of this construction

3.3.b the trees and the seeds

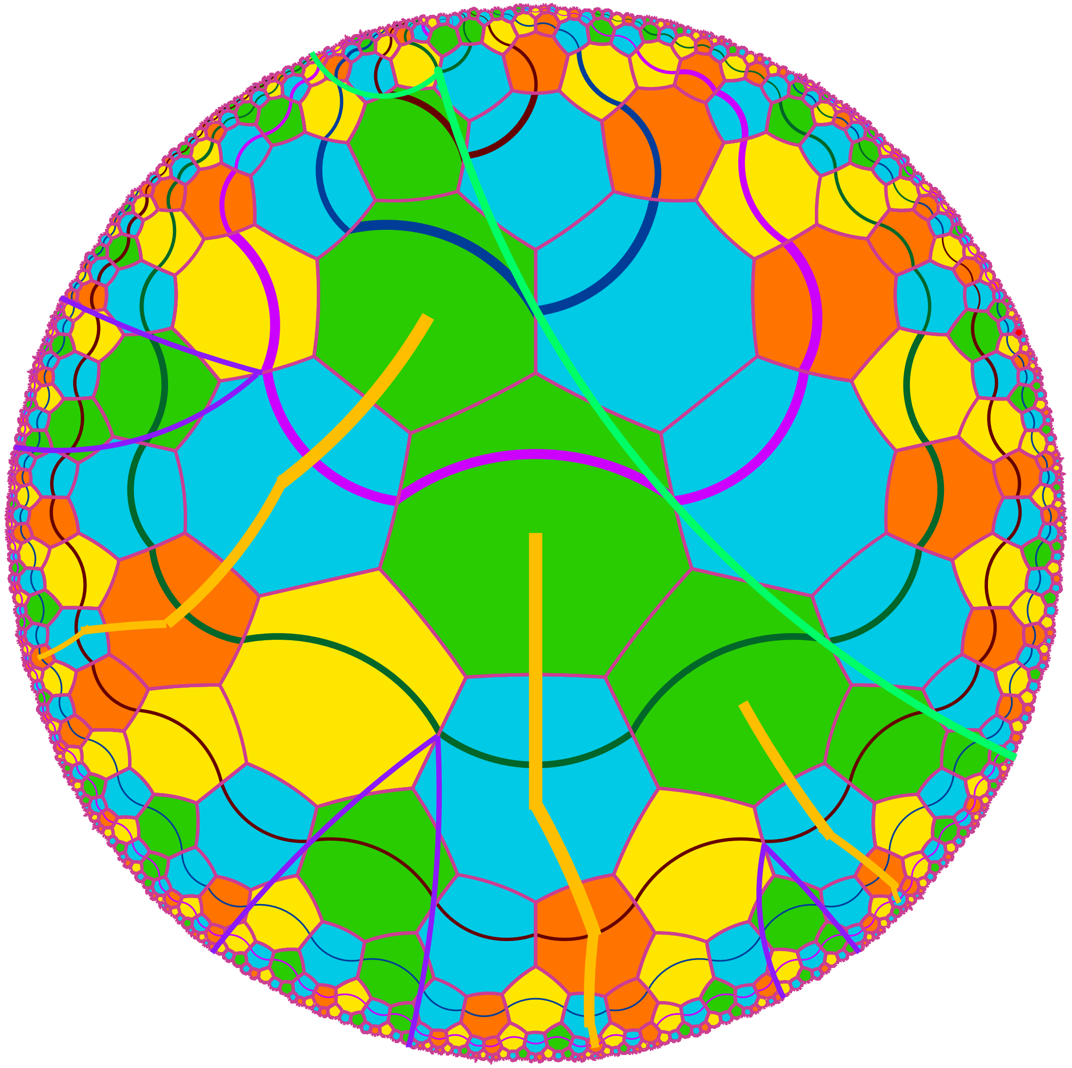
implementation in the hyperbolic plane thanks to:

- threads of successively embedded trees of the heptagrid and
- synchronization of the implementation on each thread with that of the others

the trees with the green roots

key property:

either
embedded
or
disjoint



avoided
by the
verticals

trees of the heptagrid

we select the trees:

say a G-node with Y-father and G-grand father is a **root**

a root generates a **tree of the heptagrid**:

V is the vertex of the root, e the edge between the Y-father and its B-uncle

let A be the mid-point of e
from e, draw the two mid-point rays which cross the root
they define the **borders** of the tree

main property of the trees of the heptagrid

two trees of the heptagrid are either embedded or disjoint

trees of the heptagrid can be gathered into **sequences** of consecutively embedded elements, indexed by $\mathbb{N}$ or $\mathbb{Z}$, called **threads**

isoclines

the horizontals we defined

number them periodically from 0 to 7

this defines the direction

from up to bottom

seeds:

roots which are on an even isocline

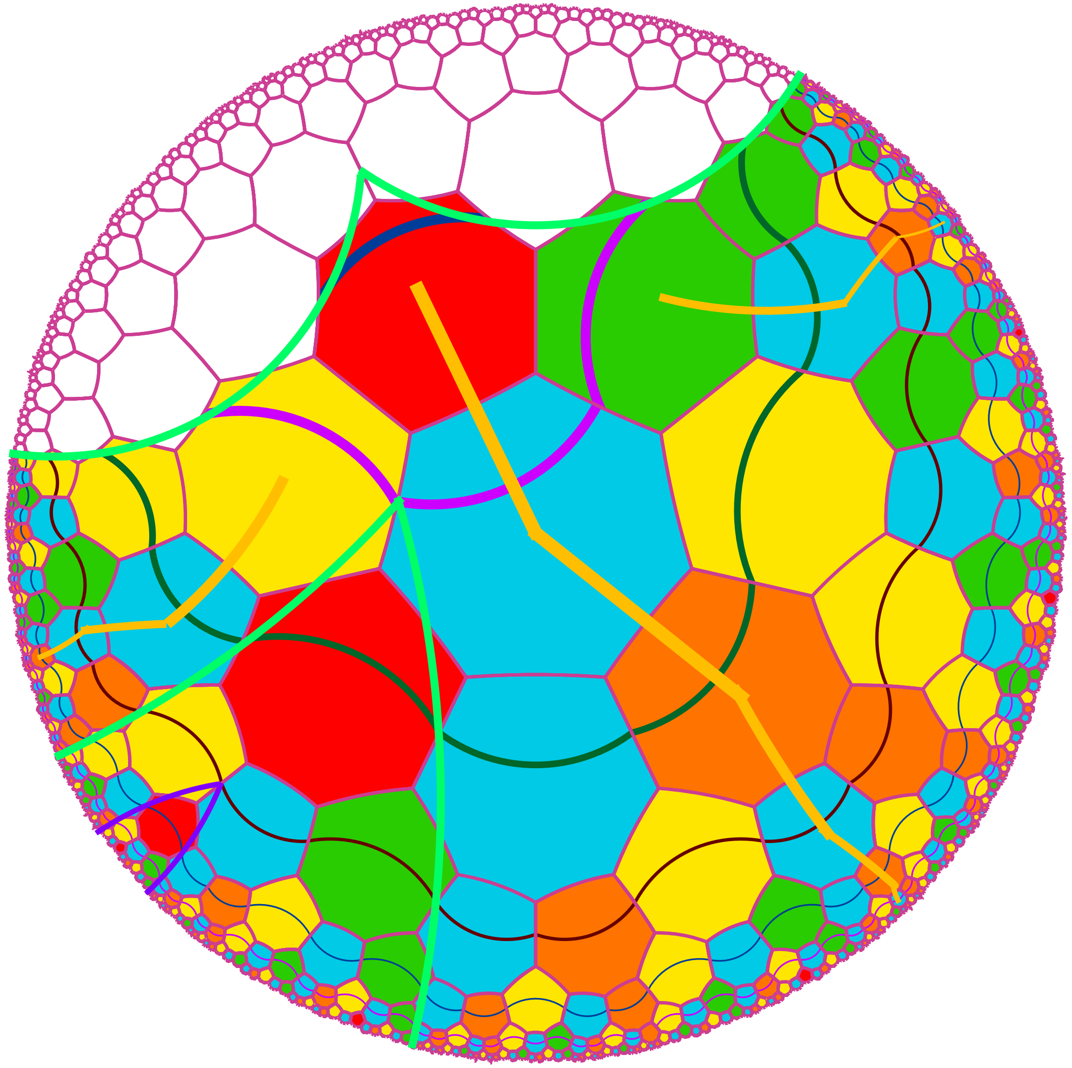
the seeds on an isocline 0 are **active**

by definition, seeds within a tree rooted at an active seed σ and lying on the 2nd isocline below σ are also **active**

the set of the seeds is **dense enough** in $I\!H^2$:

for any tile T of the heptagrid, there is an active seed within a ball of radius 10 around T

the seeds, the isocines and the verticals



the verticals
avoid the trees

the interwoven triangles in the hyperbolic plane:

the **triangles** and **phantoms** are defined by the active seeds

- the legs are along the extremal branches of the tree rooted at the seed
- the basis runs along an isocline

- the generation 0, blue-0 triangles, have their vertices on the isoclines 0 and their basis on the next isoclines 10,

  then red and blue generations alternate

- in each generation, the active seeds on a basis of a triangle generate a **phantom**, the same figure as a triangle but distinguished, whose basis is on the isocline of the vertices of the next triangles

the verticals:

they are defined by the sequence of alternating B- and O-nodes issued from a the root of a tree or from a G- or a Y-node of its border

key property:

verticals never meet a tree of the heptagrid rooted at an active seed

the horizontals:

red triangles contain isoclines 0 and 4 which never meet any inner red triangle, call them **free rows**

in a red triangle T of the generation $2n+1$ there are 2^{n+1} free rows which never meet an inner red triangle of T

3.3.c simulations of a Turing machine

the just defined free rows and verticals inside a red triangle define a **grid** in which an initial segment of the computation of a Turing machine can be defined

in each red triangle, the same computation of the same Turing machine starting from an empty tape is simulated

thanks to the properties of the trees, the different computations do not interfer with each other

simulations of a Turing machine

as there are infinitely many triangles of all admissible sizes which makes an increasing sequence,

as the same Turing machine with the same data is simulated from the beginning in each red triangle,

we get:

the set of prototiles tiles the plane if and only if the machine does not halt

4. application to cellular automata in the hyperbolic plane

4.1 characterization of hyperbolic CA's

4.2 injectivity problem of their global function

4.3 small universal hyperbolic CA's

4.4 beyond the Turing barrier

4.1. characterization of hyperbolic CA's

we place α sectors around a central tile

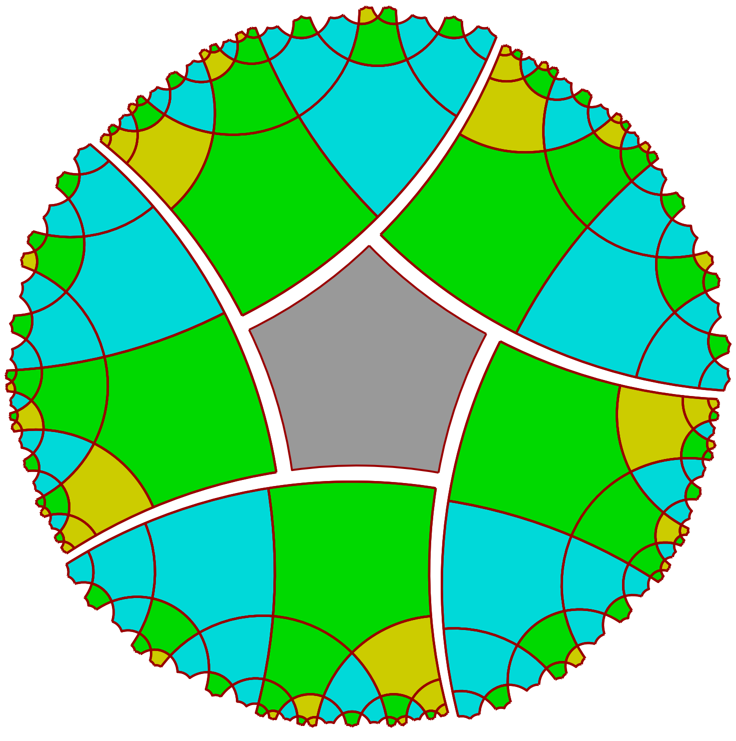
coordinates of a cell:

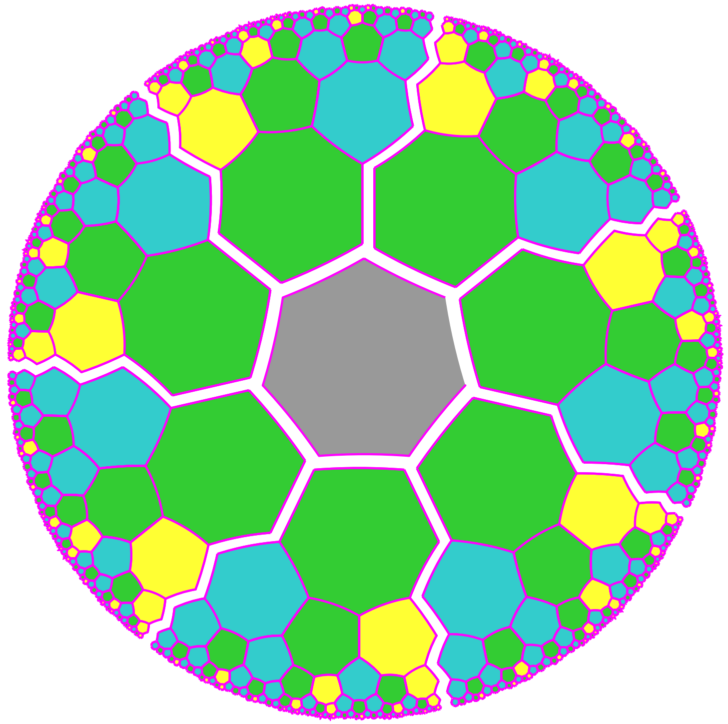
central cell: 0, otherwise, $\nu(\sigma)$

with:

ν coordinate of the cell in its quarter

$\sigma \in \{1..\alpha\}$: number of the sector remember the coordinate system





numbering the neighbours

neighbour 1:

- central cell: fixed once and for all
- other cells: the father

all cells:

- neighbours $2..\alpha$:
  - increasing numbers while counter-clockwise turning around the cell, starting from neighbour 1

format of a rule

η_0: current state of the cell

η_i: current state of neighbour i,
$i \in \{1..\alpha\}$

η_0^1: new state of the cell

format: $\eta_0\eta_1\eta_2..\eta_\alpha \longrightarrow \eta_0^1$

for short: $\underline{\eta_0}\eta_1\eta_2..\eta_\alpha\underline{\eta_0^1}$

$\eta_0\eta_1\eta_2..\eta_\alpha ==$ **context** of the rule

characterization of these CA's

remind the classical results in the Euclidean case:

global function of a CA A:

set of configurations: $Q^{\mathbb{Z}^2}$, Q: states of A

global function: $G_A : Q^{\mathbb{Z}^2} \mapsto Q^{\mathbb{Z}^2}$ defined by $G_A(x)(c) = f(N(c))$,

f: transition function of A, the **rules**

$x \in Q^{\mathbb{Z}^2}$, $c \in \mathbb{Z}^2$

and $N(c)$: neighbourhood of c

Hedlund's theorem

classical characterization theorem of the global function of a CA:

theorem (Hedlund, 1969)
Let $F : Q^{\mathbb{Z}^2} \mapsto Q^{\mathbb{Z}^2}$, where Q is a finite set. Then, F is the global function of a CA A with states in Q if and only if F is continuous, when $Q^{\mathbb{Z}^2}$ is fitted with the product topology, and if F commutes with the shifts σ_1 and σ_2.

shift σ_1: $(x, y) \mapsto (x+1,y)$
shift σ_2: $(x, y) \mapsto (x, y+1)$

global function of a CA on the penta- and the heptagrids

let $\mathcal{F}_\alpha$, $\alpha \in \{5,7\}$, be the set constituted of a central cell O and the union of α sectors around O, spanned by the Fibonacci tree

space of configurations:

$Q^{\mathcal{F}_\alpha}$, where Q set of states of the CA A

global function:

$G_A : Q^{\mathcal{F}_\alpha} \mapsto Q^{\mathcal{F}_\alpha}$ given by:
$G_A(x)(c) = f(x(N_c), x(c)$, $x \in Q^{\mathcal{F}_\alpha}$,
$c \in \mathcal{F}_\alpha$, N_c: neighbours of c

properties of the shifts on the penta- and the heptagrids

lemma 1

the shifts of the hyperbolic plane which leave the pentagrid globally invariant are generated by two shifts and their inverses

lemma 2

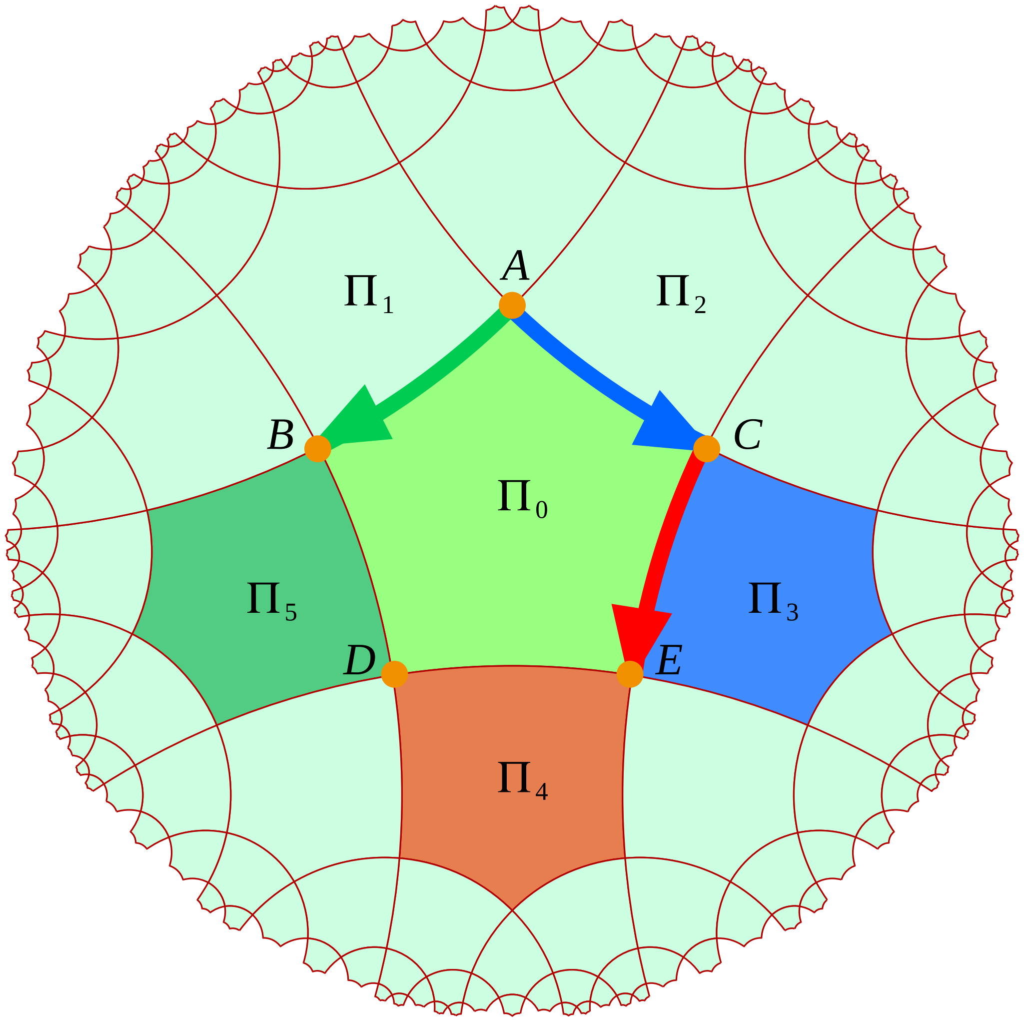
the shifts of the hyperbolic plane which leave the ternary heptagrid globally invariant are generated by two shifts and their inverses

basic lemma:

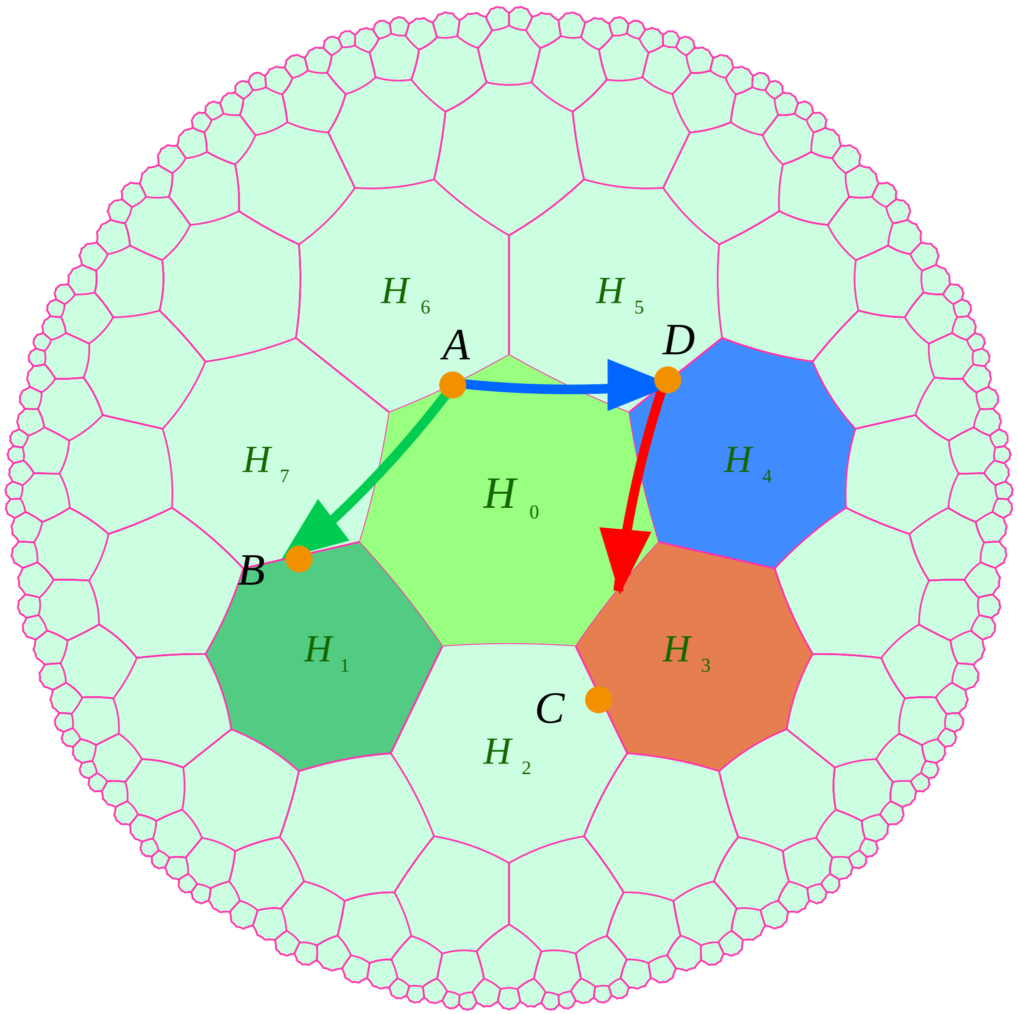
let τ_1 *and* τ_2 *be two shifts along the lines* ℓ_1 *and* ℓ_2 *respectively; then,* $\tau_1 \circ \tau_2 \circ \tau_1^{-1}$ *is a shift along the line* $\tau_1(\ell_2)$*, with the same amplitude as* τ_2 *and in the same direction*

notation: $\tau_2^{\tau_1} == \tau_1 \circ \tau_2 \circ \tau_1^{-1}$

shifts in the pentagrid:



shifts in the heptagrid:



rotation invariant CA's

assume that N_c is a ball or radius k around c, $k \geq 1$, fixed once for all, let $\alpha \in \{5,7\}$

let π be a circular permutation on $[1..\alpha]$;
it induces a rotation on N_c, denote it by $[\pi(1)..\pi(\alpha.u_k)]$; say that π is **extended** to $[1..\alpha.u_k]$

say that a CA A is **rotation invariant** if and only if for any rule $\underline{\eta_0}\eta_1..\eta_{\alpha.u_k}\underline{\eta_0^1}$ and any circular permutation π on $[1..\alpha]$ extended to $[1..\alpha.u_k]$, the rule $\underline{\eta_0}\eta_{\pi(1)}..\eta_{\pi(\alpha.u_k)}\underline{\eta_0^1}$ is also in the table of A

theorems

theorem 1 (MM, 2007)
A CA on the pentagrid or the heptagrid commutes with the shifts if and only if it is rotation invariant

theorem 2 (MM, 2007)
A mapping $F : Q^{\mathcal{F}_\alpha} \mapsto Q^{\mathcal{F}_\alpha}$ *is the global function of a rotation invariant CA if and only if it is continuous on* $Q^{\mathcal{F}_\alpha}$*, fitted with the product topology, and if it commutes with the shifts leaving the grid invariant.*

note that the product topology can be defined by a distance, as in the Euclidean case:

$$\mathrm{dist}(x,y) = \sum_{i \in \mathcal{F}_\alpha} \frac{\mathrm{dist}\big(x(i), y(i)\big)}{\alpha . u_{|i|}} 2^{-|i|}$$

where $|i|$ is the index of the level of the tree on which i is

here, $u_k = f_{2k+1}$ in both cases

the proof is very similar to the Euclidean case, up to rotation invariance

it is non-constructive: compacity argument

4.2 injectivity problem of the global function of a cellular automaton

theorem (MM, 2008)
the injectivity of the global function of a cellular automaton on the heptagrid is undecidable

plan of the proof:

the mauve triangles

the path

reduction of the halting problem

4.2.a the mauve triangles

definition of these triangles

intersection properties

particular points and isoclines

the β-clines

the β-points and the γ-points starting point of the construction:

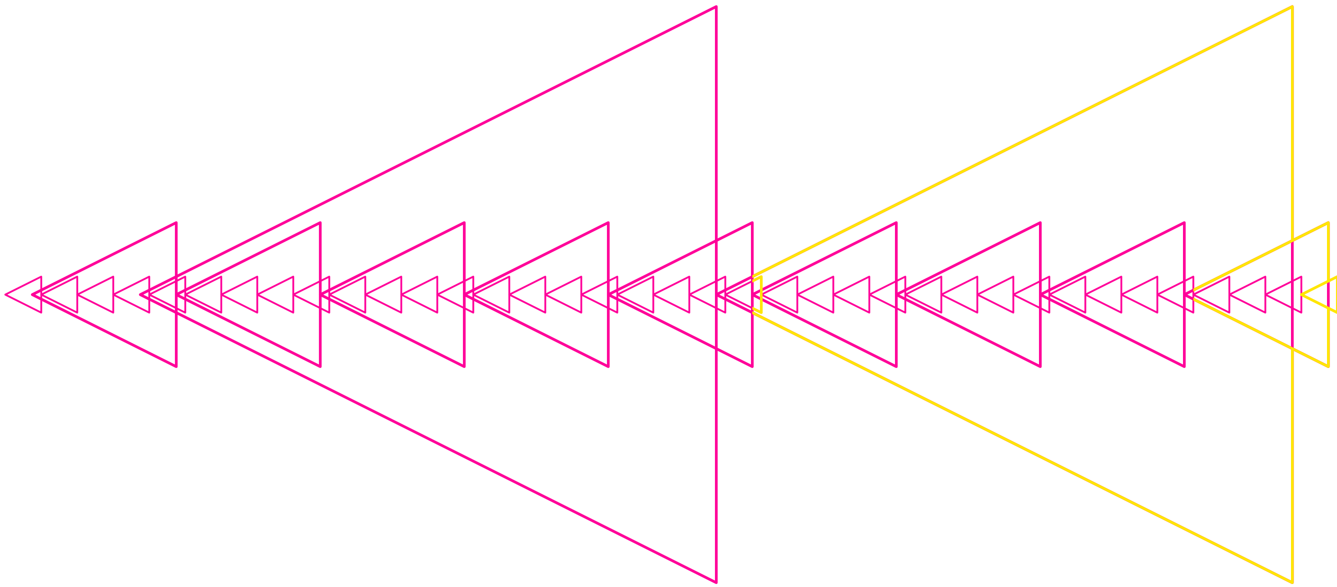
consider red triangles only
each red triangle R of the generation $2n+1$ defines a **mauve triangle** T of the generation n:

vertex of T = vertex of R
legs of T on those of R but twice longer
basis along an isocline again

from the doubling, mauve triangles intersect between themselves, but as interwoven triangles of opposite colour:

the leg of one with the basis of the other

representation of the mauve triangles



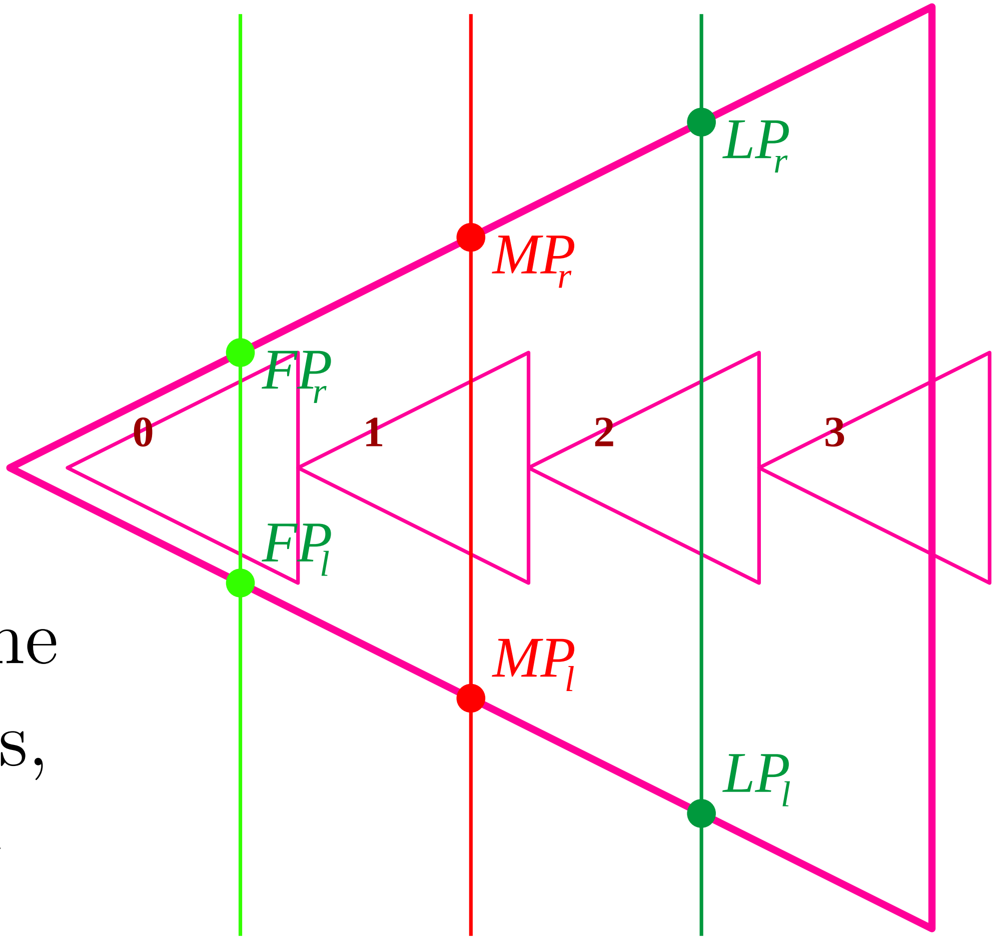
particular points:
on the leg of a mauve triangle T :

from top to bottom, h is the height of T:

- the high-point, HP:
  close to the vertex, defined later
- the first point, FP:
  at $\dfrac{h}{4}$ from the vertex
- the mid-point
- the low-point, LP:
  at $\dfrac{3h}{4}$ from the vertex

all of them constructible by the tiles

the special points in a mauve triangle



remark the i-triangles, $i \in \{0..3\}$

the FP-, MP and LP's define the 0, 1 and 2-**clines** and the basis defines the 3-cline

intersection properties:

let T be of generation $n+1$:

the i-clines of T cuts the legs of its inner i-triangles of generation n and those of the 2-tiangles of these i-triangles and, recursively, the legs of the 2-triangles of generation m of the already cut 2-triangles of generation $m+1$ for $m+1 < n$,

all legs being cut at their LP's

the β-clines

from the overlapping between mauve triangles, define a new notion:

define T: mauve triangle of generation $n+1$

its basis is cut by a 3-triangle of the generation n ;

by recursion, this defines $\{T_i\}_{i\in\{0..n+1\}}$, with:

$T_{n+1} = T$, T_i, $i \in \{0..n\}$, is a 3-triangle,
T_i is a mauve triangle of the generation i,
T_i cuts the basis of T_{i+1}

β-cline $==$ the isocline of the basis of T_0

its **type:** rank of T_{n+1}

from the existence of the β-clines:

β-points and γ-points

define, for a triangle T of the generation $n+1$:

β-point: the intersection of the leg of T with the β-cline of its 2-triangles of the generation n
the β-point is below the LP's of T

the β-points emit **lateral signals outside** T
opposite lateralities can be joined once in between two consecutive triangles of the same generation and latitude

γ-point:

for a triangle T of the generation $n+1$:

intersection of the leg with the β-cline of its hat

> the hat of T may not exist, but there are copies of T on the same latitude which have a hat

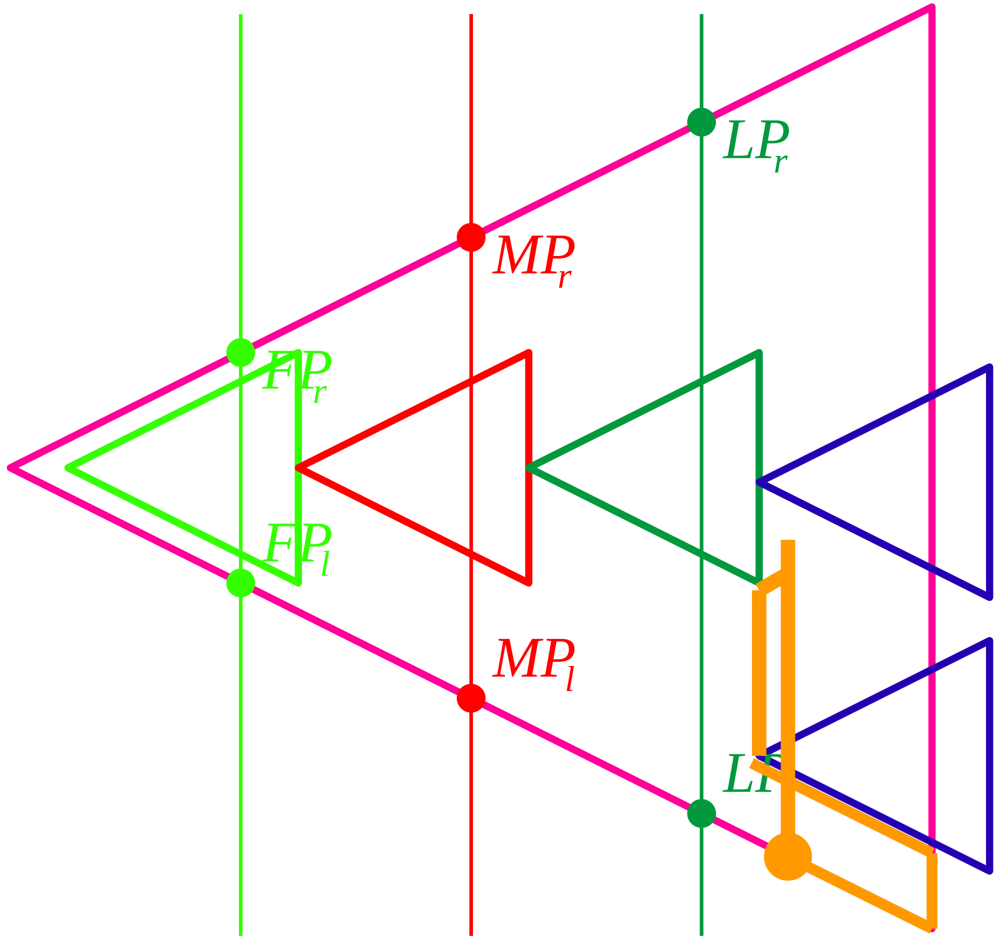
$\Rightarrow$ the γ-point can always be defined

construction of the β-points

key points:

- from the corner of T, take the first 3-triangle of the generation n which cuts the basis of T
- from there reach a 2-triangle D of the generation n inside T
- the β-cline of D cuts the legs of T at the expected β-point

construction of the β-points



construction of the γ-points

in a triangle T of the generation $n+1$, their 0-triangles are hatted

same β-cline as that of the hat of T

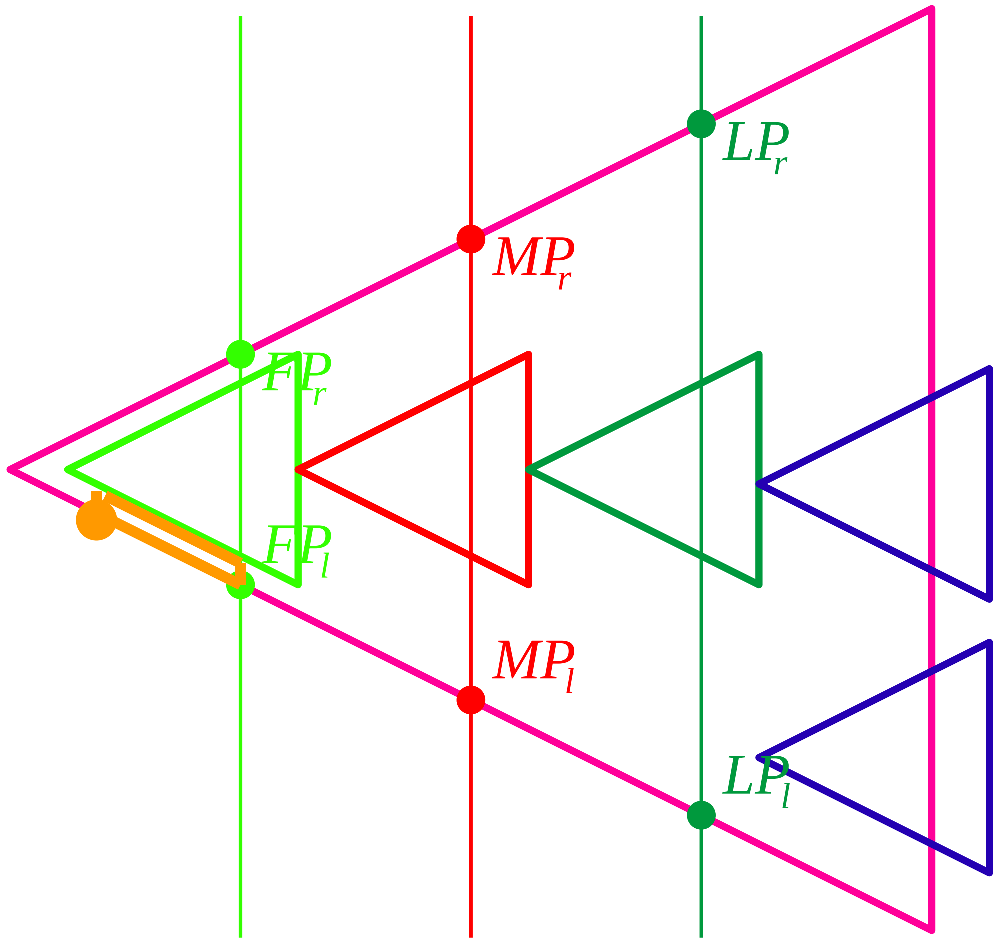
construction by a recursive algorithm:

start from the mid-point of the red triangle supporting T

go on the isocline to the first 0-triangle

find its γ-point and go back to the leg of T

construction of the γ-point



the HP's of a mauve triangle

starting from the FP, in the direction of the vertex:

if β-cline of type 2 at the γ-point,
then $HP = \gamma$-point

otherwise:
$HP =$ intersection with first basis if any

if no basis between the FP and the vertex,
HP's on the legs, on the isocline below the vertex

4.2.b **the path**

basic step for the injectivity theorem:

> construction of a frame guaranteeing a plane-filling path, or at least a half-plane filling path

construction of the path

definition by induction on the generations of the mauve triangles:

generation 0:

entry into the triangle:

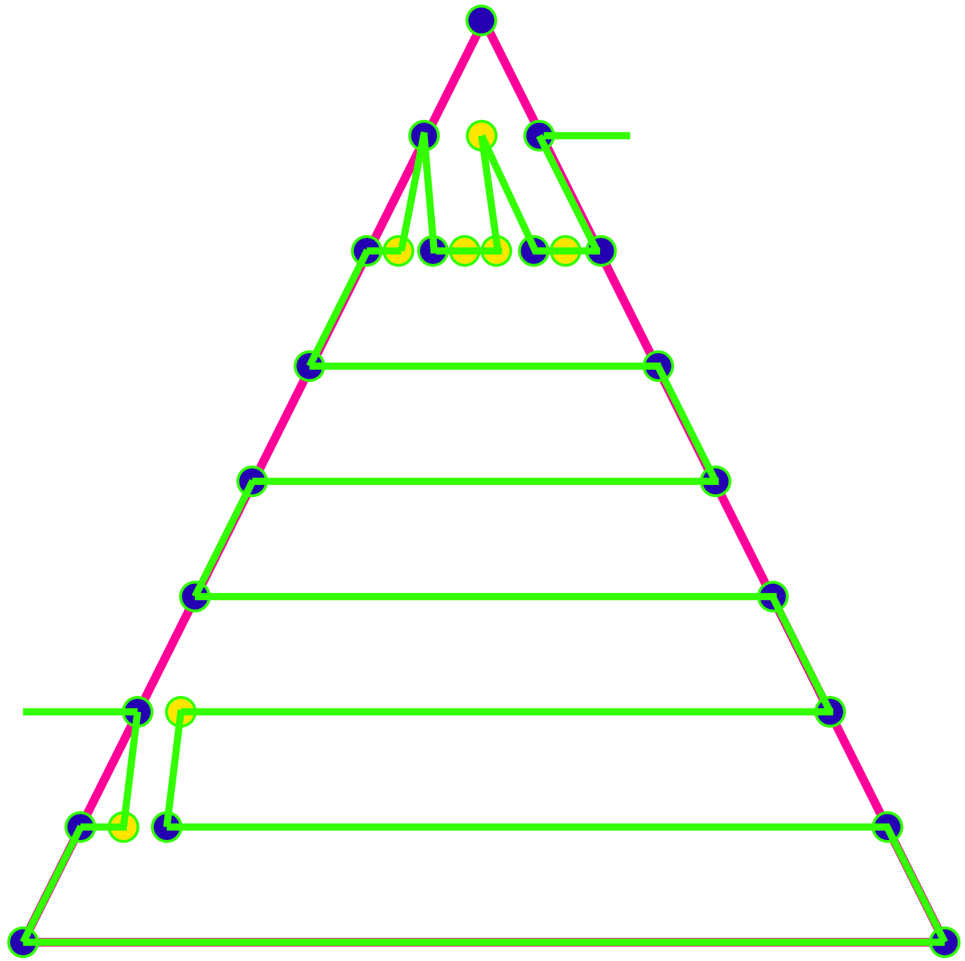
at the LP on one side

exit from the triangle:

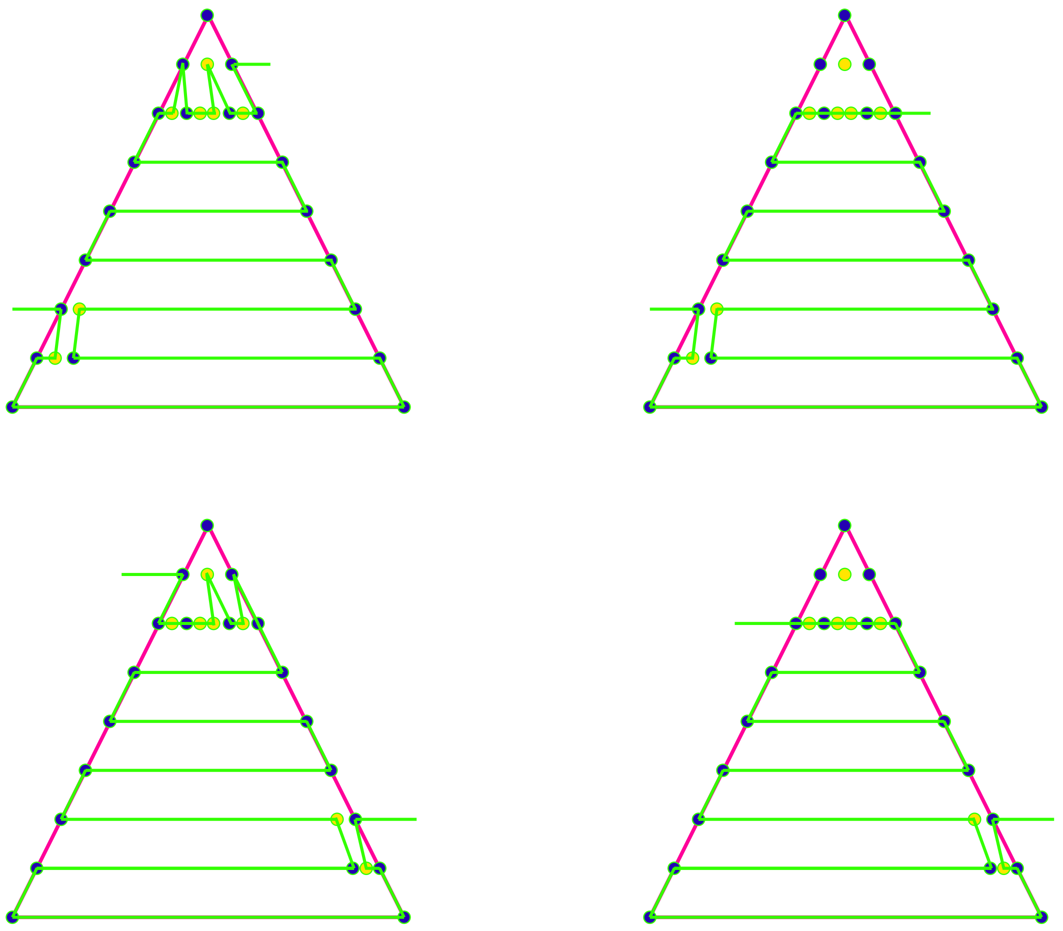
at the vertex of T or the isocline just below, towards the other side

in-between : zig-zags

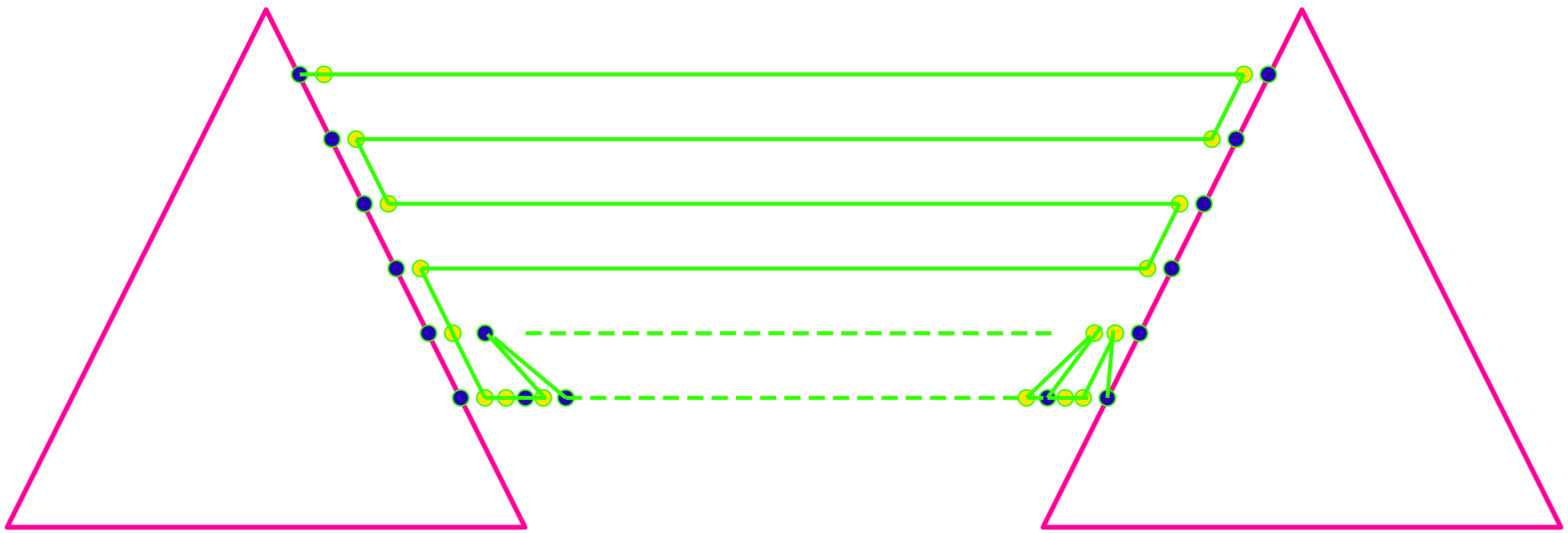
illustration:



illustrating all cases:

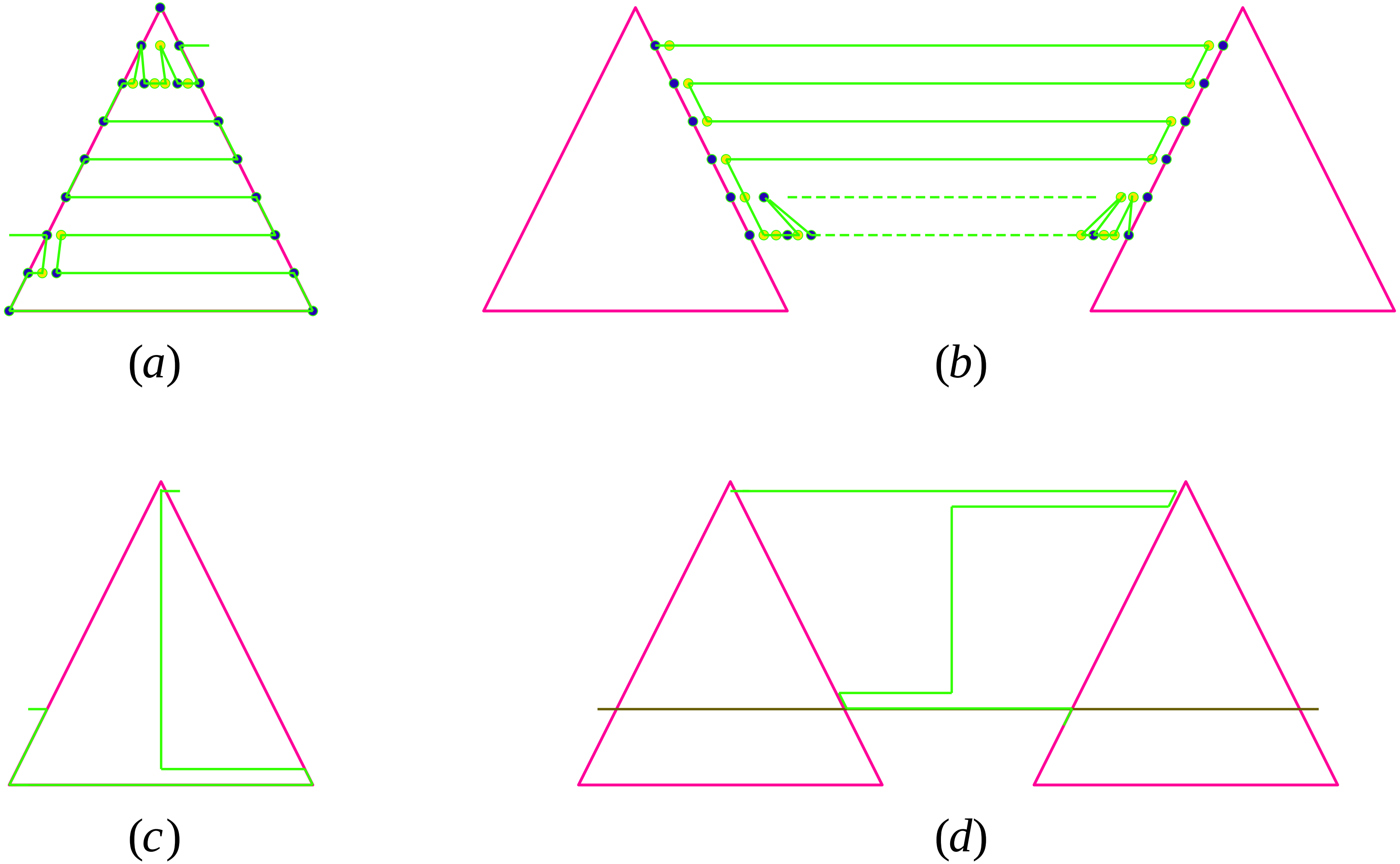


in between mauve-0 triangles:



and similar figures for the other cases

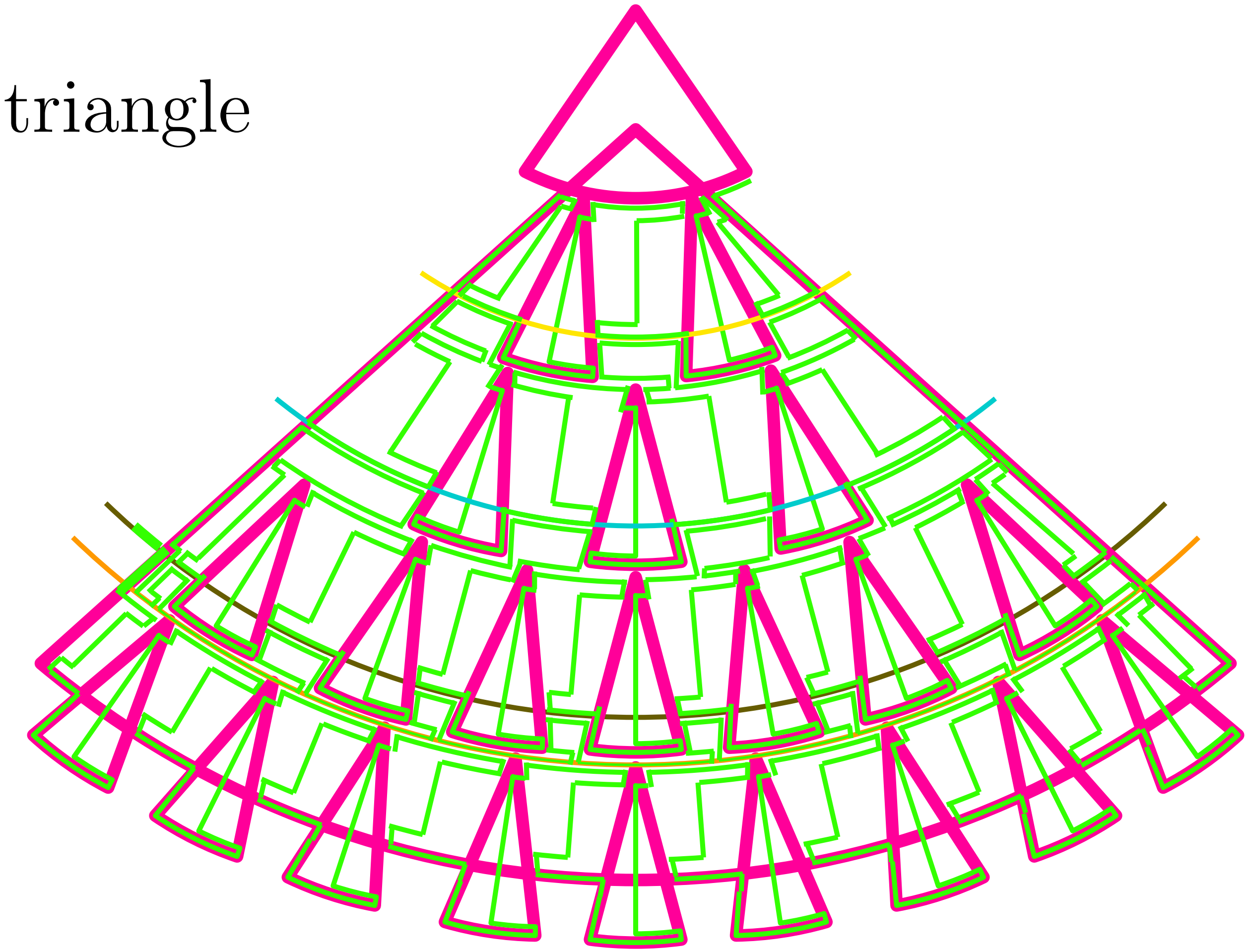
schematic representation for the generation 0



(*a*) (*b*) (*c*) (*d*)

from the generations n to $n+1$

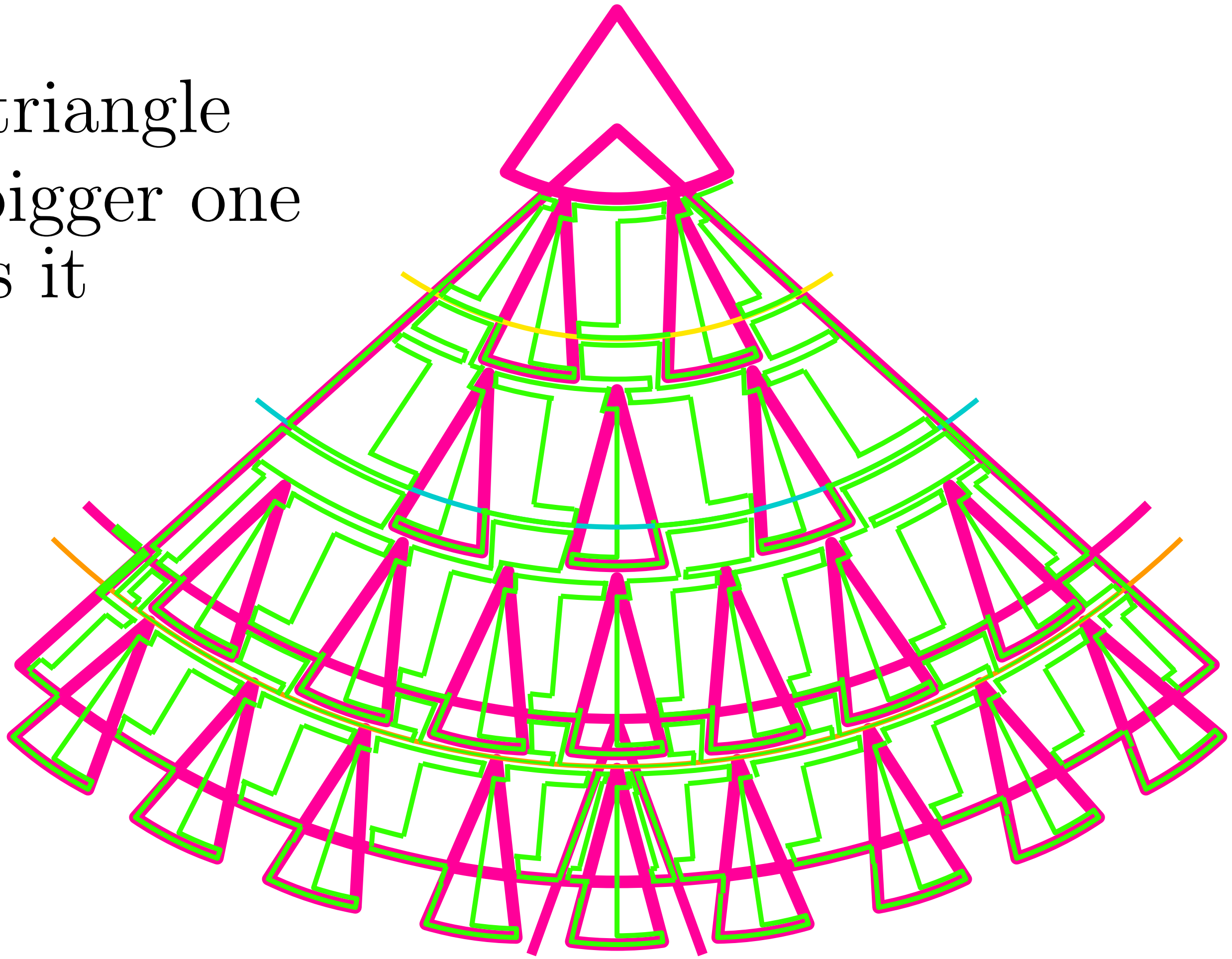
inside a triangle



namely,
a 0- or a 1-triangle

from the generations n to $n+1$

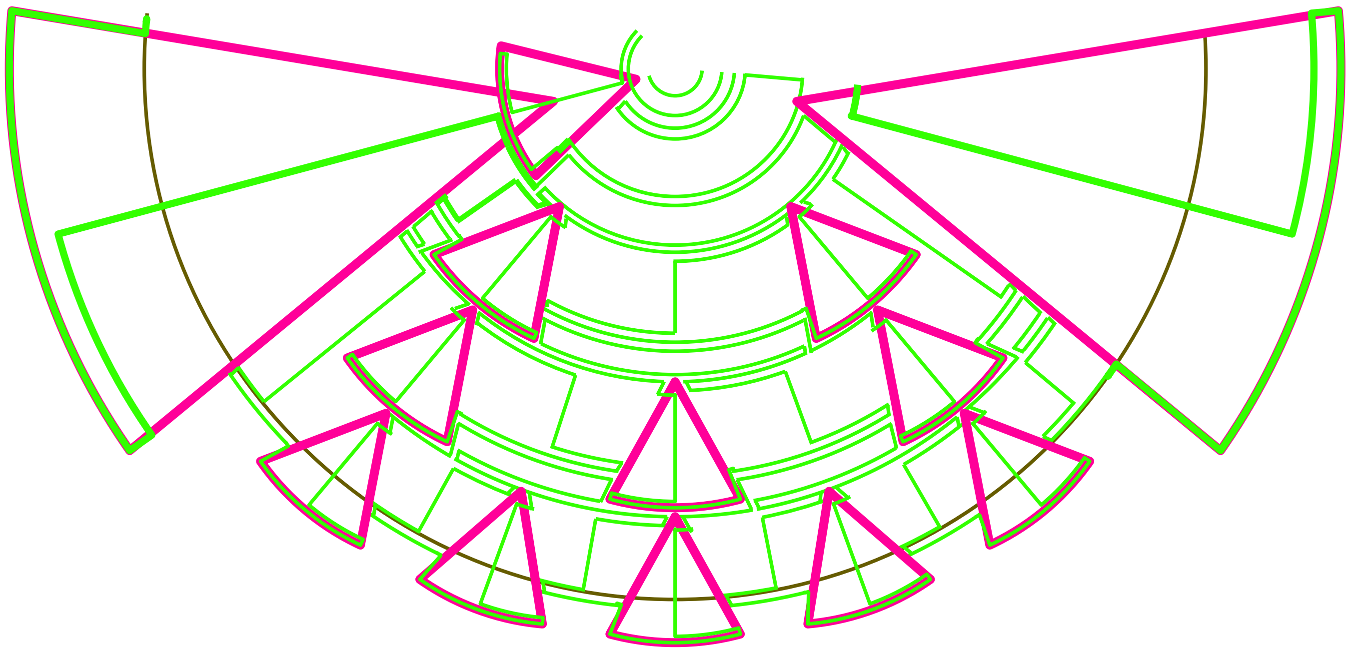
inside a triangle
when a bigger one
intersects it



namely,
a 3- or a 2-triangle

from the generations n to $n+1$

in-between two consecutive
triangles of the same generation



as in the previous figure:

- motion within a latitude in between legs of a higher generation:
  - either legs of a triangle
  - or consecutive legs of two consecutive triangles of the same generation
- this defines the **basic areas** of the path for each generation

special role of β-clines of 2-triangles:

they require that the path cuts the top of the encountered triangles when in an open part of the β-cline

mechanism needed to ensure the global direction

the same 2-β-cline used inside a triangle T and then, in between two consecutive triangles of the same latitude but below T

all elements above indicated:

LP's, HP's, mid-points, rank,
β- and γ-points, β-clines and their types,
correspondence entry/exit in a triangle

can be fixed by a finite set of prototiles

by induction on the generation, this forces the path for next generation as a corollary,

a basic area cannot contain a cycle of the path

and so the path contains no cycle

in most cases, the path consists of one component: it visits all the tiles

this is the case when there is no infinite mauve triangle

the exceptional case: the infinite triangle

this case is yelded by the case of the butterfly model for the interwoven triangles

case when there is an isocline which is never cut by a any red or blue **triangle**, whatever the generation

a possibility only:

it cannot be forced algorithmically

there may be infinite red triangles

when there are infinite red triangles, this entails the same for mauve triangles

in this case:

there is an infinite mauve triangle T_∞ with a basis cutting infinitely many 2-triangles:

the infinite basis of T_∞ contains infinitely many vertices of infinite mauve triangles in this case:

there are infinitely many paths:

one over the infinite basis,
it also fills up the 2-triangles T_i's crossed by the basis

and one path in each infinite triangle which visits also a contiguous zone in-between the next infinite triangle and itself in all cases:

one component
or infinitely many components

basic lemma

any component of the path fills up an infinite sequence of triangles of increasing sizes

4.2.c reduction of the halting problem

theorem
the injectivity of the global function of a CA on the heptagrid is undecidable

proof

easy to define an orientation of the path in each component:

define three colours in a given order used by the path only

rest of the proof:
argument similar to the Euclidean proof with a difference

proof, continued

let $\mathcal{M}$ be the set of Turing machines starting from an empty tape

let T_M be a finite set of prototiles of the heptagrid associated to $M \in \mathcal{M}$ with the interwowen triangles:

T_M tiles the plane if and only if M does not halt, starting from the empty tape

let D be the finite set of prototiles of the mauve triangles with the oriented paths

proof, continued

for $M \in \mathcal{M}$, define A_M as a CA on the heptagrid by:

states: $T_M \times D \times \{0,1\}$

transition function f, addresses the bit only:

if T_M or D not correct at c, no change

if both correct,

$f(c,t{+}1) = \mathrm{xor}(f(c,t), f(d(c),t))$,

where $d(c)$ is the next tile on the path after c

$G = G_{A_M}$ == global function of A_M

proof, continued

then, G is not injective if and only if M does not halt

indeed:
if M does not halt:
then T_M tiles the plane ;

let ξ be a configuration corresponding to a correct tiling in T_M and in D ;

we may chose ξ in such a way that the path has a single component under D

proof, continued

define x_0 at c by:

ξ for the tile, 0 for the bit

similarly define x_1 at c by:

ξ for the tile, 1 for the bit

then, by the xor, which applies, the next transition is always x_0

hence, G is not injective

proof, continued

if G not injective: there are x_0, x_1 and c with $x_0(c) \neq x_1(c)$ and $G(x_0)(c) = G(x_1)(c)$

as G changes only the bits, same situation of the tilings at c ; easy to see that necessarily, both T_M and D are correct at c and then, also at $d(c)$;

proof, continued

by induction,

T_M and D correct along the path starting from c;

now, the component of the path through c visits an infinite sequence of triangles with increasing sizes, also after c;

by the construction of T_M, then M cannot halt

4.2.d about Gardens of Eden

natural questions:

what about surjectivity ?

what about bijectivity and reversibility ?

in the Euclidean setting:

theorem 1 (J. Kari, 1994, B. Durand 1996)
it is undecidable to know whether the global function of a CA on the Euclidean plane is surjective

theorem 2 (J. Kari, 1994)
it is undecidable to know whether the global function of a CA on the Euclidean plane is bijective

basis of these theorems:

in the Euclidean setting, the Garden of Eden theorem (Moore, Myhill, 1963) says, for the global function G of a CA that:

G surjective
$\Leftrightarrow$ G injective on finite configurations

Hedlund's characterizations of CA's
+ compacity of the space of configurations entails:

$$\text{bijectivity} \Leftrightarrow \text{reversibility}$$

for the global function of a CA

and so, G injective $\Leftrightarrow$ G reversible

theorem 1 is based on the proof of J. Kari's theorem, 1994, of the undecidability of the injectivity of the global function of a CA in the hyperbolic plane:

> an analoguous version of Hedlund's characterizations of CA's holds
>
> but the Garden of Eden theorem is not true

theorem 3 (J. Kari, M. Margenstern) *there are CA's in the hyperbolic plane whose global function is surjective but not injective, even on finite configurations, and there are others whose global function is injective but not surjective* theorem 3 has a partial refinment, considering CA's in the hyperbolic plane which are rotation invariant:

theorem 4 (M. Margenstern, J. Kari)
there are rotation invariant CA's in the hyperbolic plane whose global function is surjective but not injective, even on finite configurations

4.3 small universal hyperbolic CA's

4.3.a the railway model

4.3.b in the plane

4.3.c in the $3D$-space

4.3.a railway simulation

all results here: implementation of this model

introduced by Ian Stewart,

a circuit in the Euclidean plane made of:

tracks

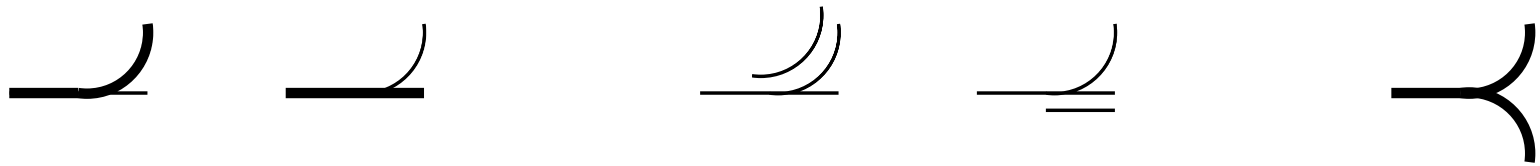
crossings

switches

a unique locomotive runs over the circuit

the switches

three types:



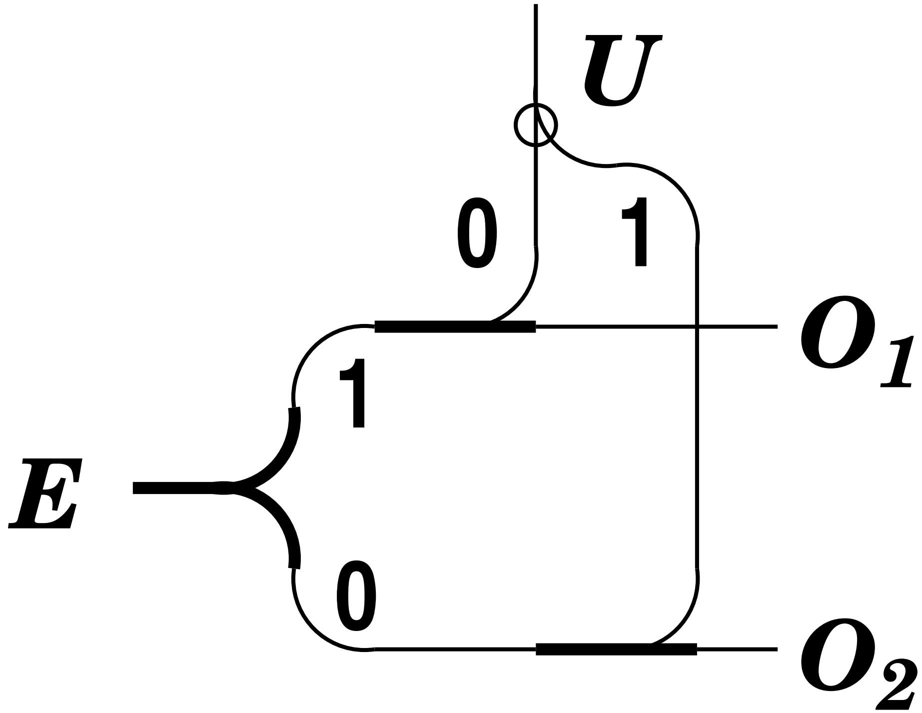
fix flip-flop memory

flip-flop: only active passage, triggers the change of the selection

memory switch:

selected track = last passive passage

the basic element

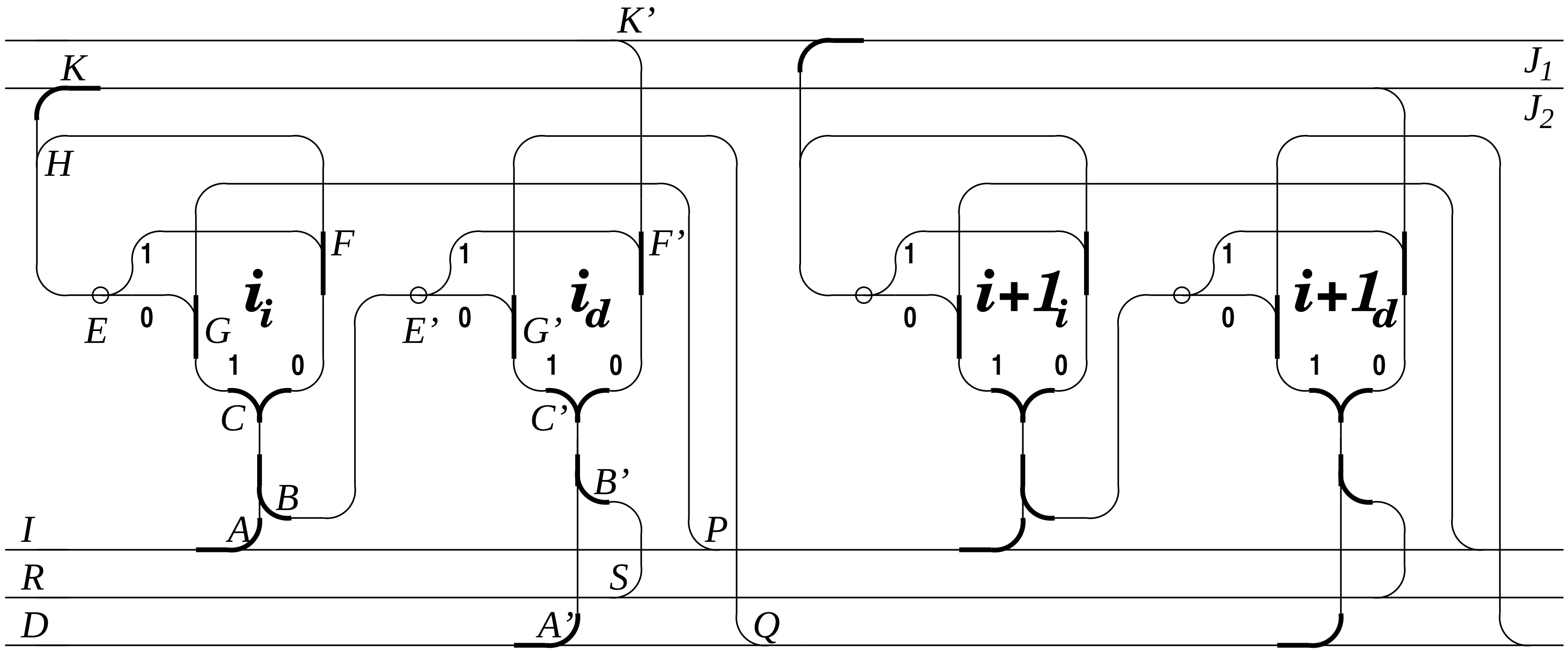


E: **reading** entry

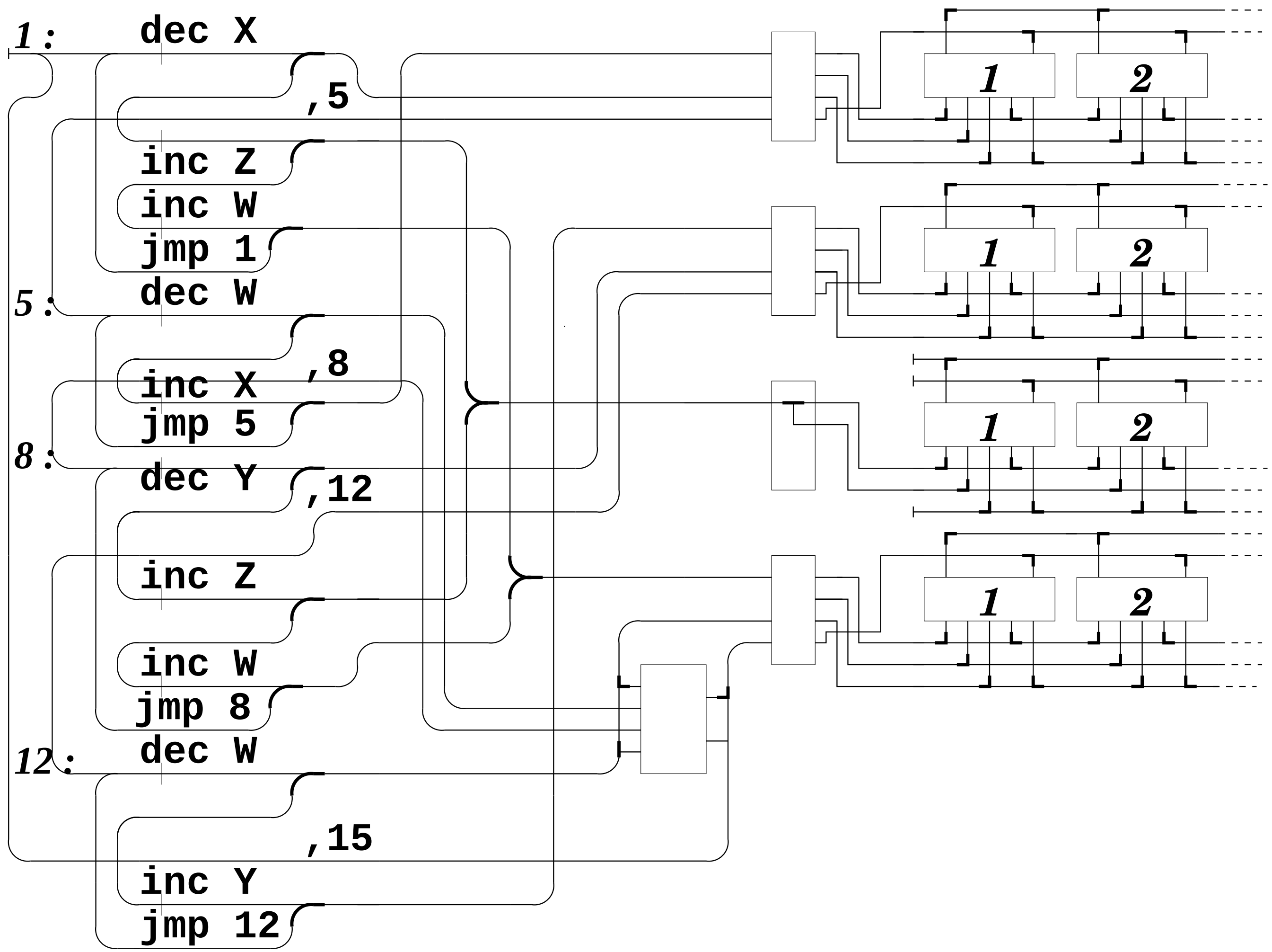
U: **writing** entry: changes the content of the element

assembly of elements: allows to construct registers or a tape of a Turing machine

a register unit:



an example:


1 :
dec X
,5
inc Z
inc W
jmp 1
5 :
dec W
,8
inc X
jmp 5
8 :
dec Y
,12
inc Z
inc W
jmp 8
12 :
dec W
,15
inc Y
jmp 12
1
2
1
2
1
2
1
2

it is known that by assembling switches and tracks, a universal computation can be simulated by the motion of the locomotive

4.4.b in the hyperbolic plane

common feature of the results:

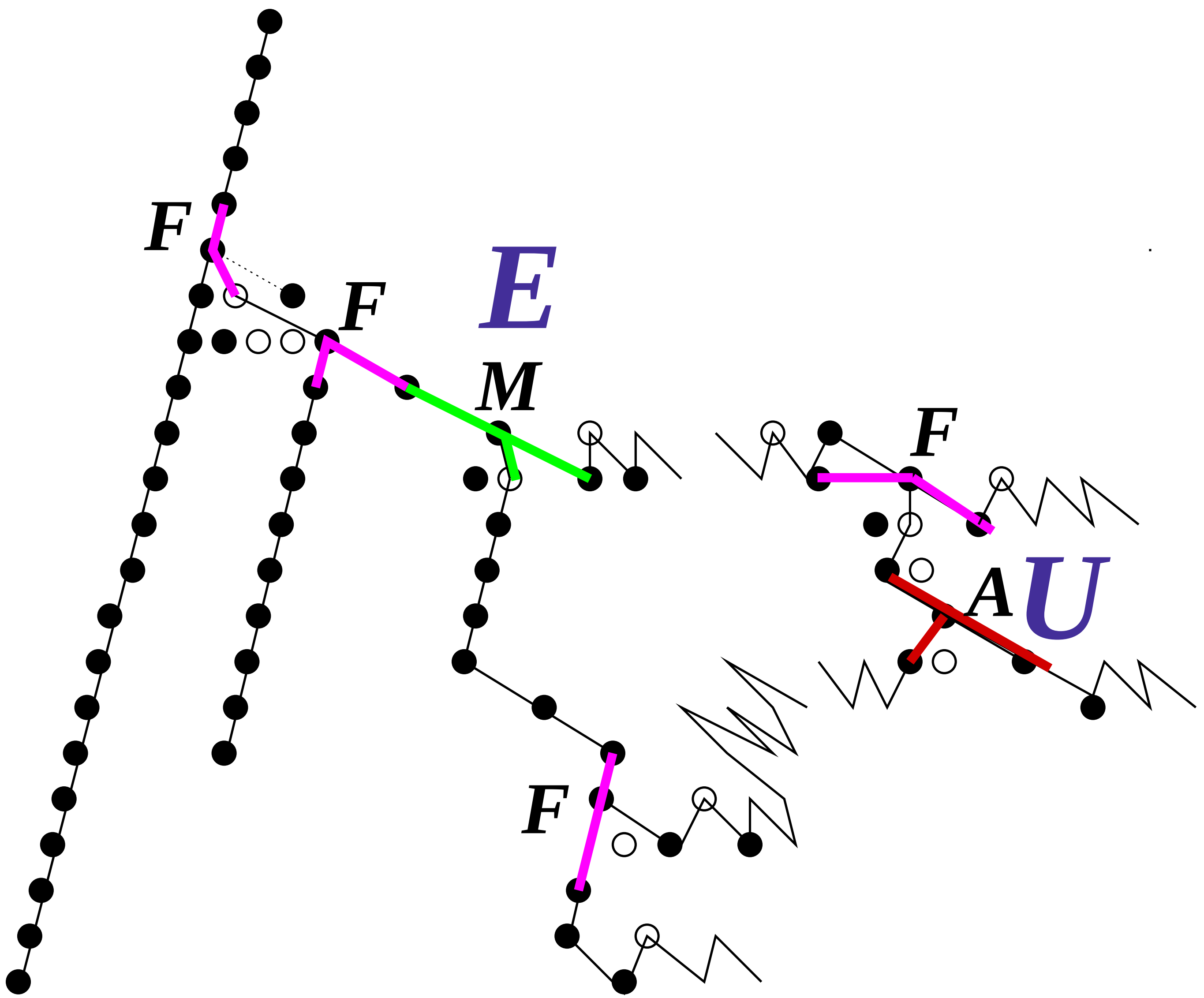
weak universality: infinite initial configuration, but not arbitrary, at large two parts globally invariant by a shift

organization of the circuit

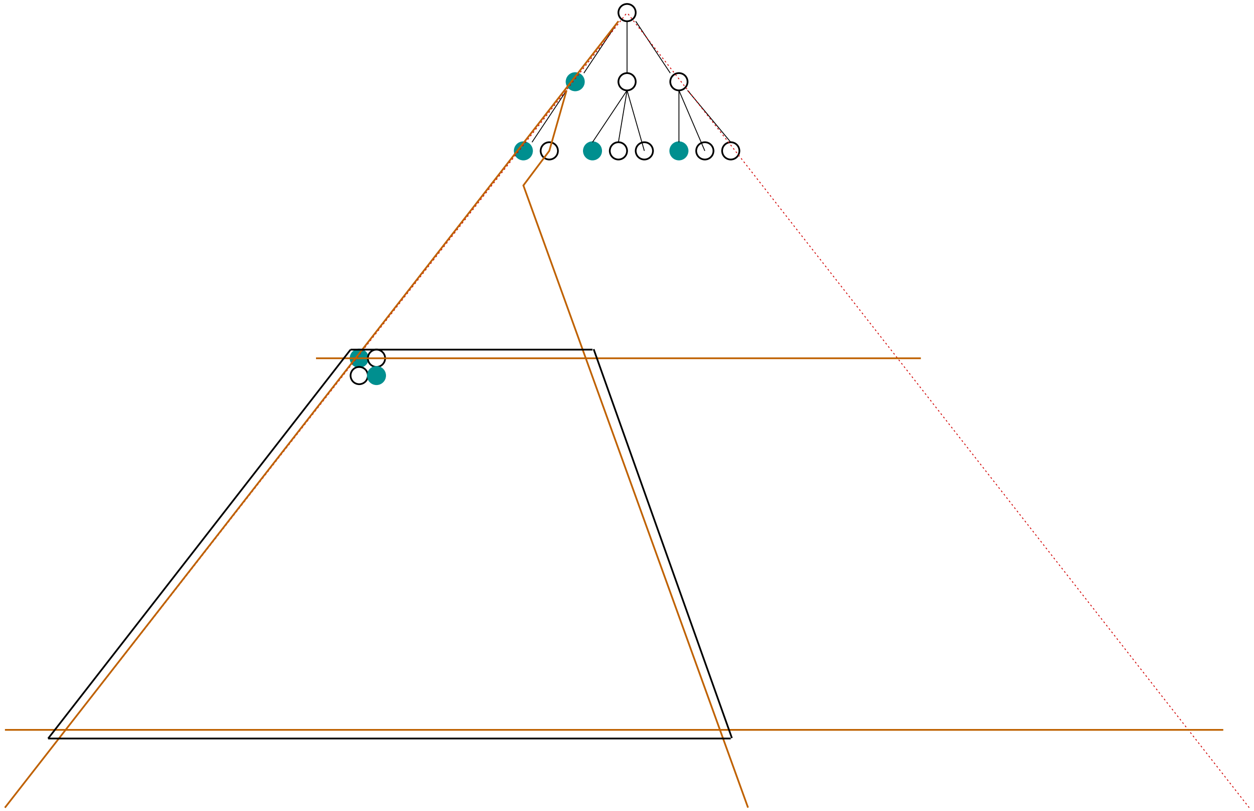
mainly, following TCS paper

- implementation of an elementary unit
- structure of the global implementation

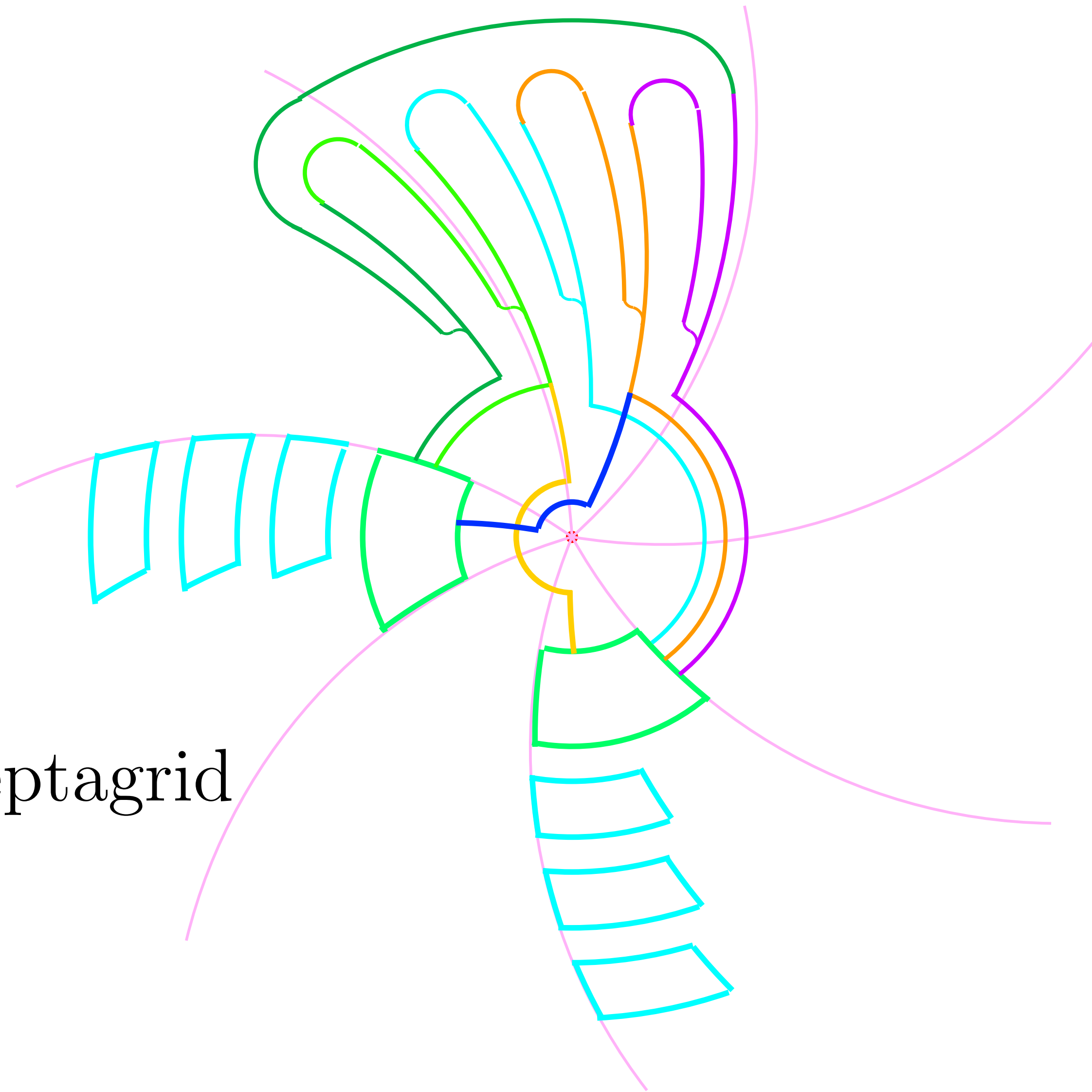
the elementary unit



a block : unit of a register



organization of the circuit on the example



here, in the heptagrid

organization of the circuit

thanks to the description of various kinds of paths:

using isoclines as **horizontals**

using **lines** following a branch in a Fibonacci tree as **verticals**

note: needed only finite parts of horizontals and verticals

the results:

22 states, pentagrid, MM-FH, 2002, first result in the hyperbolic plane

9 states, pentagrid, MM-YS, 2008

6 states, heptagrid, MM-YS, 2008, and,

very recently, 4 states, heptagrid

all of them are **rotation invariant** CA's

the HCA on the heptagrid with 6 states

we illustrate two points:

the motion of the locomotive along a track

the passive crossing of a memory switch from the non-selected track

illustration of the motion along a track

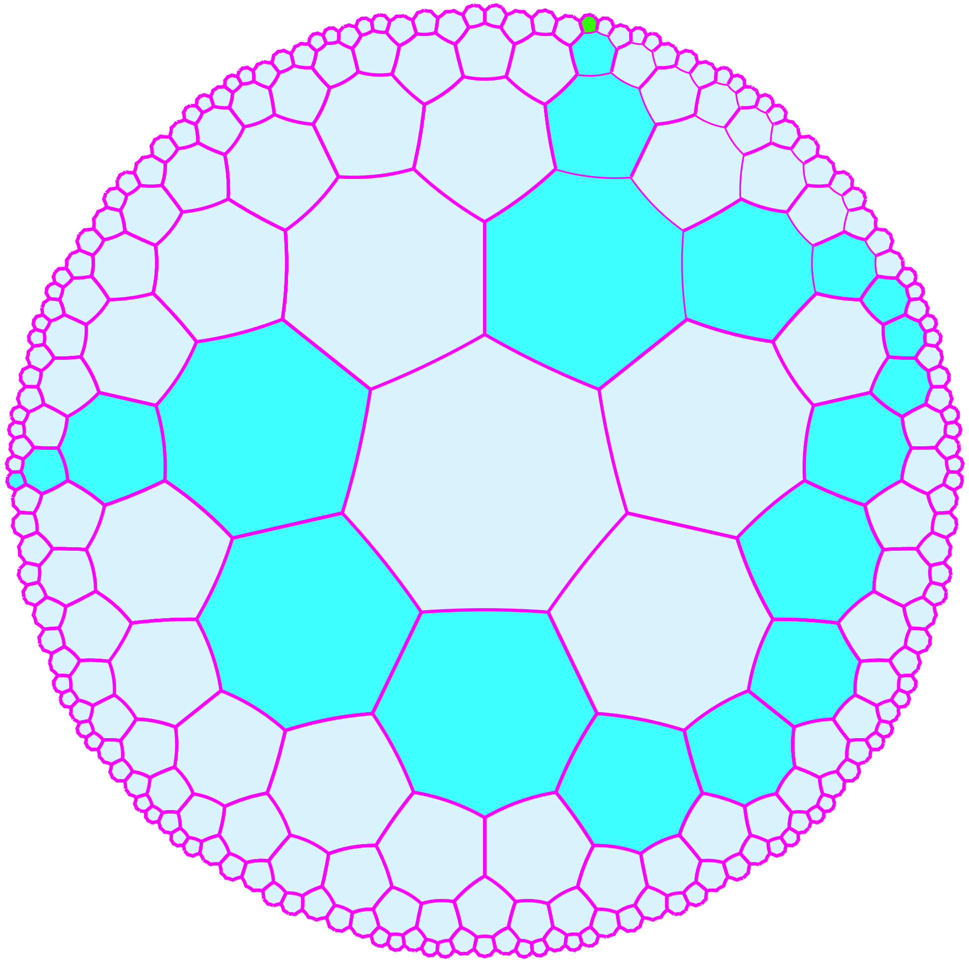


illustration of the motion along a track

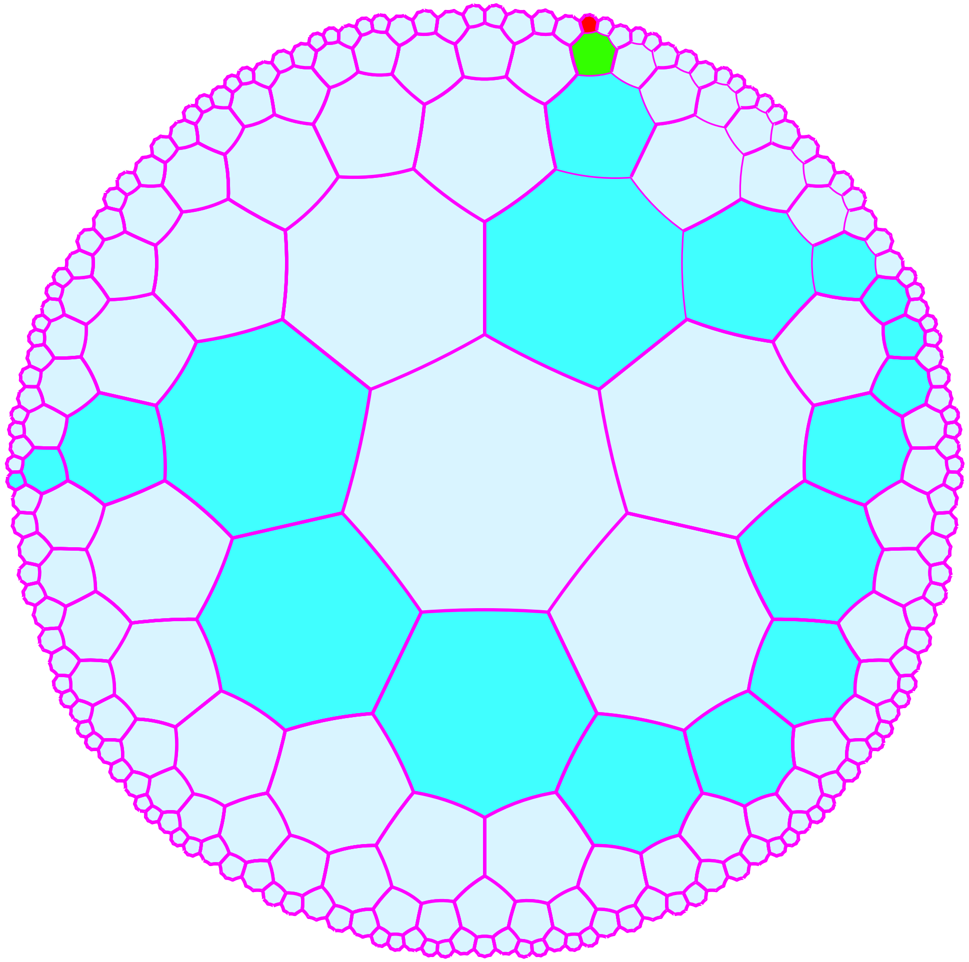


illustration of the motion along a track

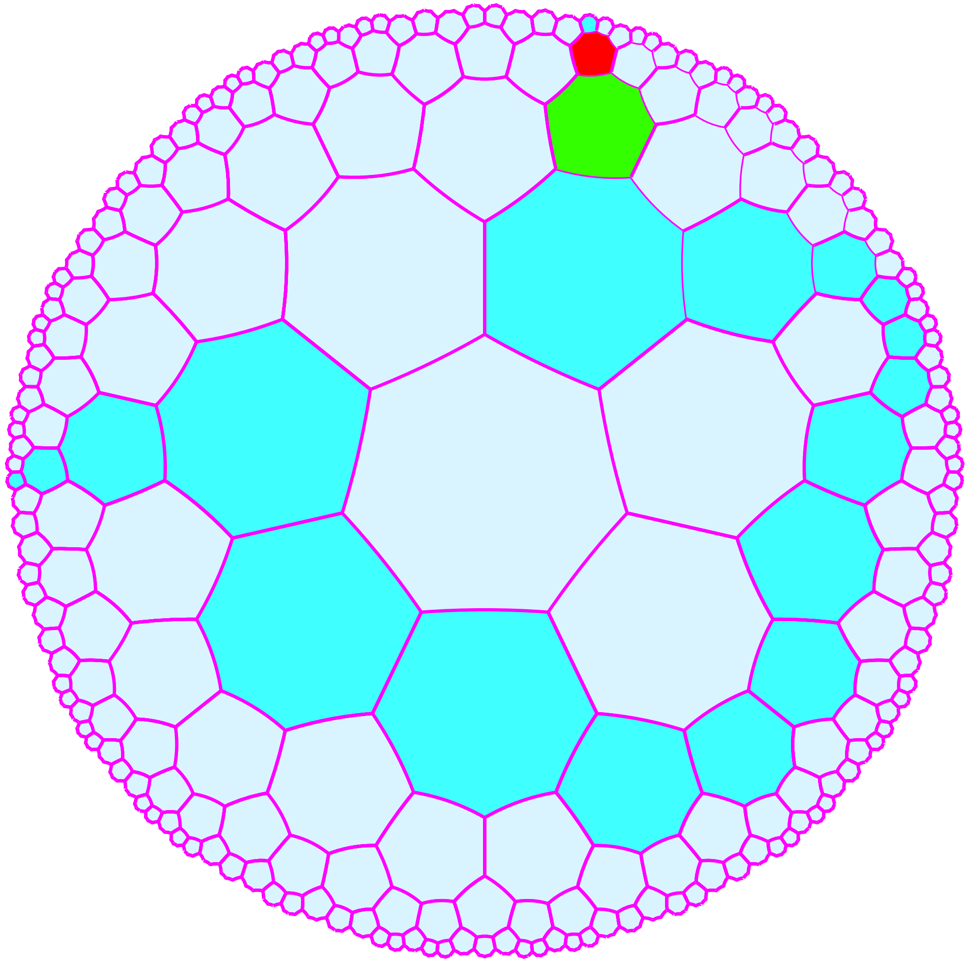


illustration of the motion along a track

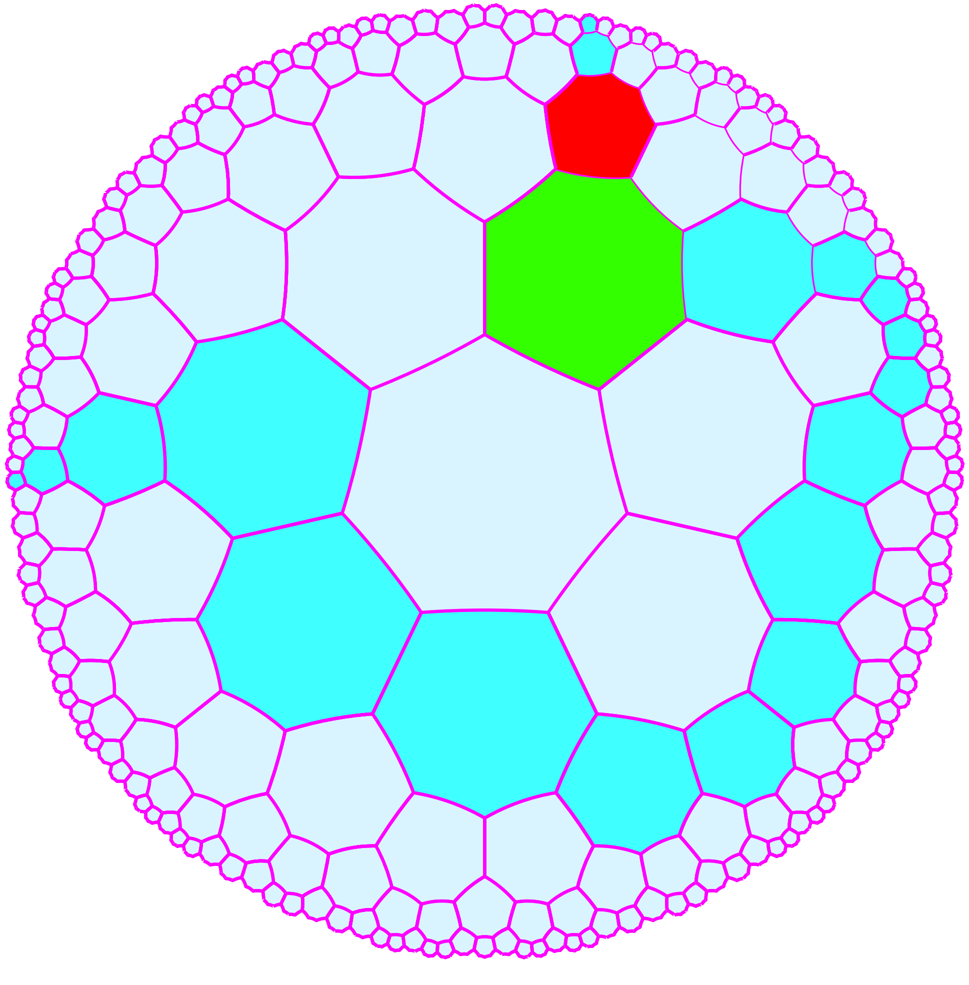


illustration of the motion along a track

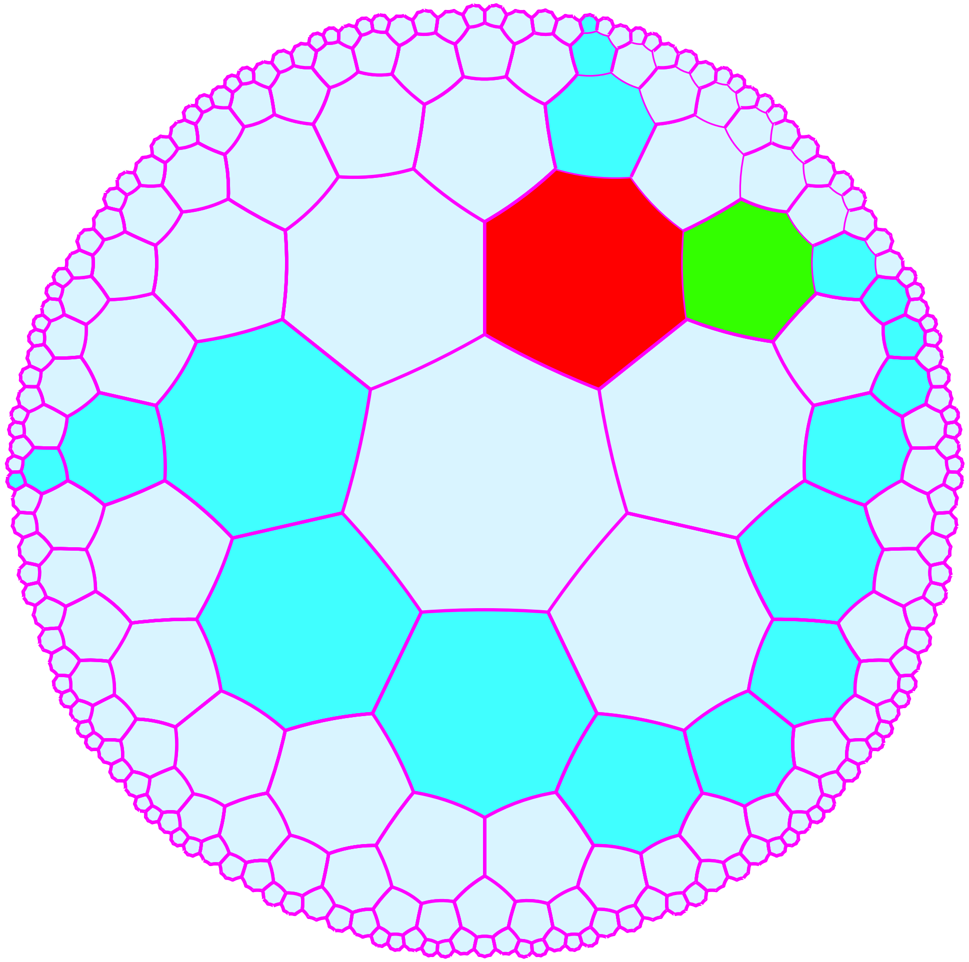


illustration of the motion along a track

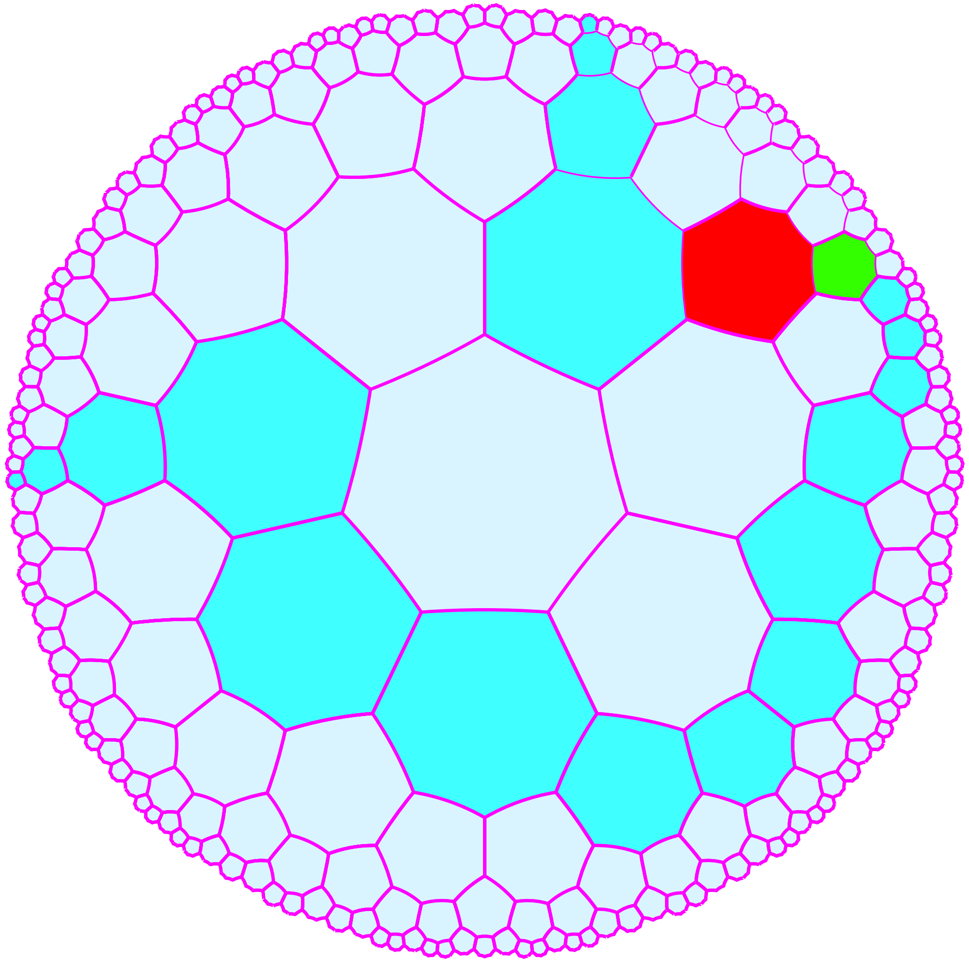


illustration of the motion along a track

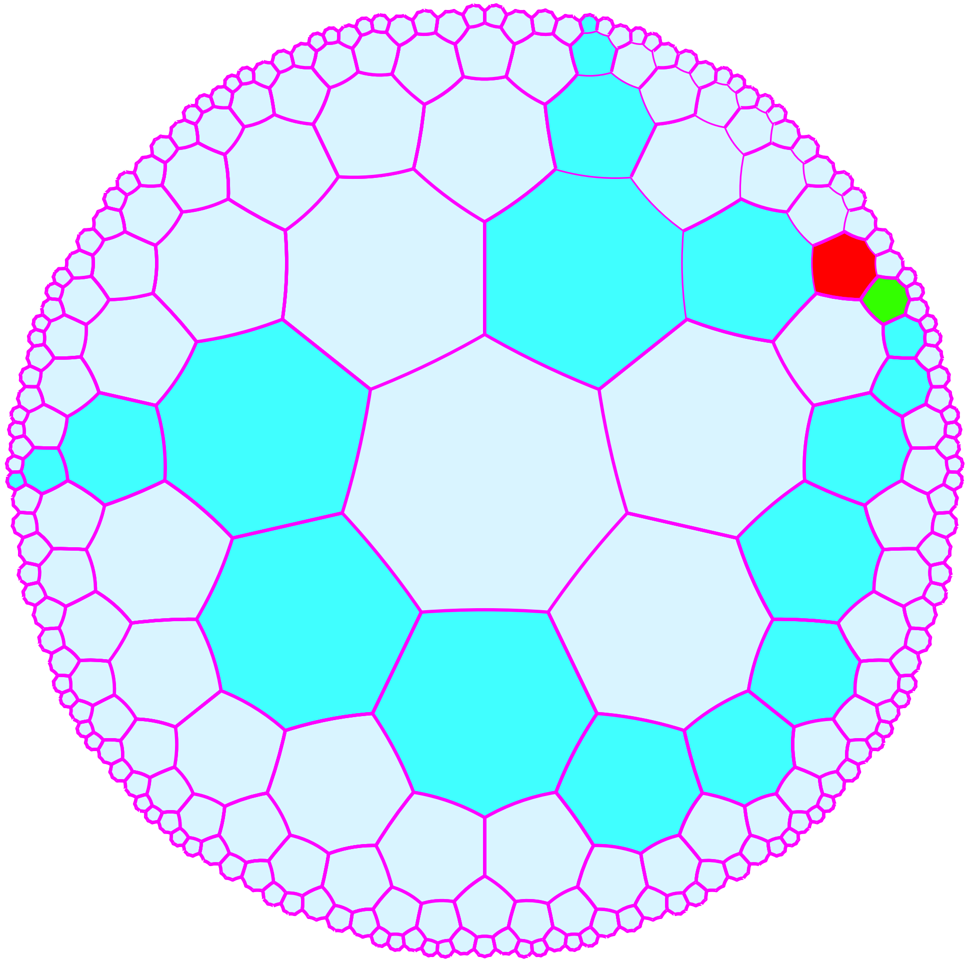


illustration of the motion along a track

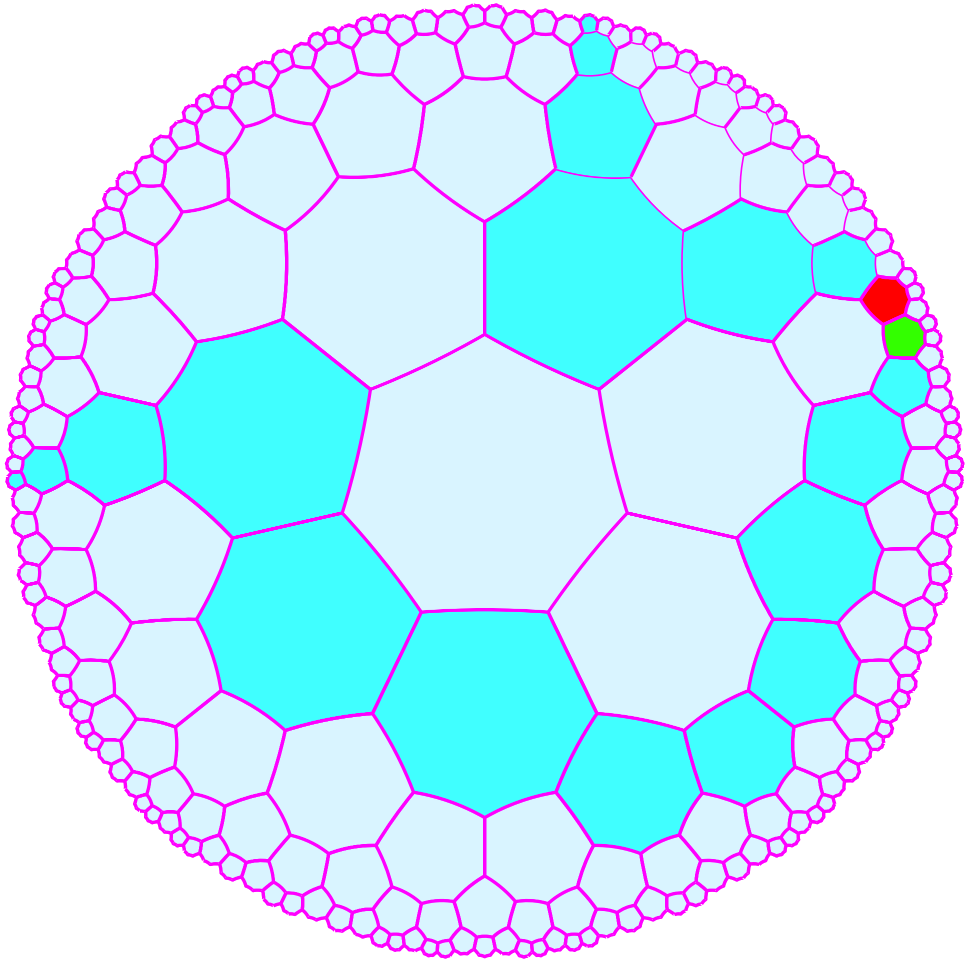


illustration of the motion along a track

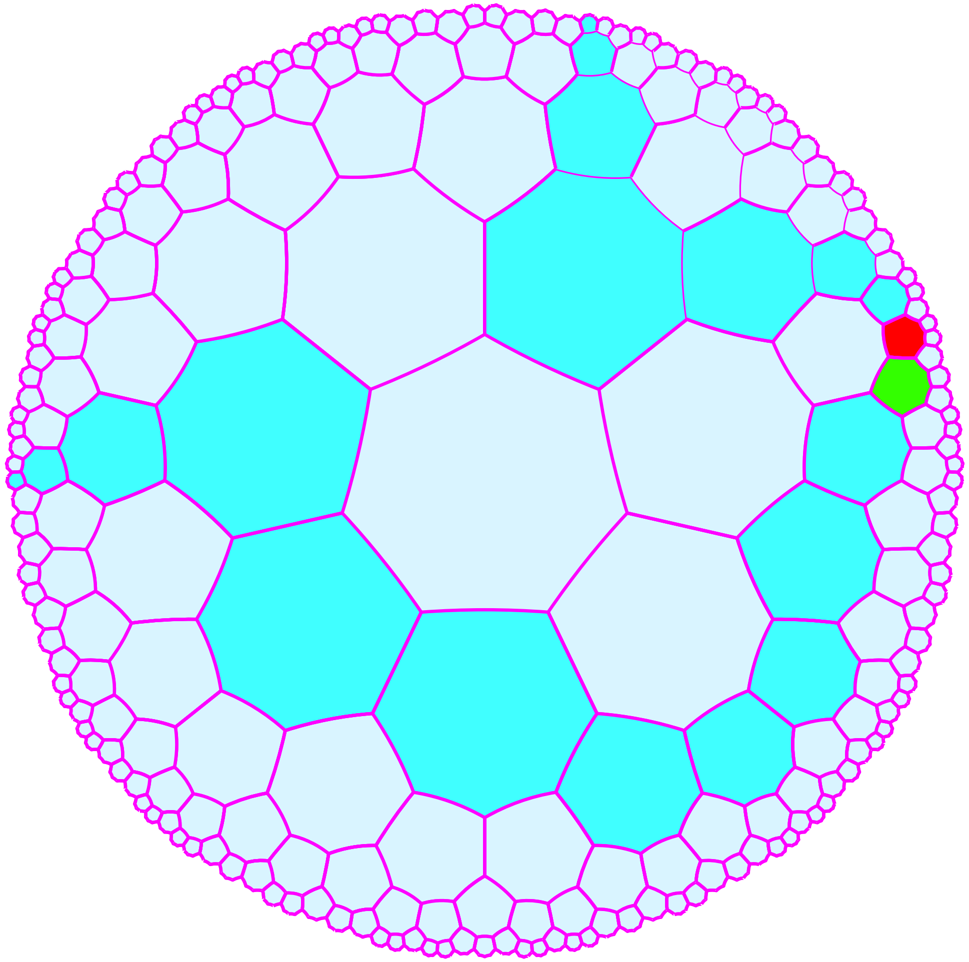


illustration of the motion along a track

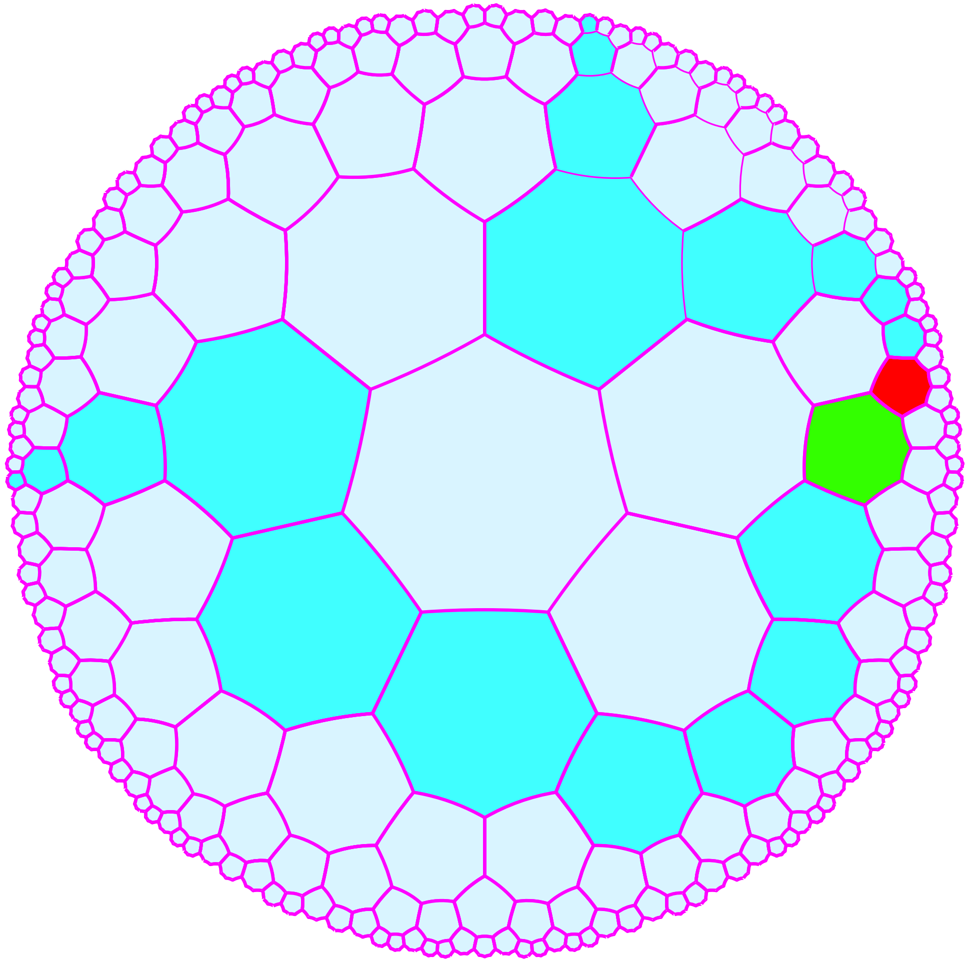


illustration of the motion along a track

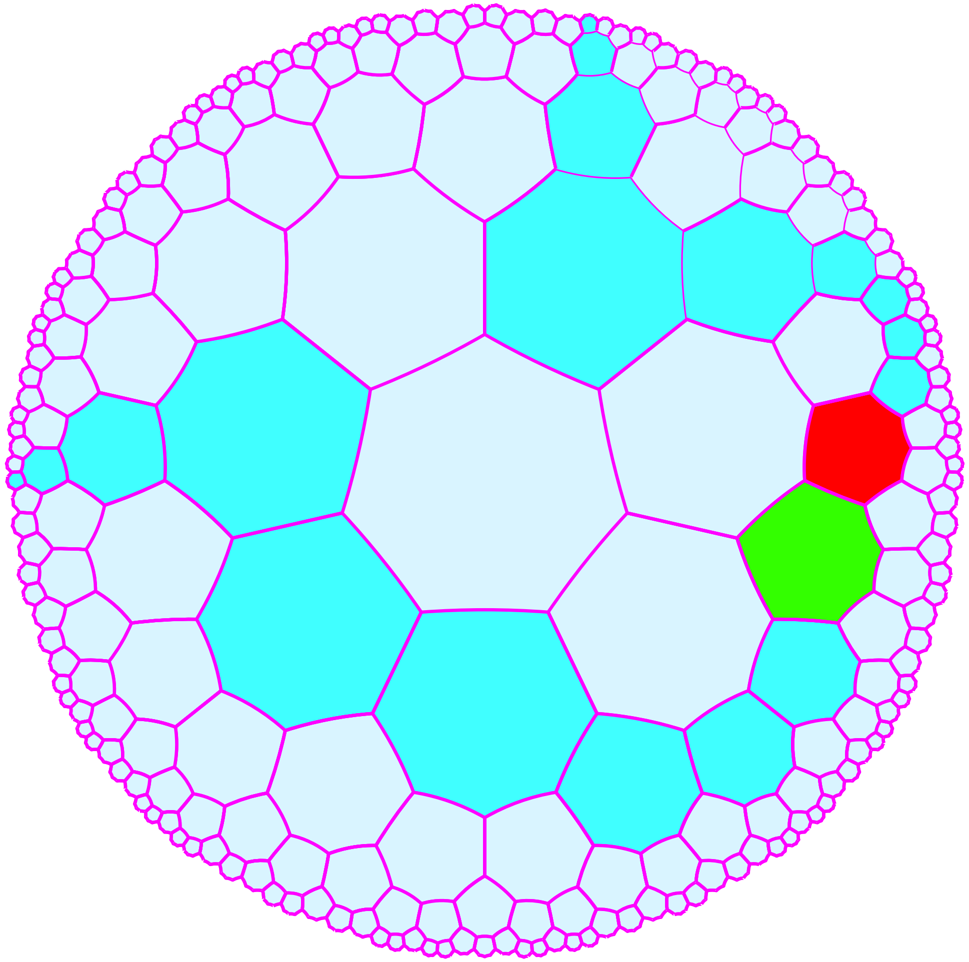


illustration of the motion along a track

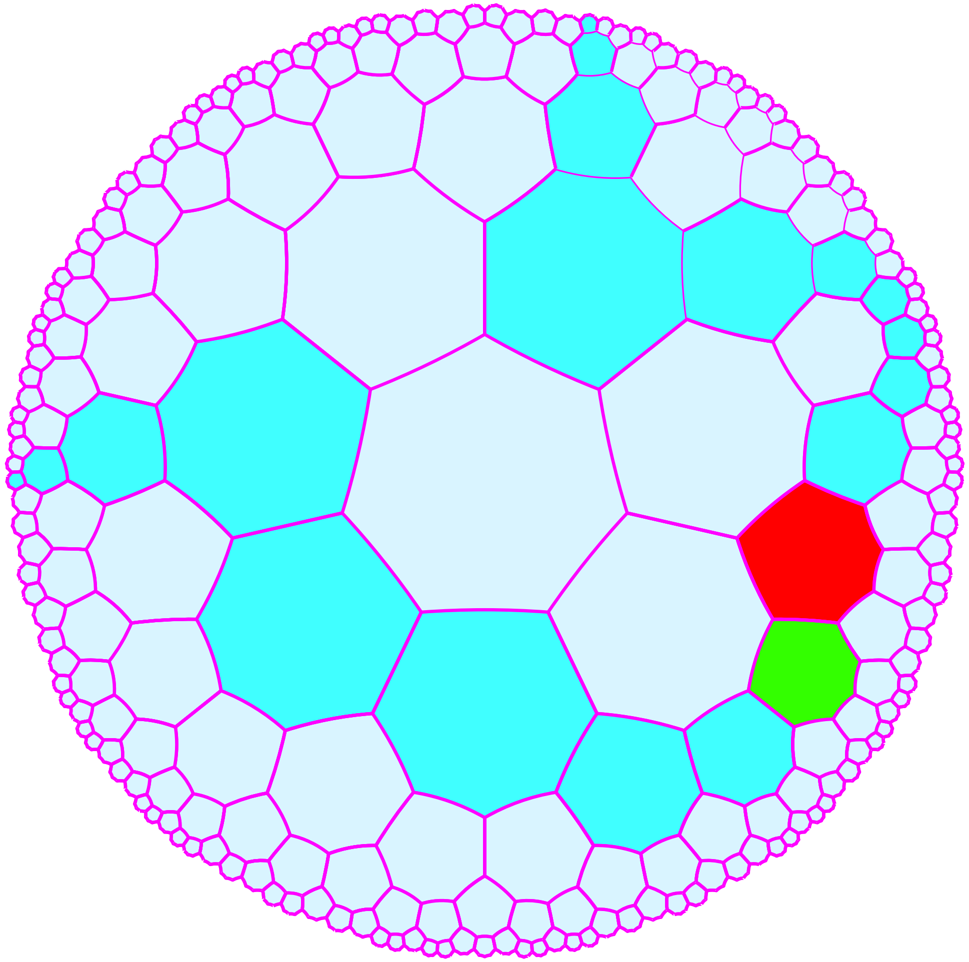


illustration of the motion along a track

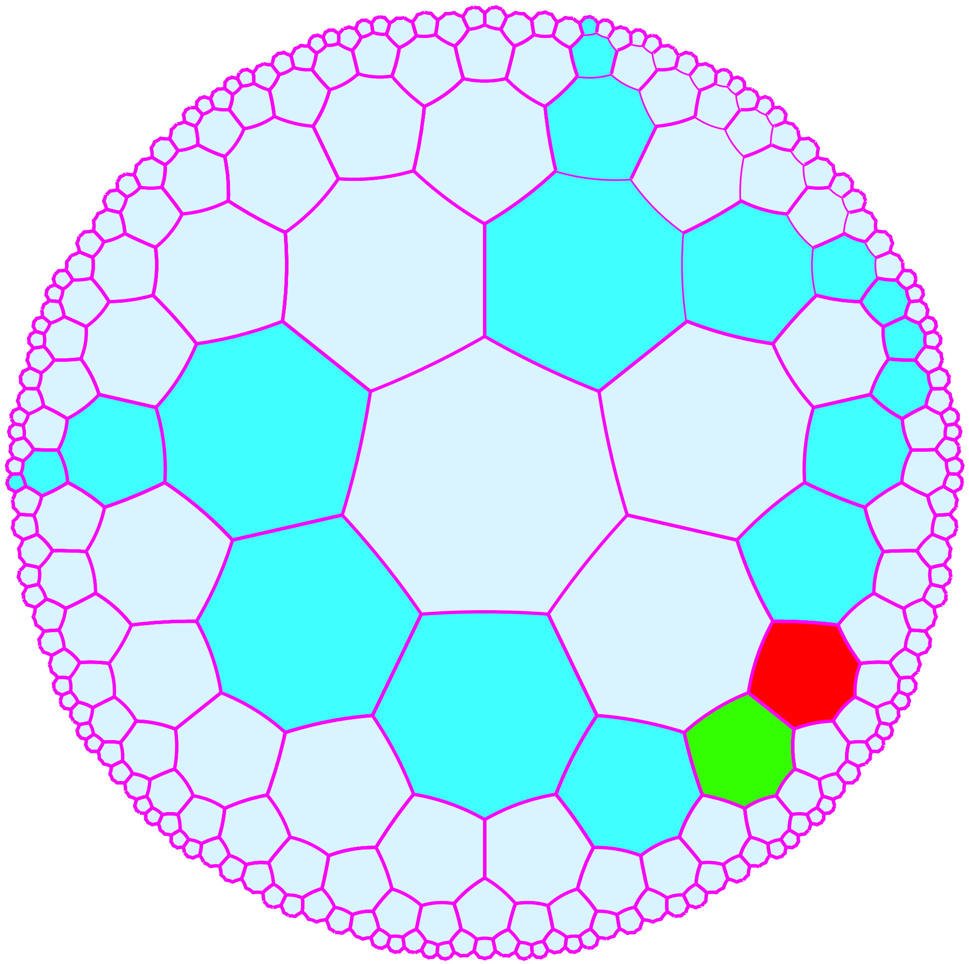


illustration of the motion along a track

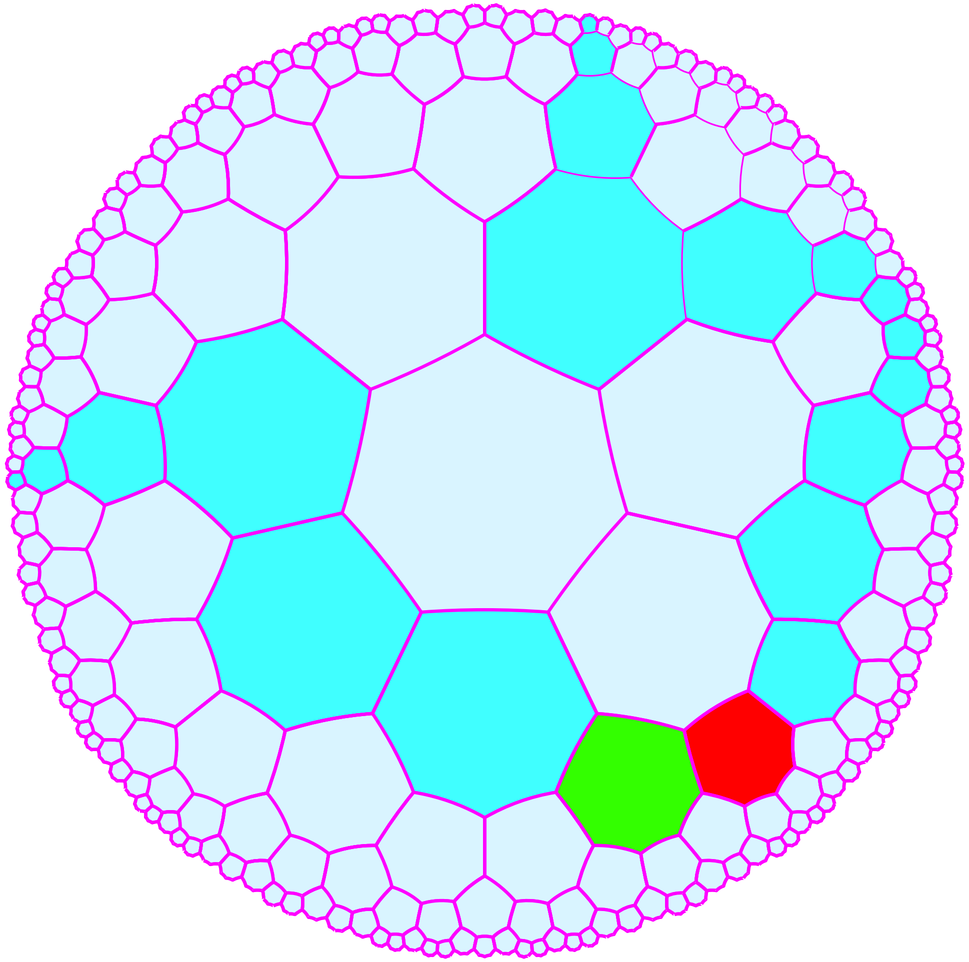


illustration of the motion along a track

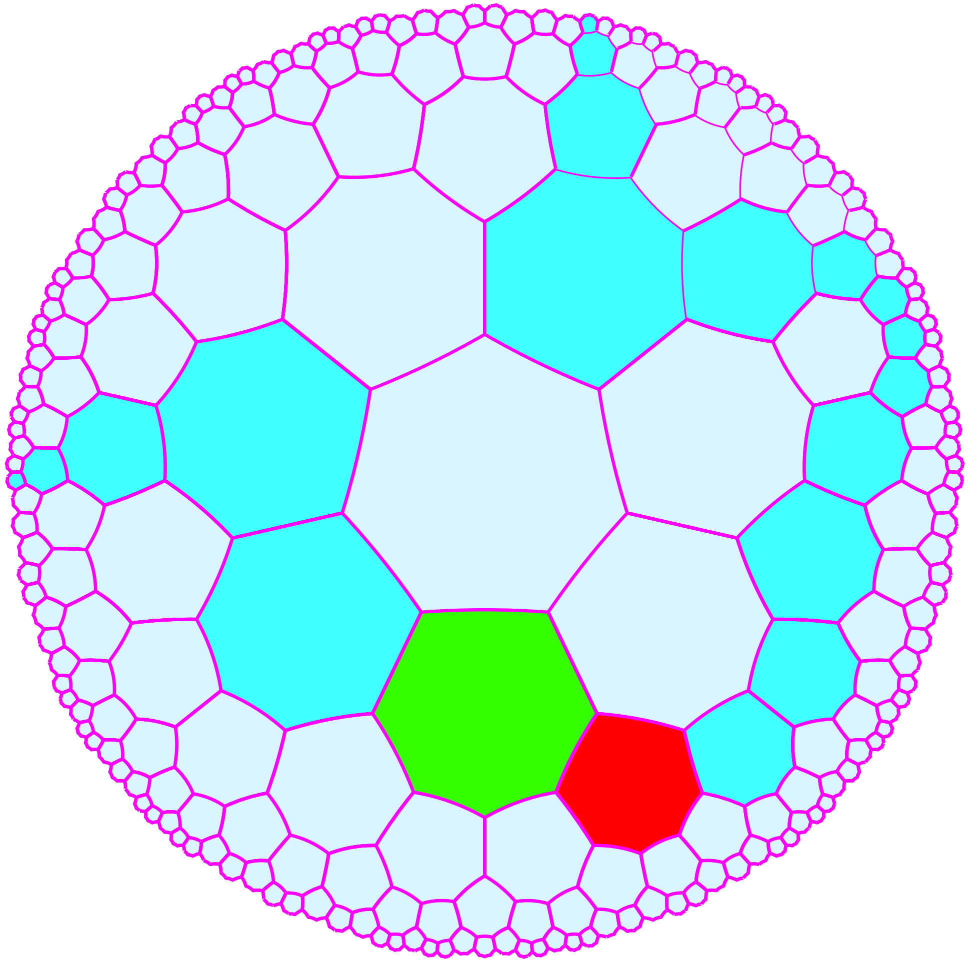


illustration of the motion along a track

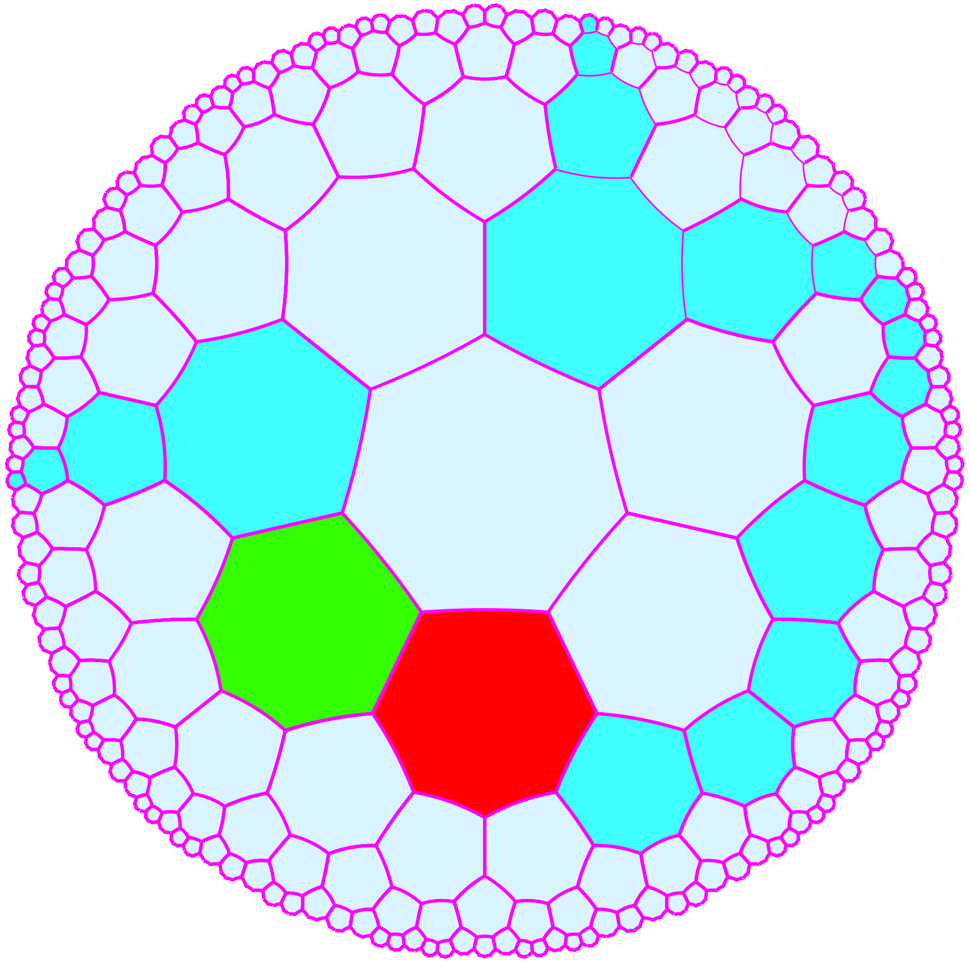


illustration of the motion along a track

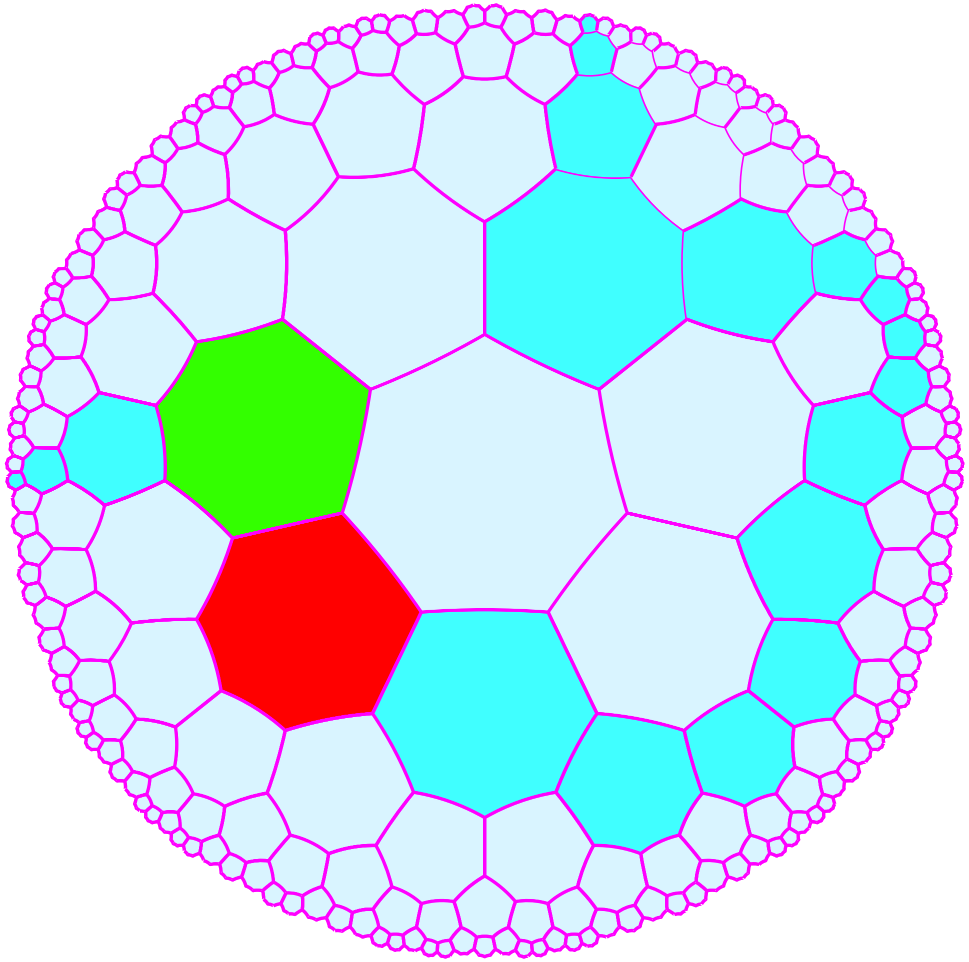


illustration of the motion along a track

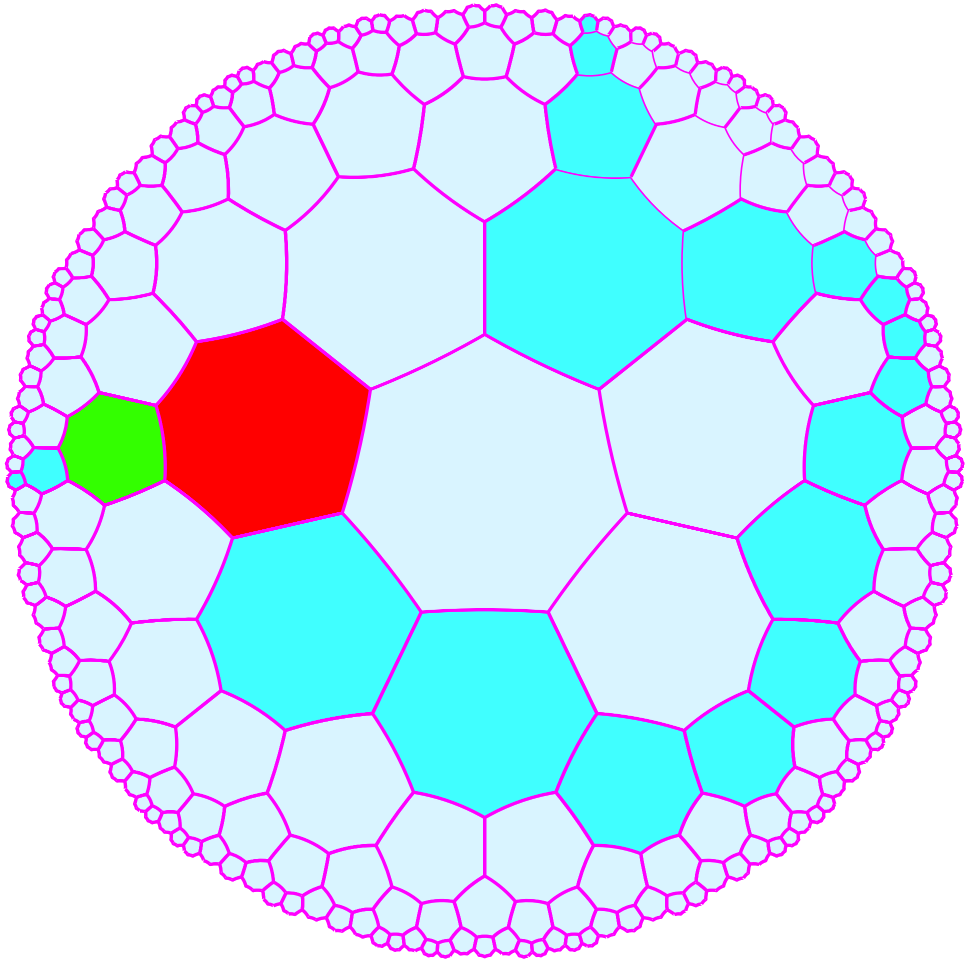


illustration of the motion along a track

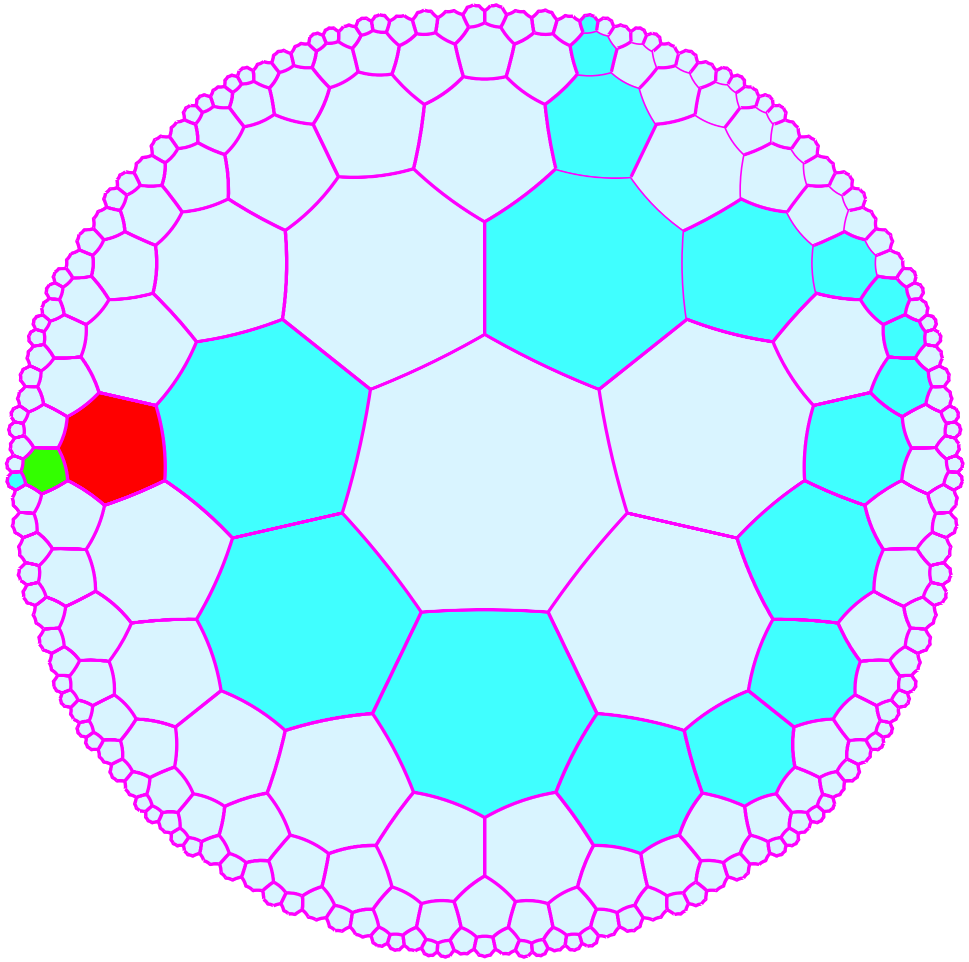


illustration of the motion along a track

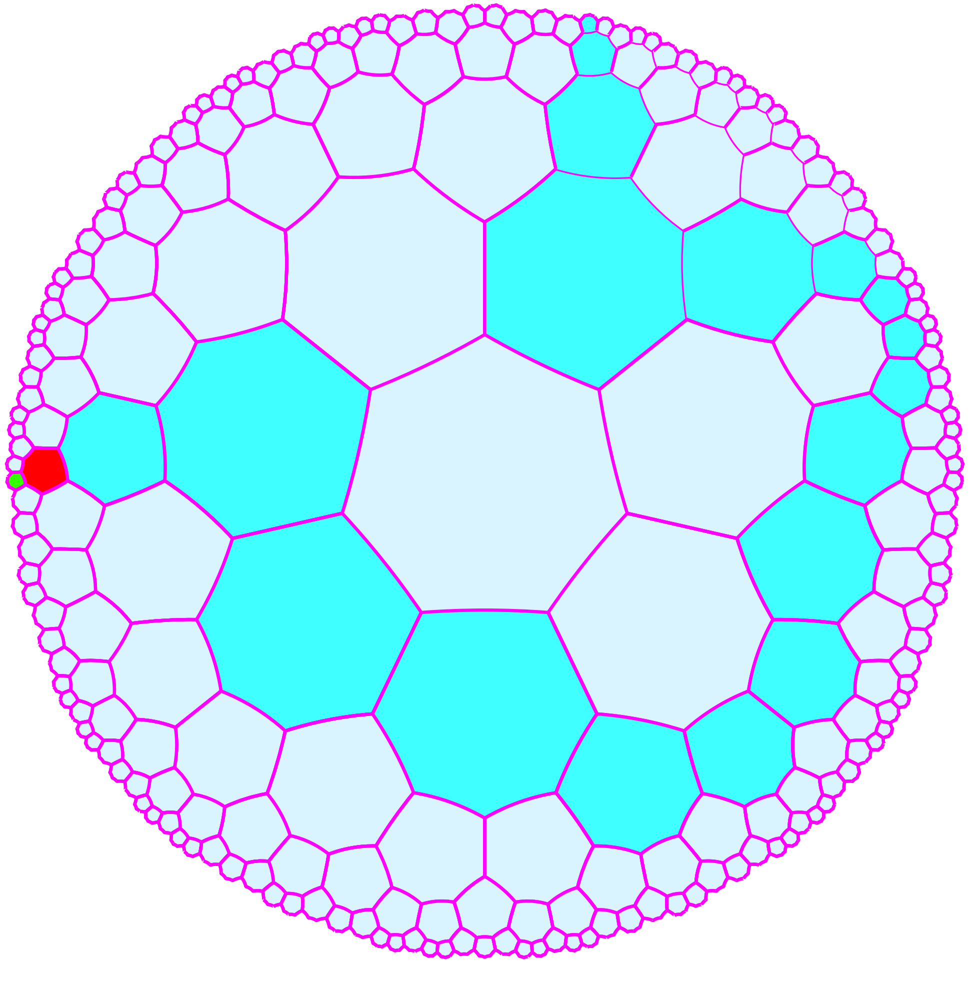


illustration of the motion along a track

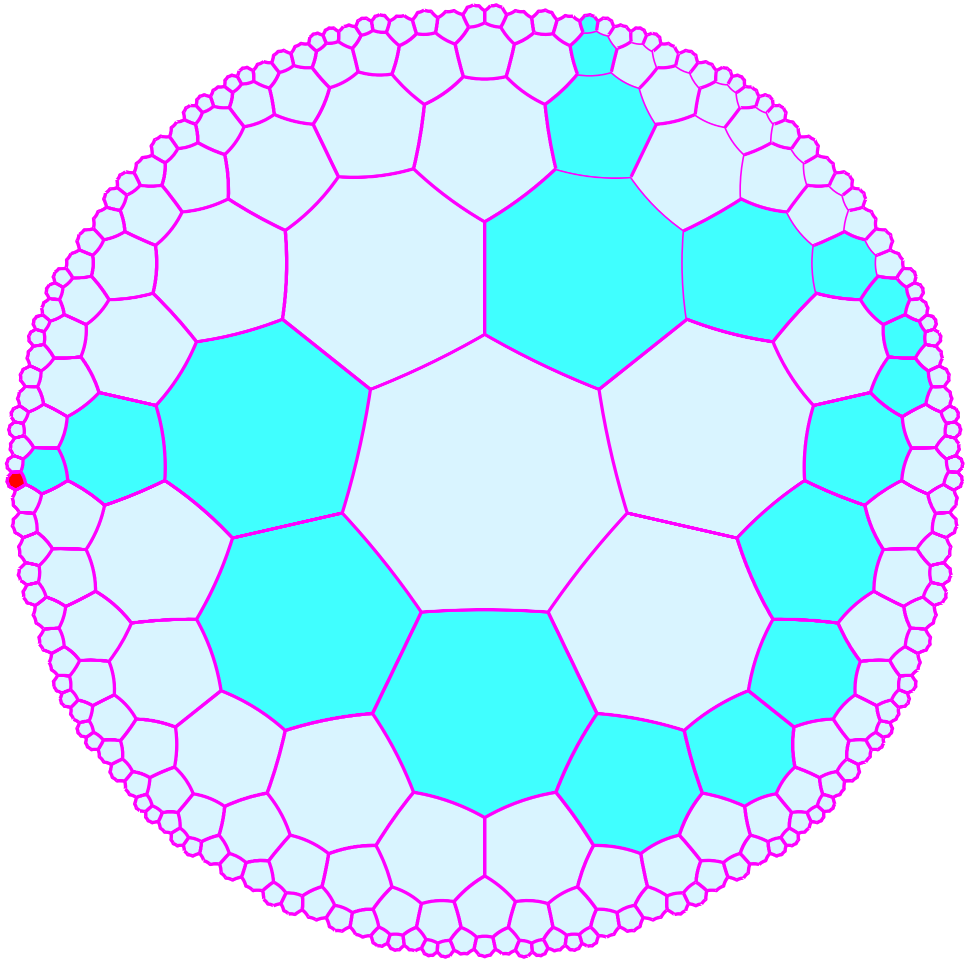


illustration of the crossing of a memory switch:

here, the passive passage through

the non-selected track, in sector 1

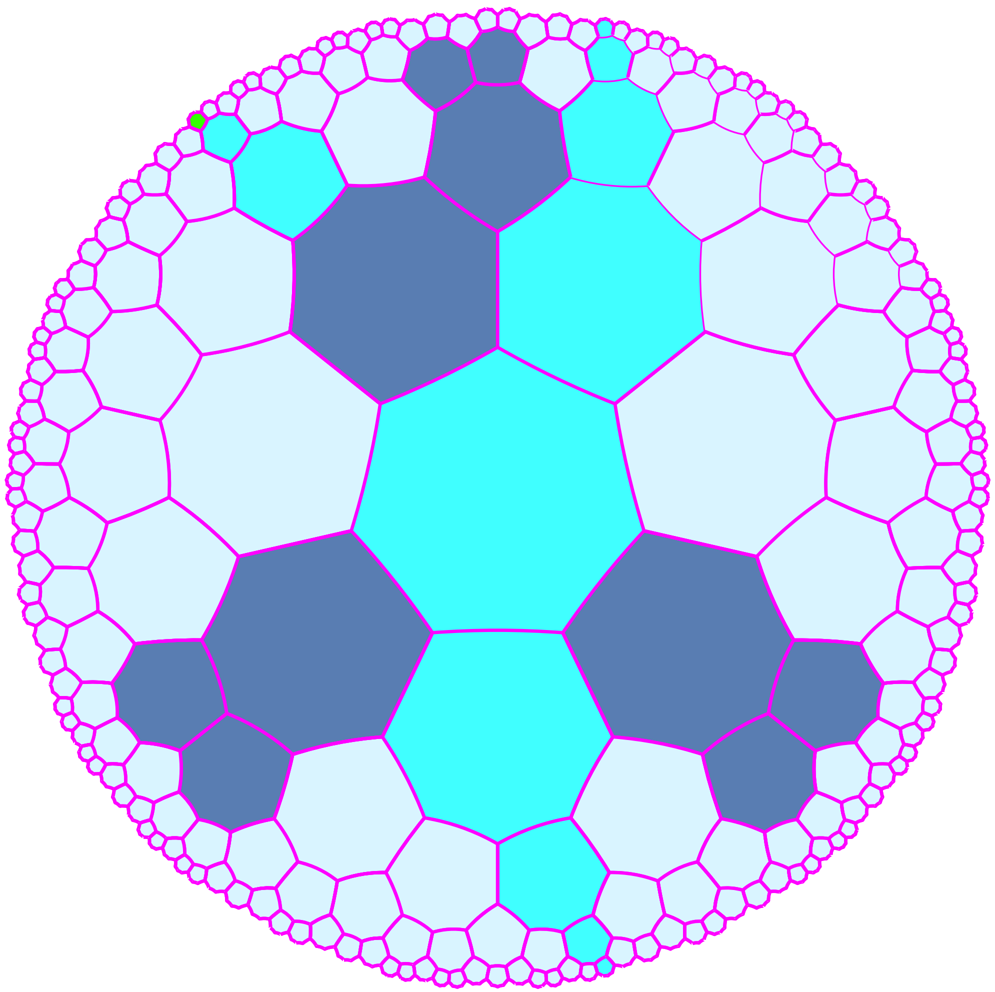
if the locomotive arrives through sector 1

it is sent to sector 4

and sector 1 becomes the **selected** track

left-hand side memory switch:

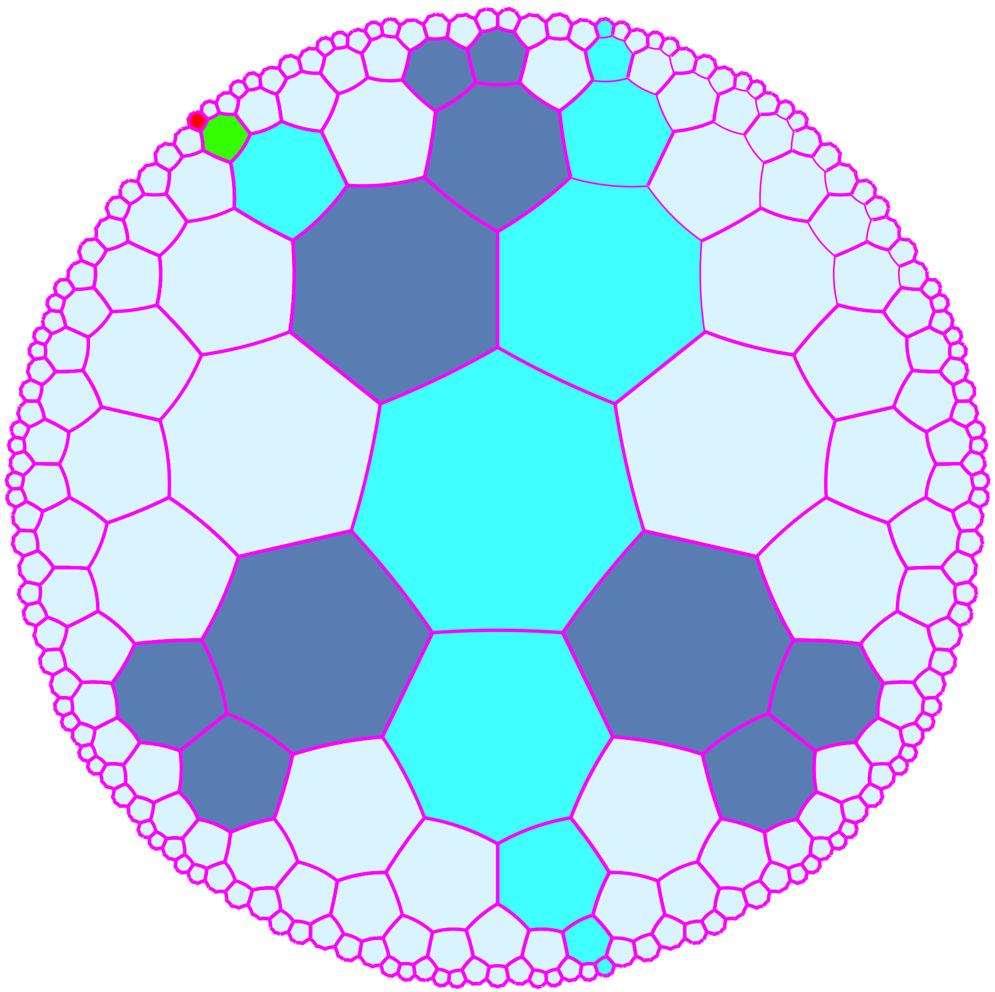
the locomotive arrives through sector 1:



front in 4(1)

left-hand side memory switch:

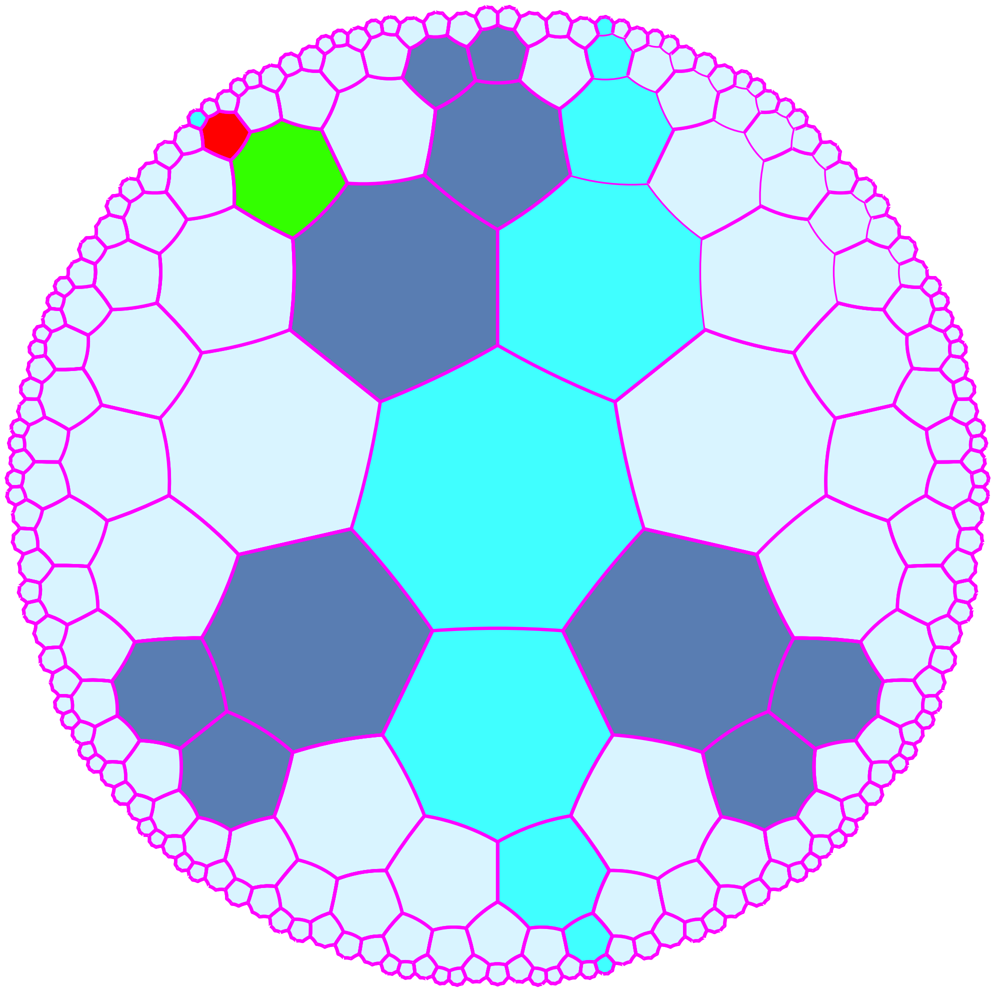
the locomotive arrives through sector 1:



front in 4(1)

left-hand side memory switch:

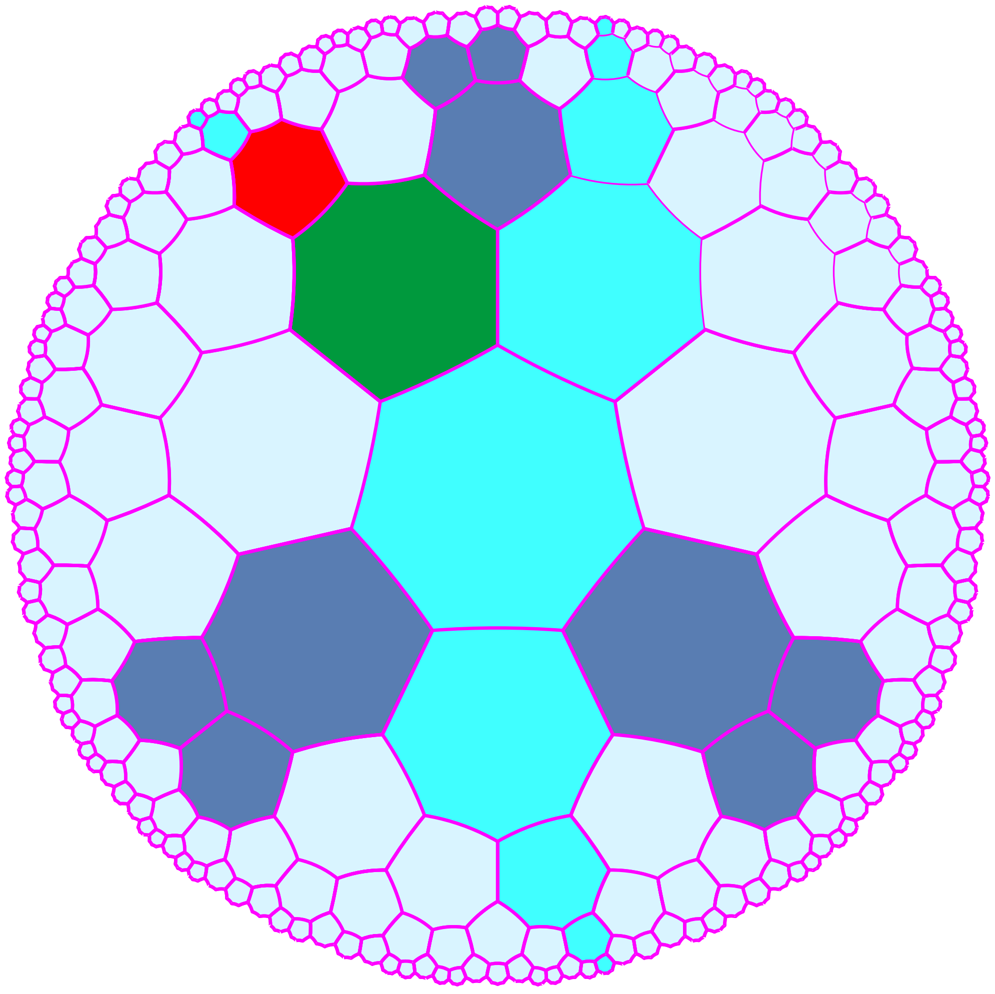
the locomotive arrives through sector 1:



front in 4(1)

left-hand side memory switch:

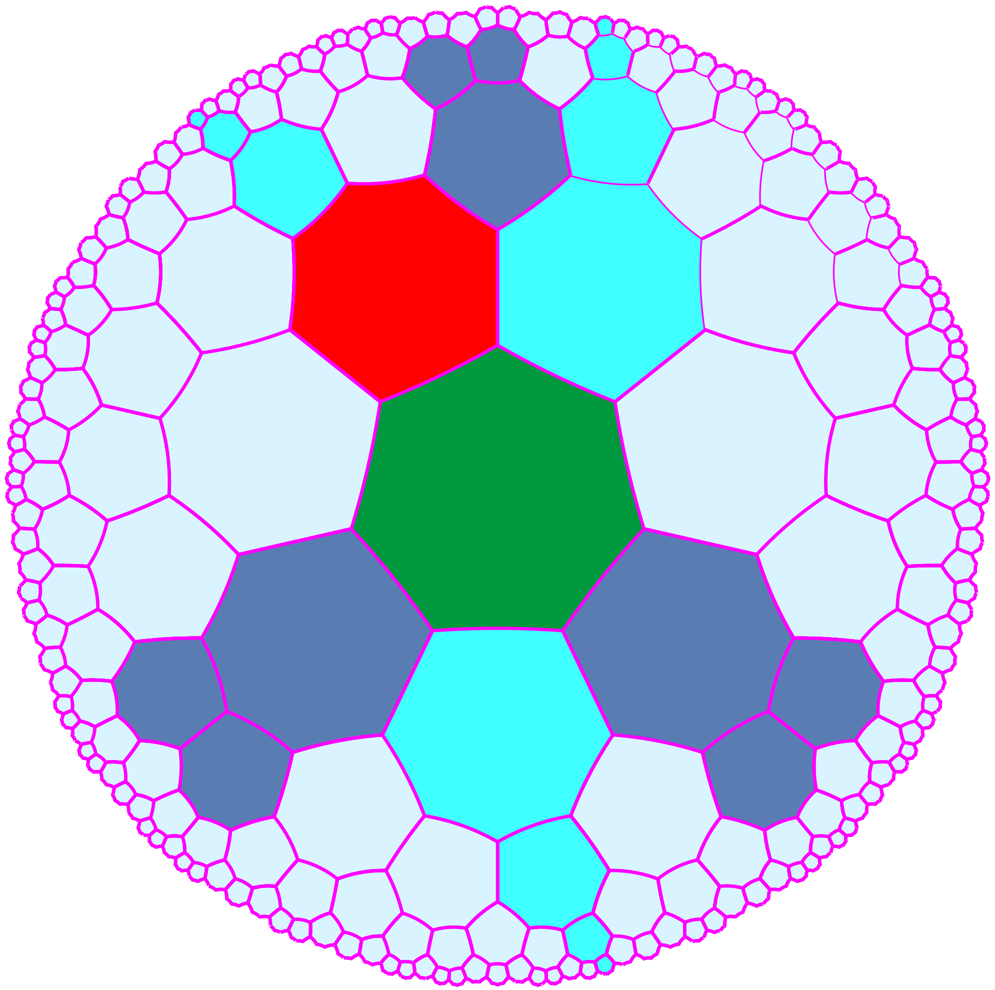
the locomotive arrives through sector 1:



front in 1(1)

left-hand side memory switch:

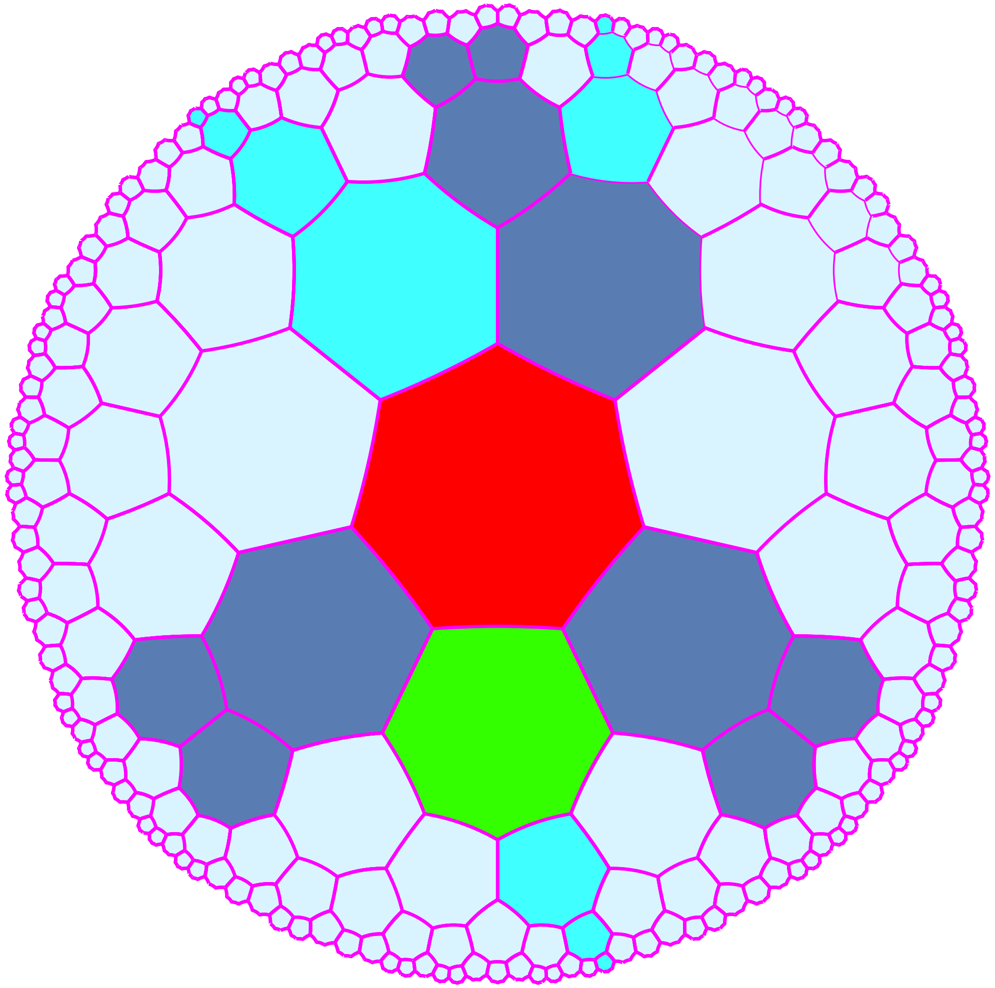
the locomotive arrives through sector 1:



front in 0

left-hand side memory switch:

and leaves through sector 4:

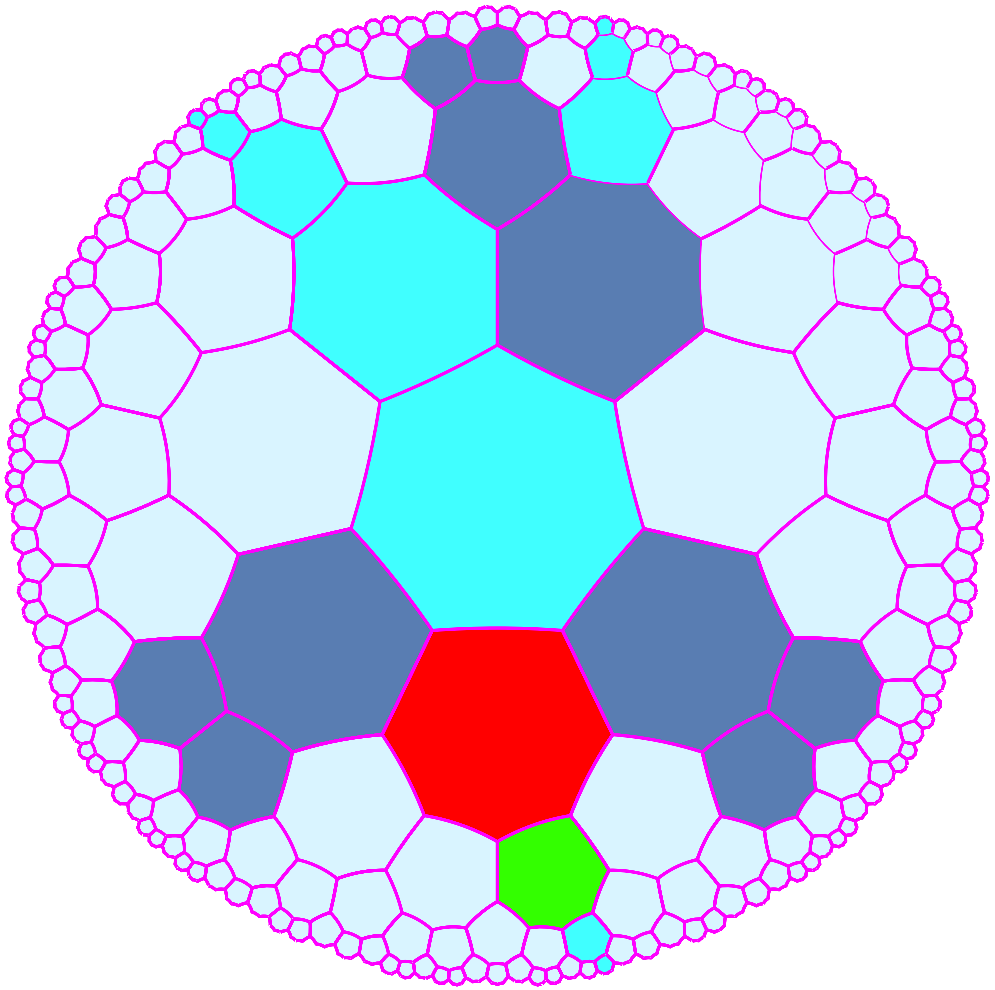


change of selection

front in 1(4)

left-hand side memory switch:

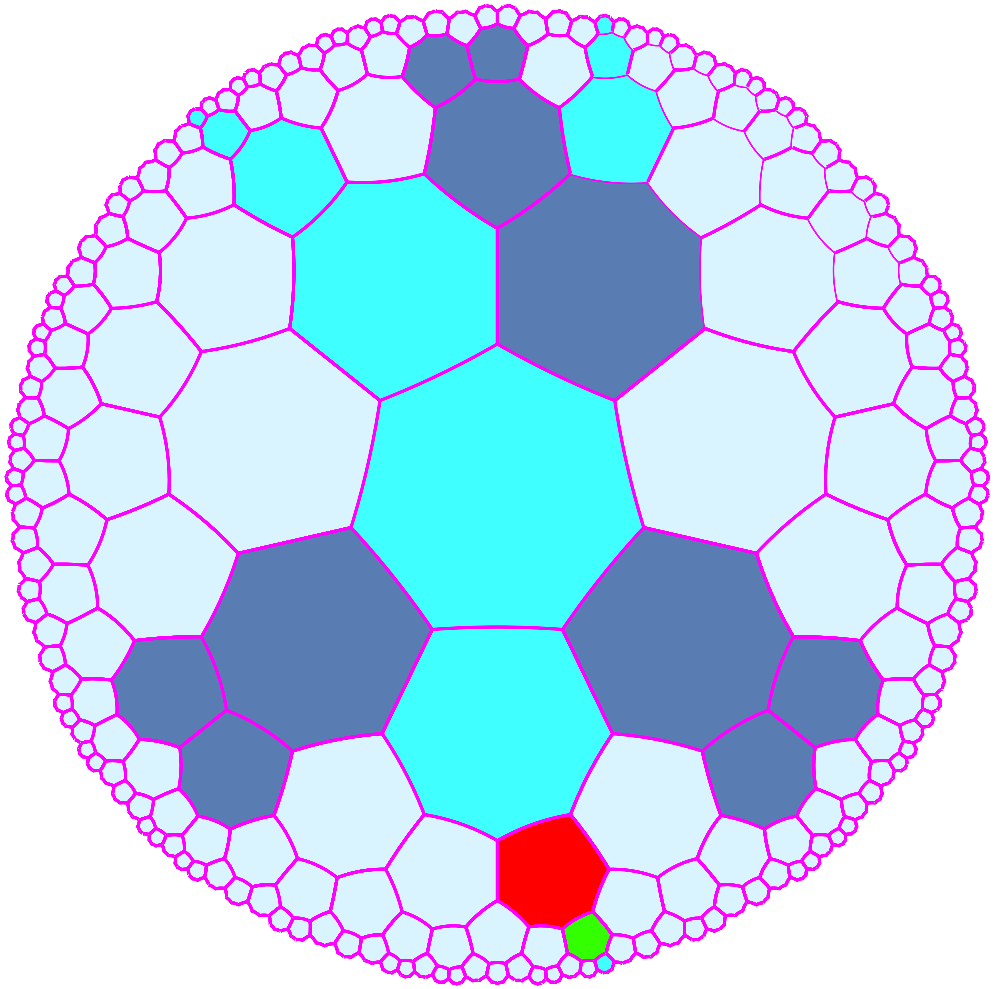
and leaves through sector 4:



front in 4(4)

left-hand side memory switch:

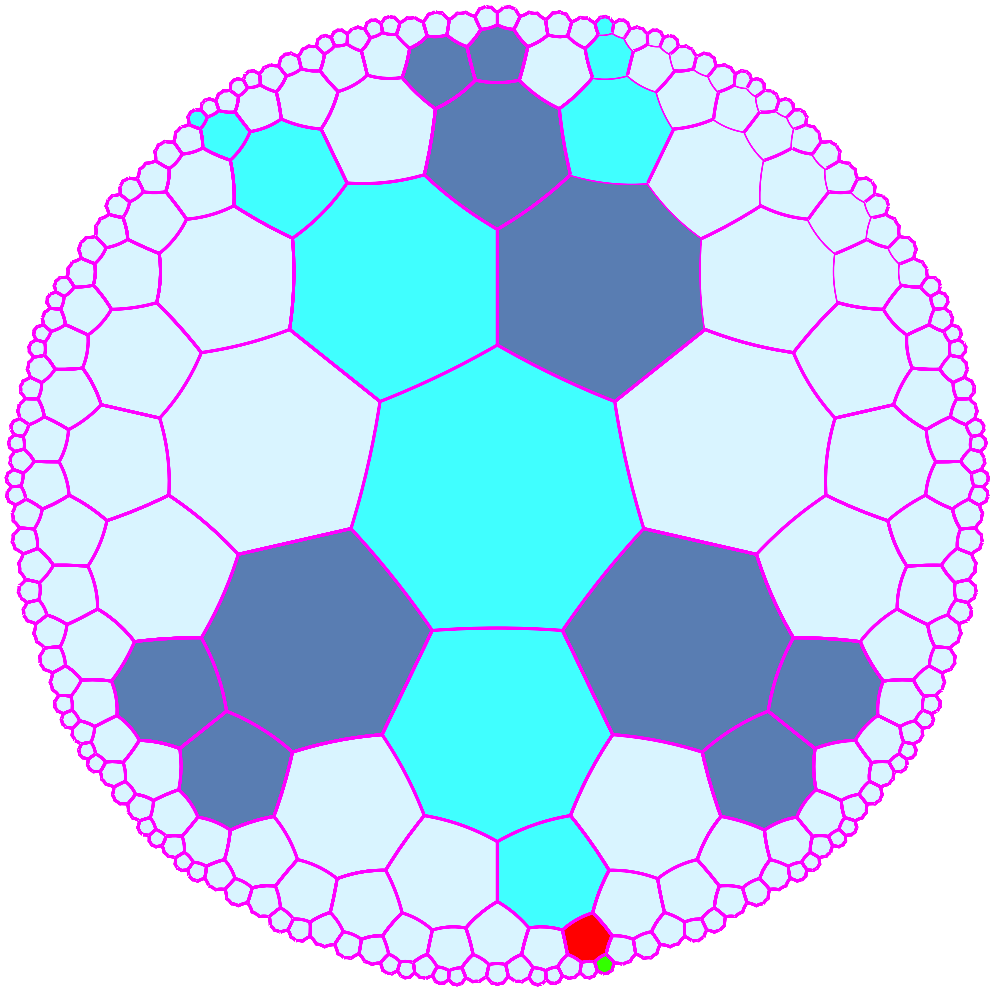
and leaves through sector 4:



front in 12(4)

left-hand side memory switch:

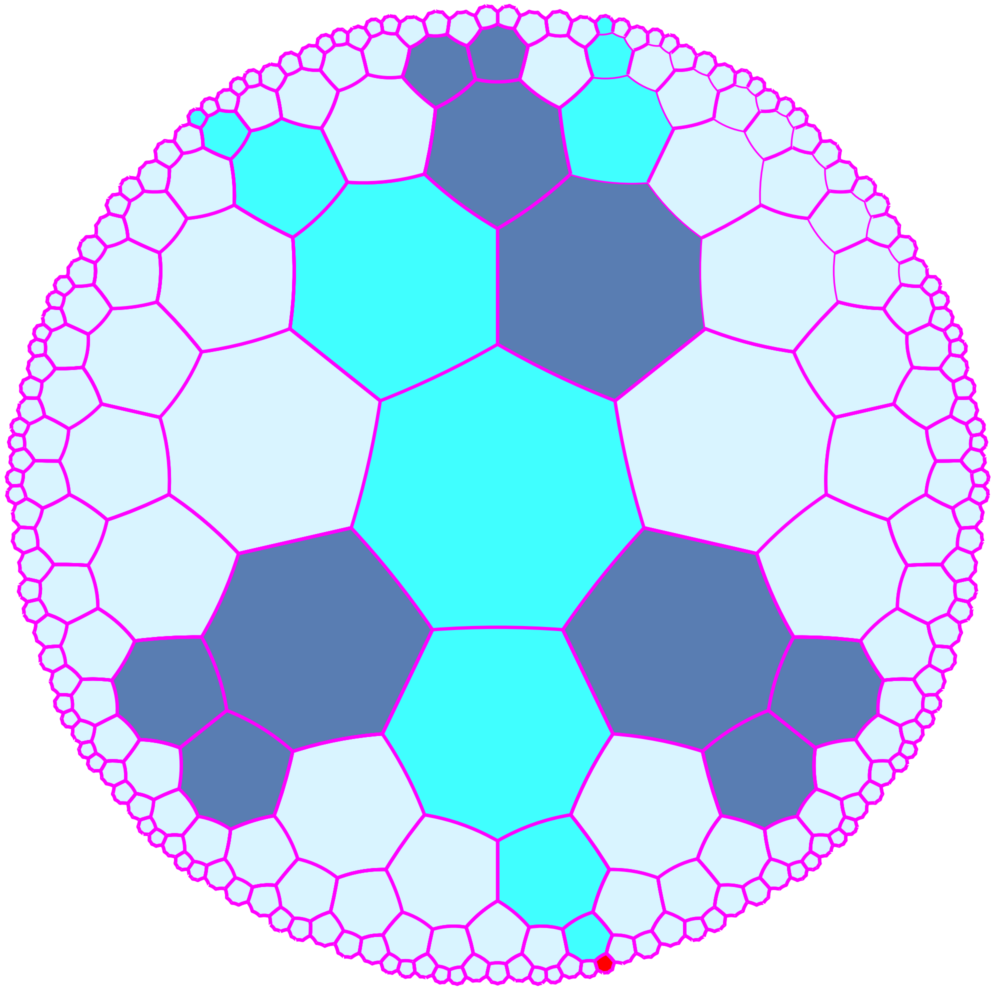
and leaves through sector 4:



front in 33(4)

left-hand side memory switch:

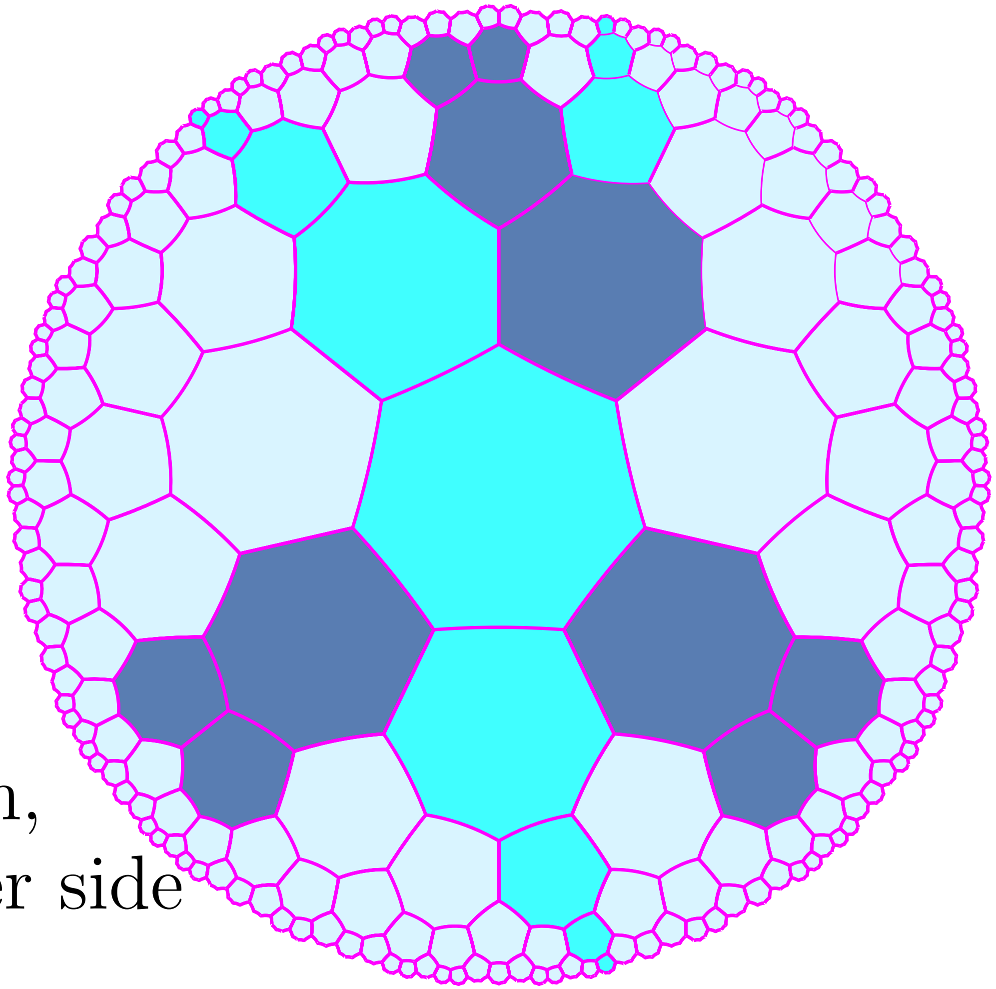
and leaves through sector 4:



front in 88(4)

left-hand side memory switch:

and leaves through sector 4:



stable again,
on the other side

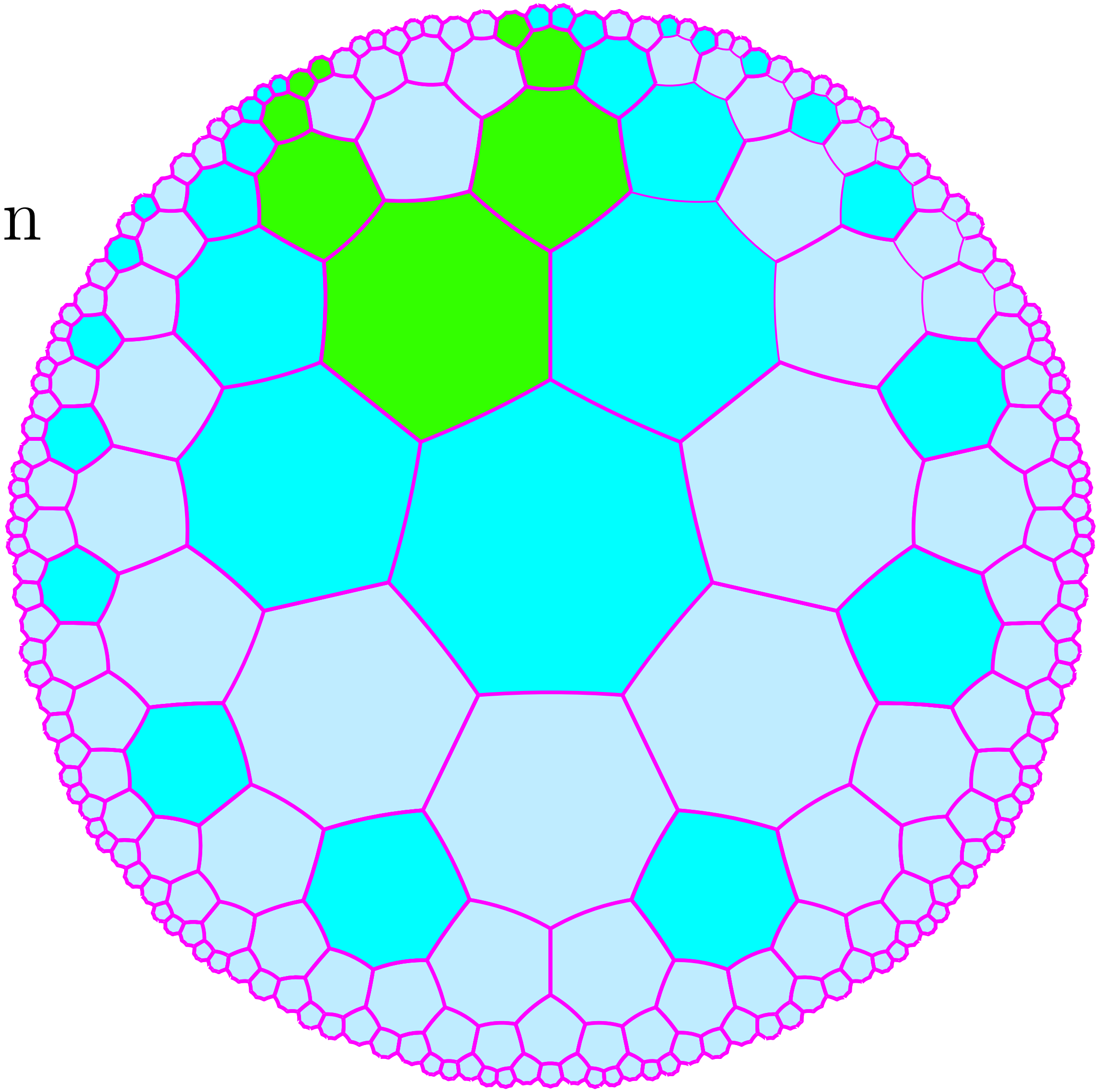
the HCA on the heptagrid with 4 states

here too, we illustrate two points:

- the motion of the locomotive along a track which follows an isocline
- the passive crossing of a memory switch from the non-selected track

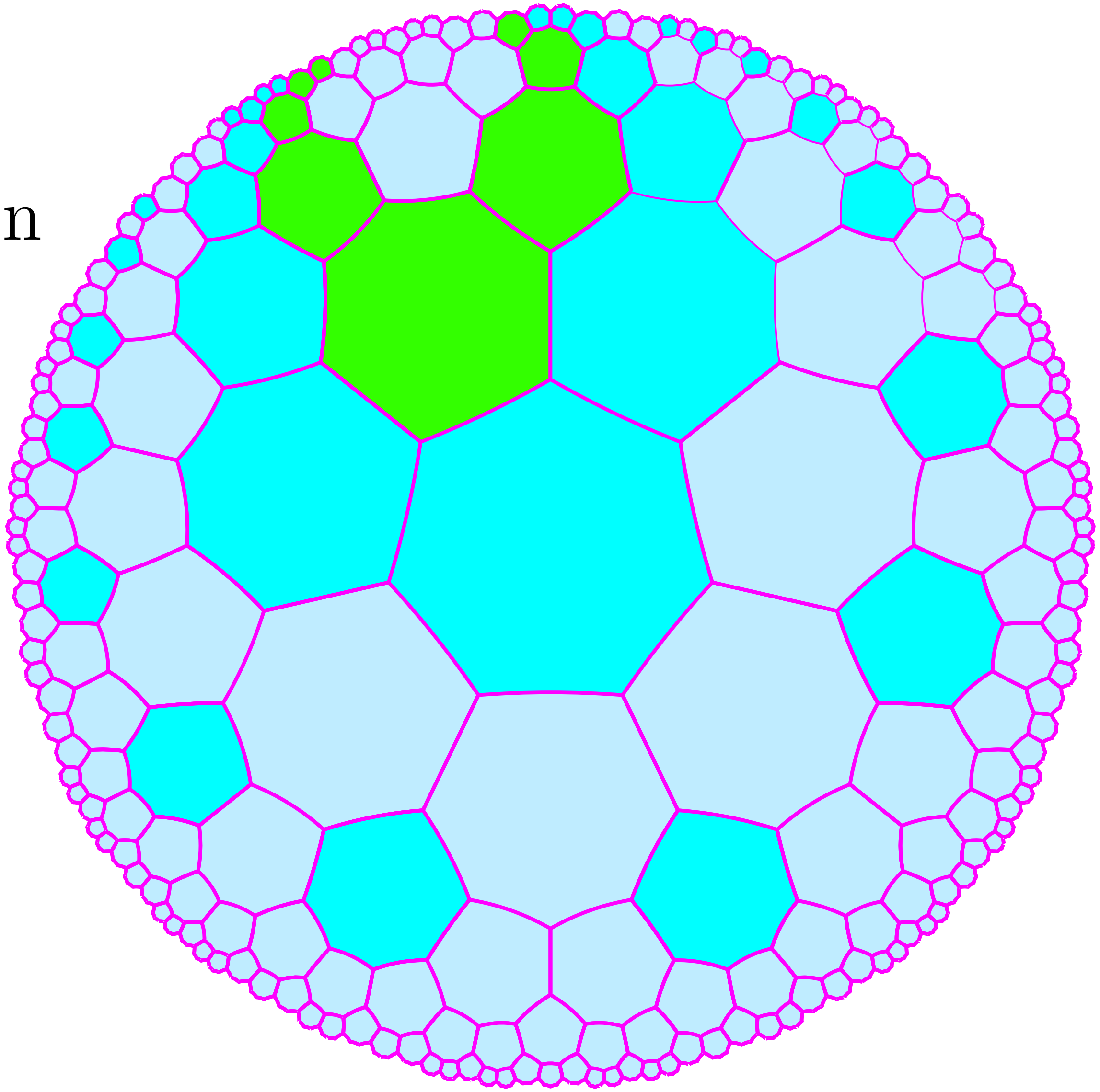
the locomotive on a track along an isocline

the idle
configuration



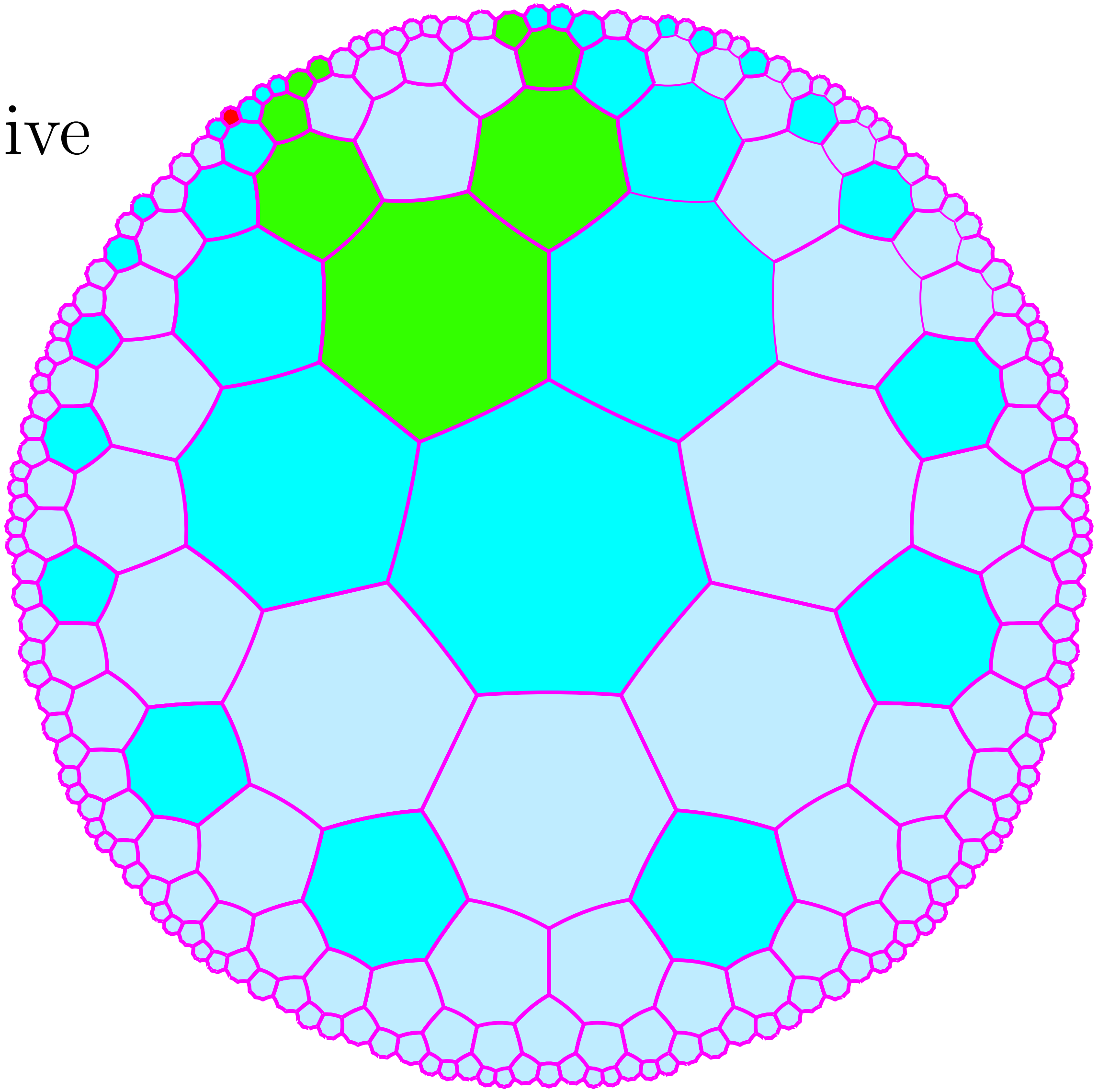
the locomotive on a track along an isocline

the idle
configuration



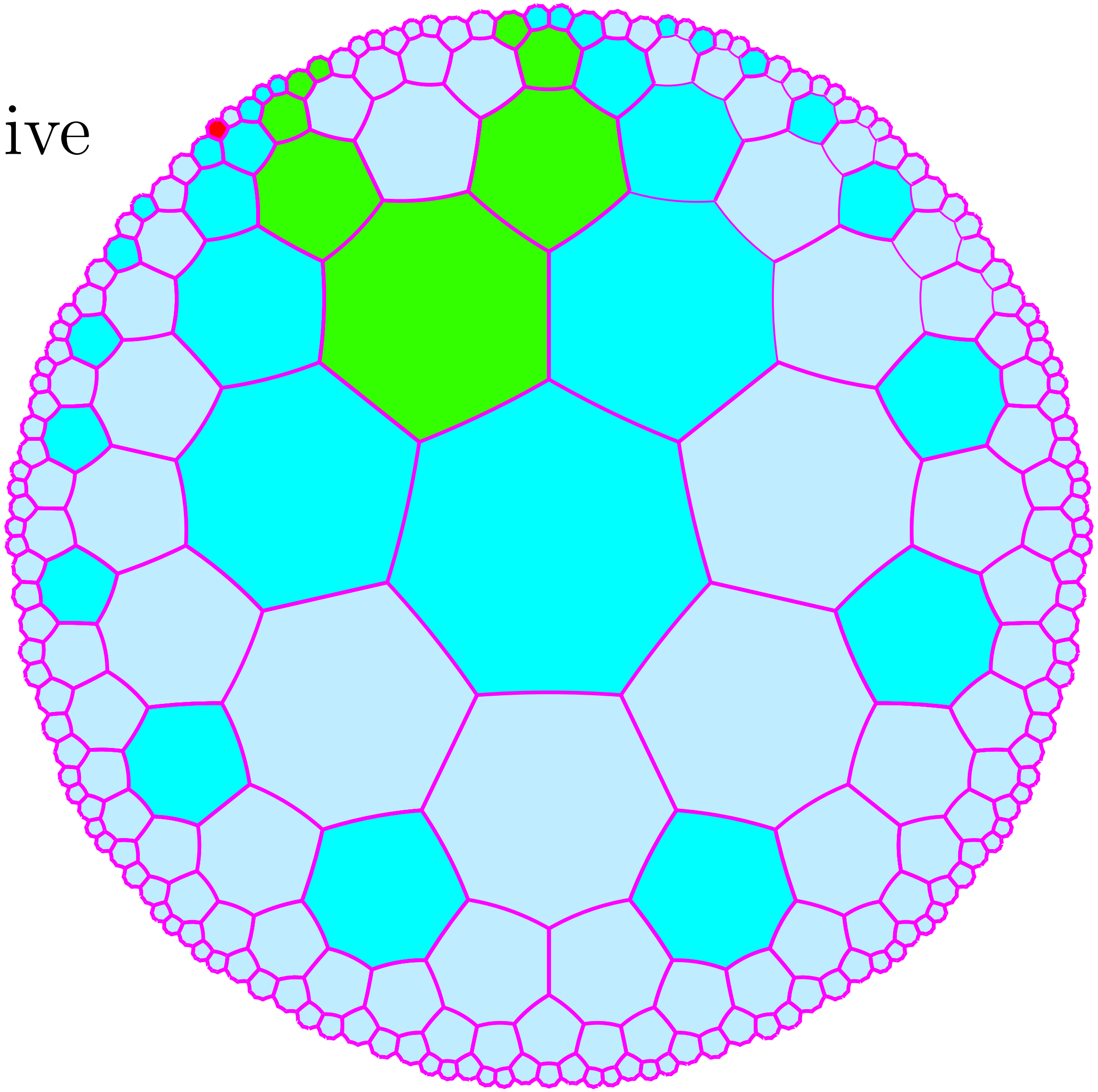
the locomotive on a track along an isocline

the locomotive
is arriving



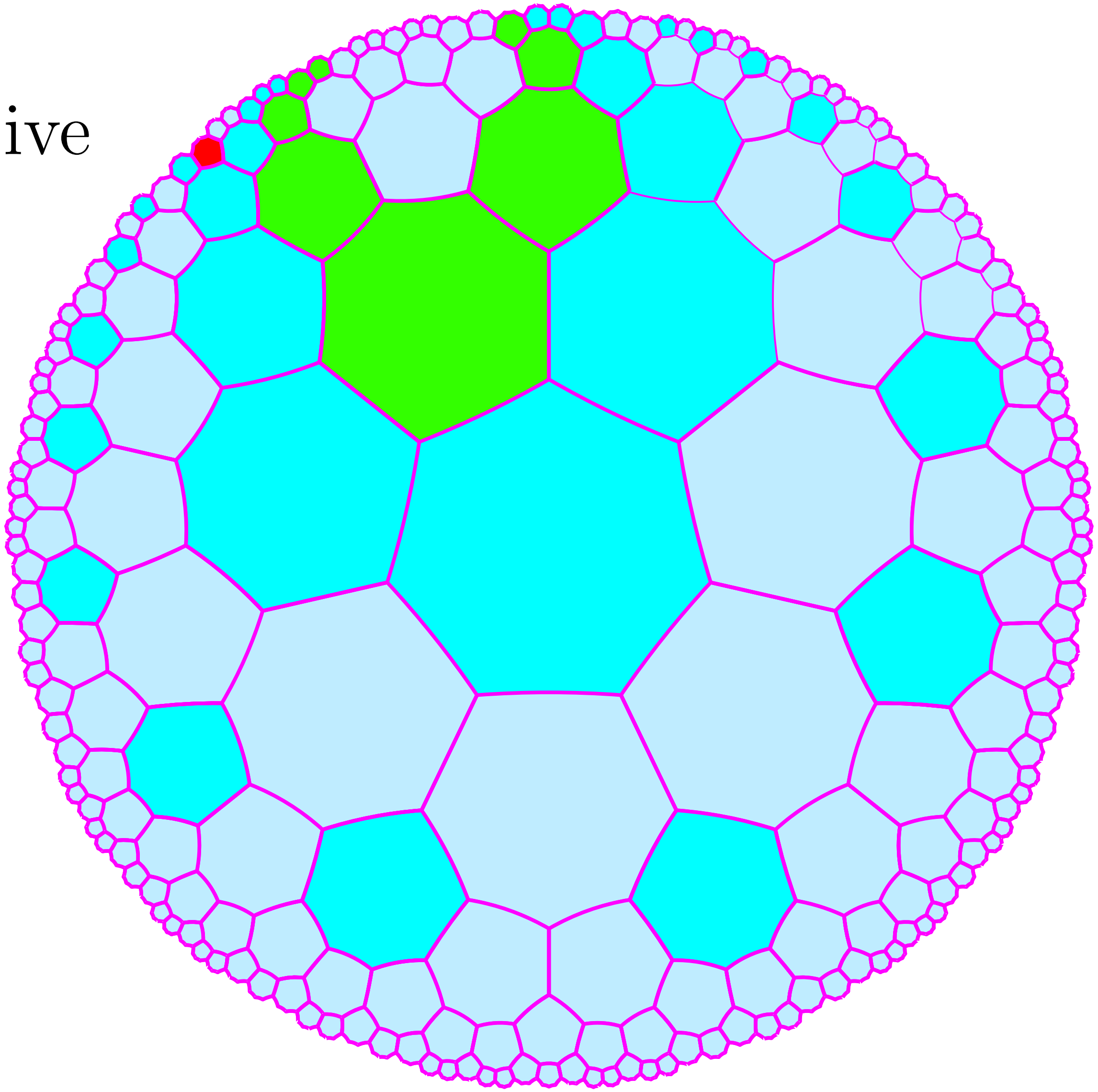
the locomotive on a track along an isocline

the locomotive
is arriving



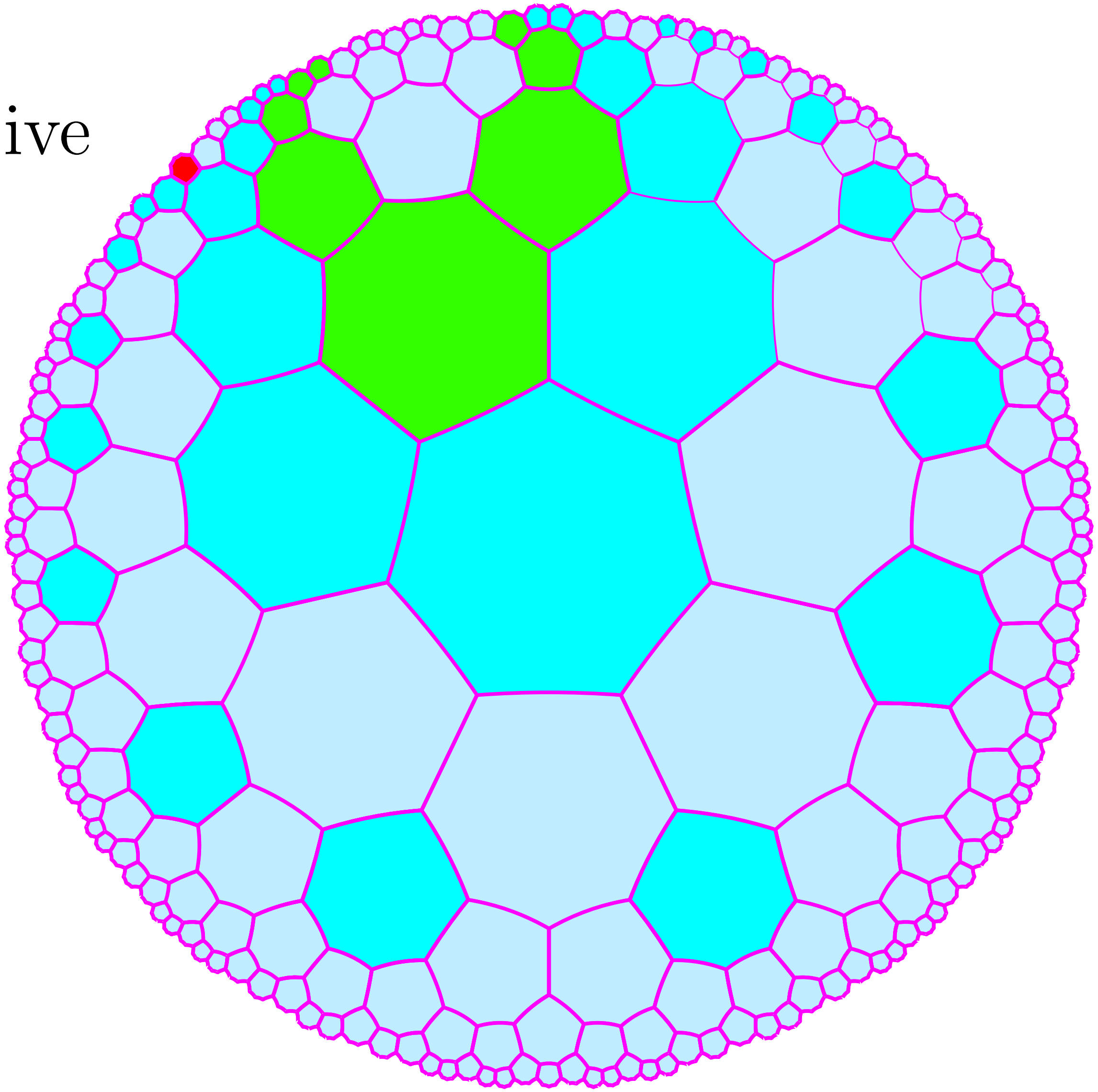
the locomotive on a track along an isocline

the locomotive
is arriving



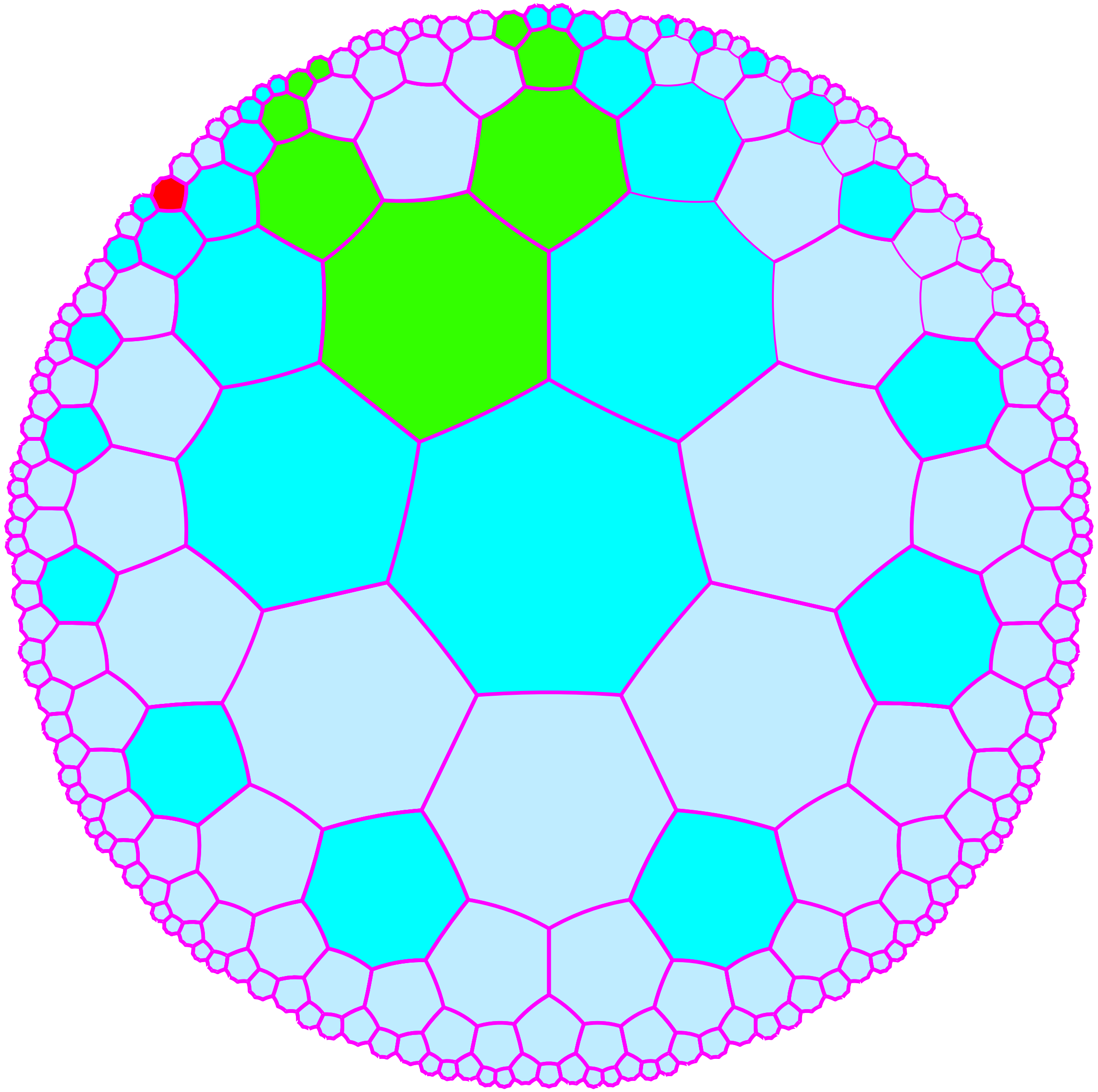
the locomotive on a track along an isocline

the locomotive
is arriving



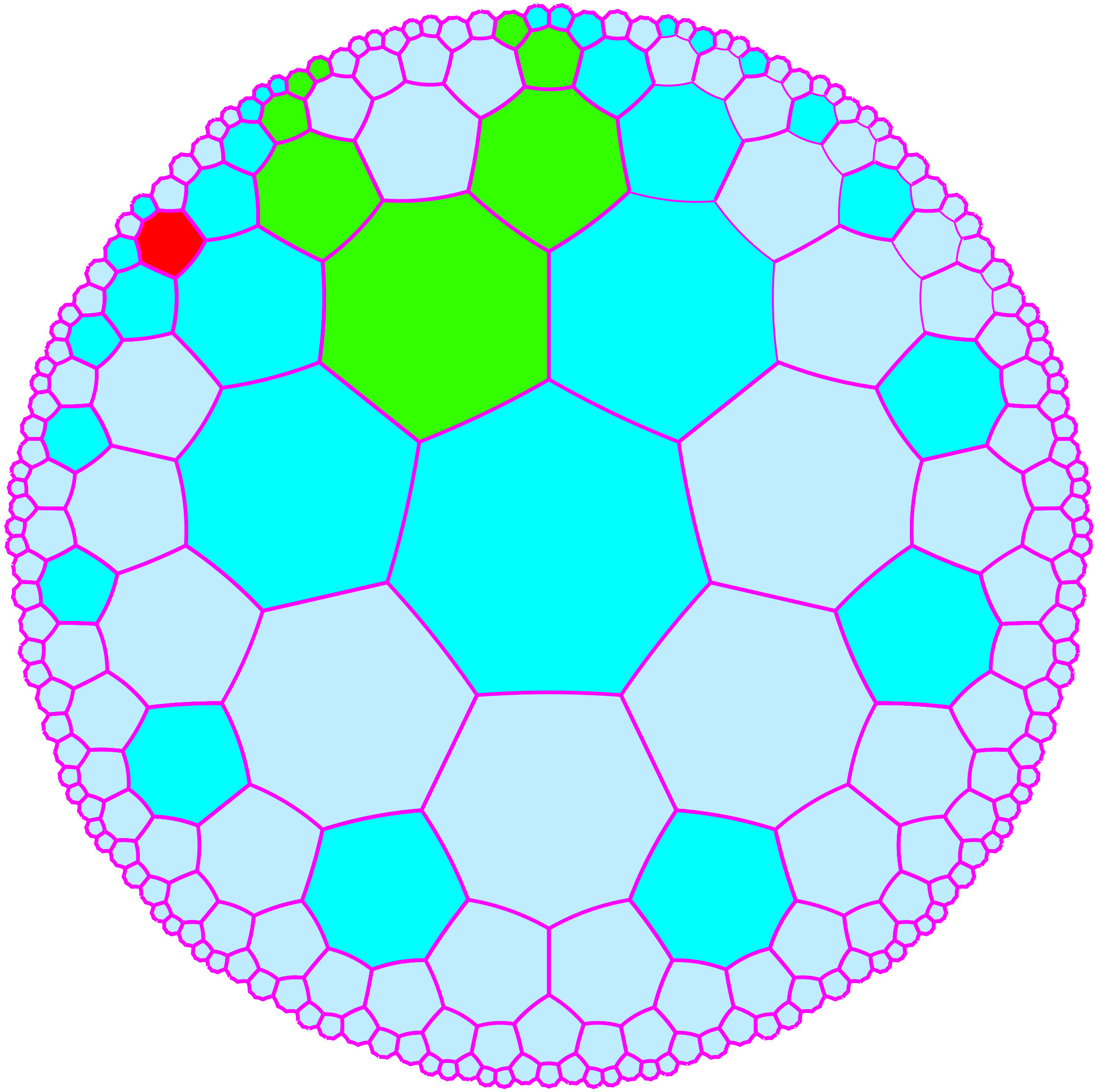
the locomotive on a track along an isocline

and it goes
along
the path



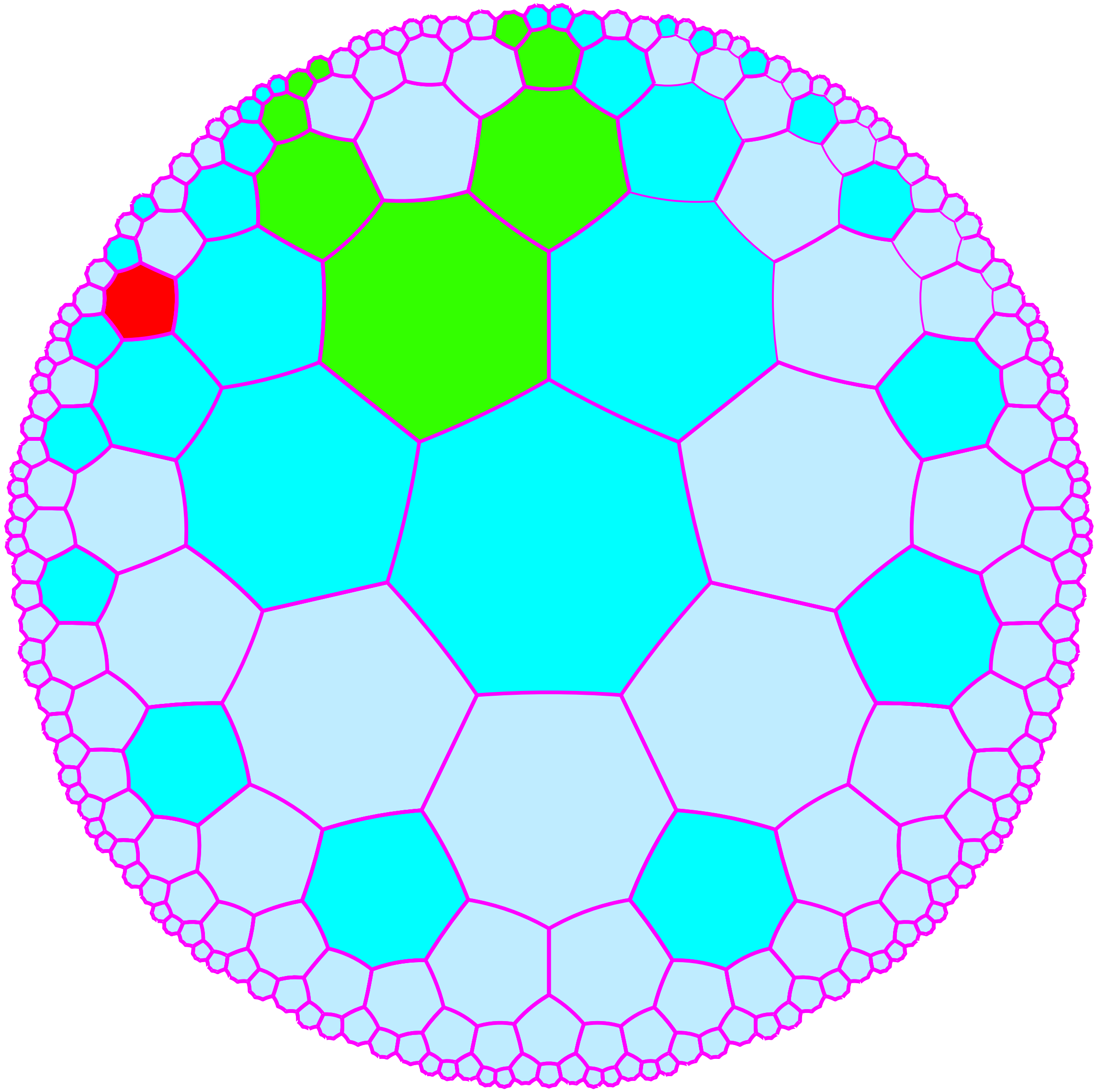
the locomotive on a track along an isocline

and it goes
along
the path



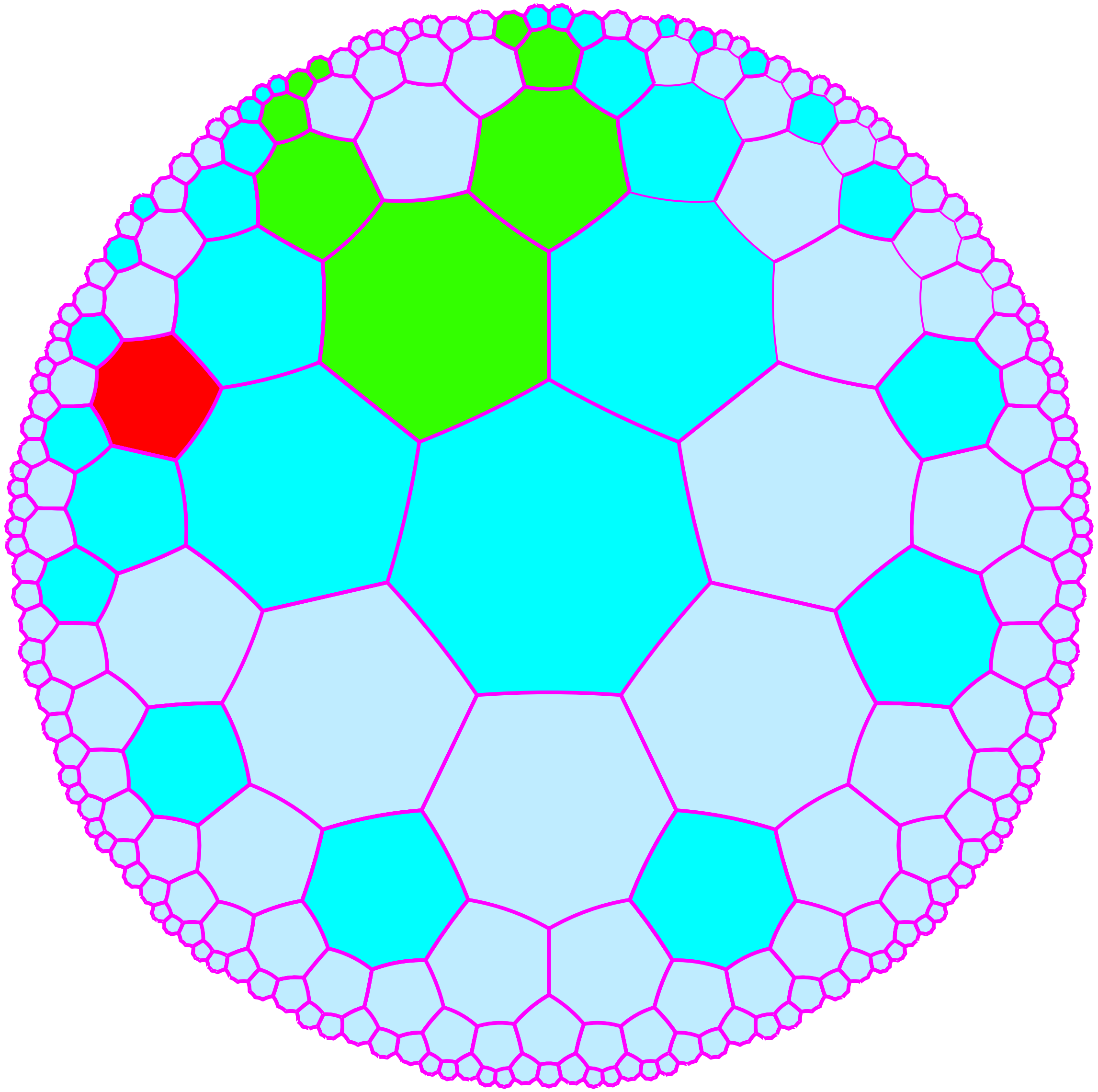
the locomotive on a track along an isocline

and it goes
along
the path



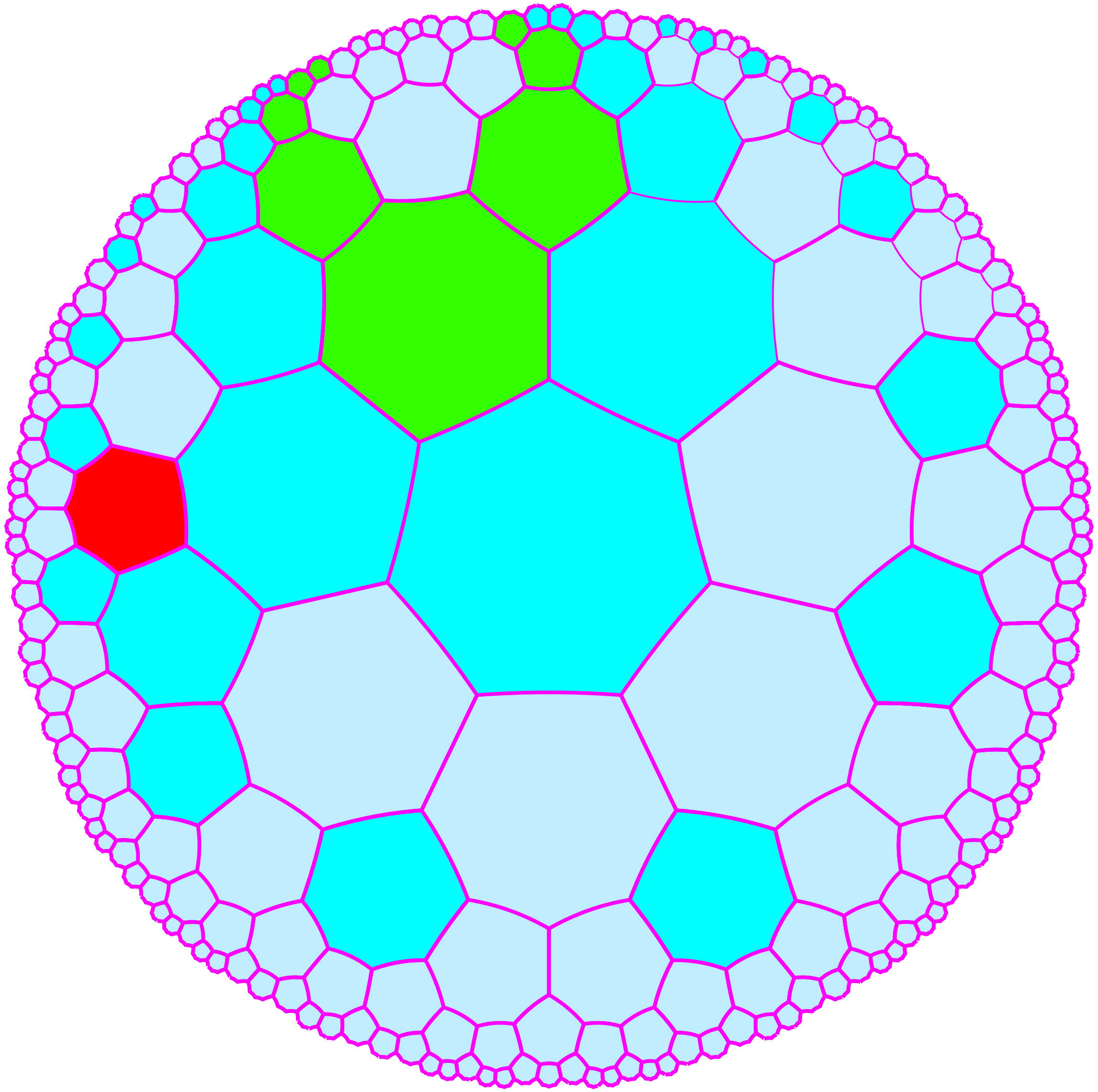
the locomotive on a track along an isocline

and it goes
along
the path



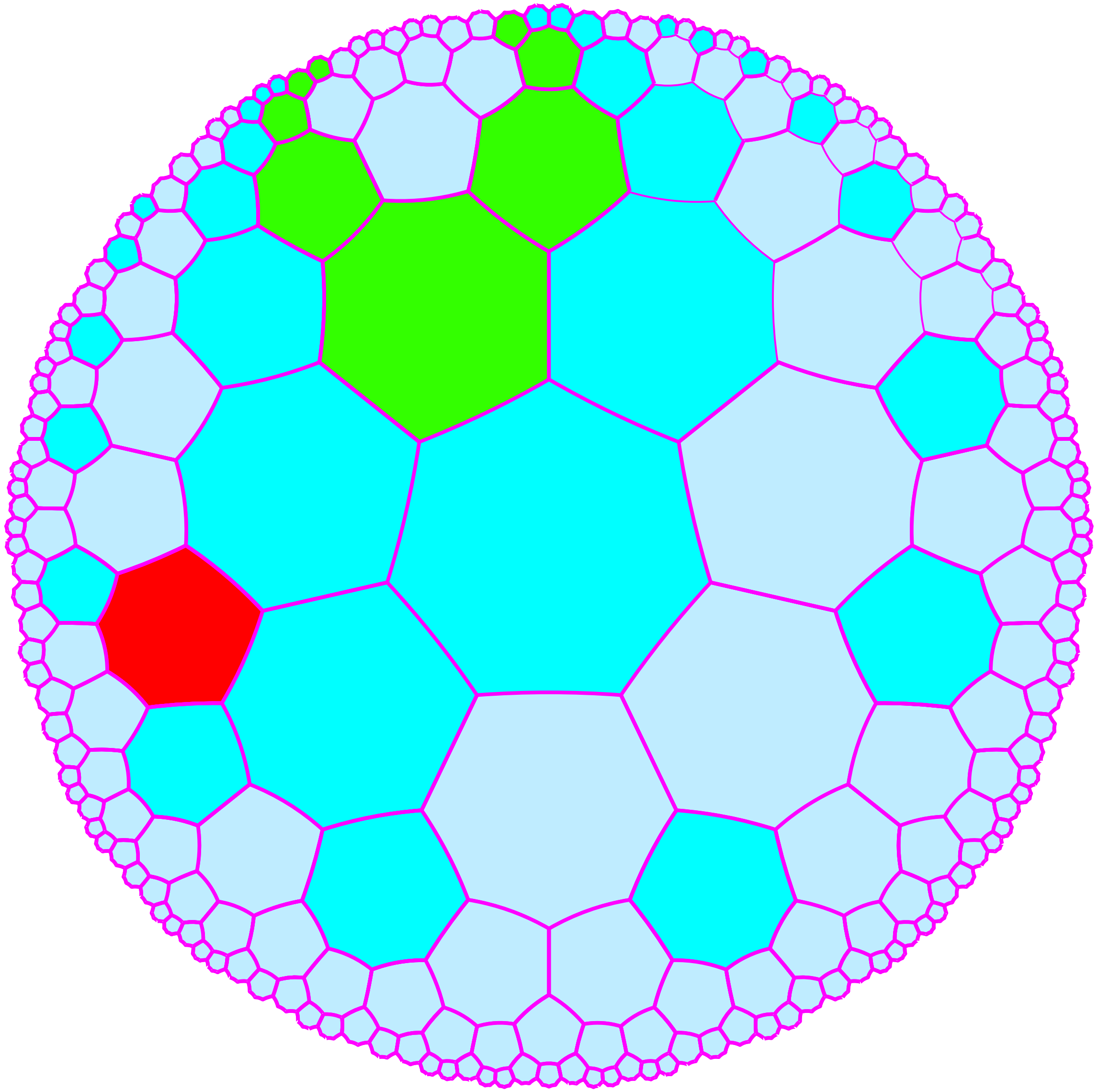
the locomotive on a track along an isocline

and it goes
along
the path



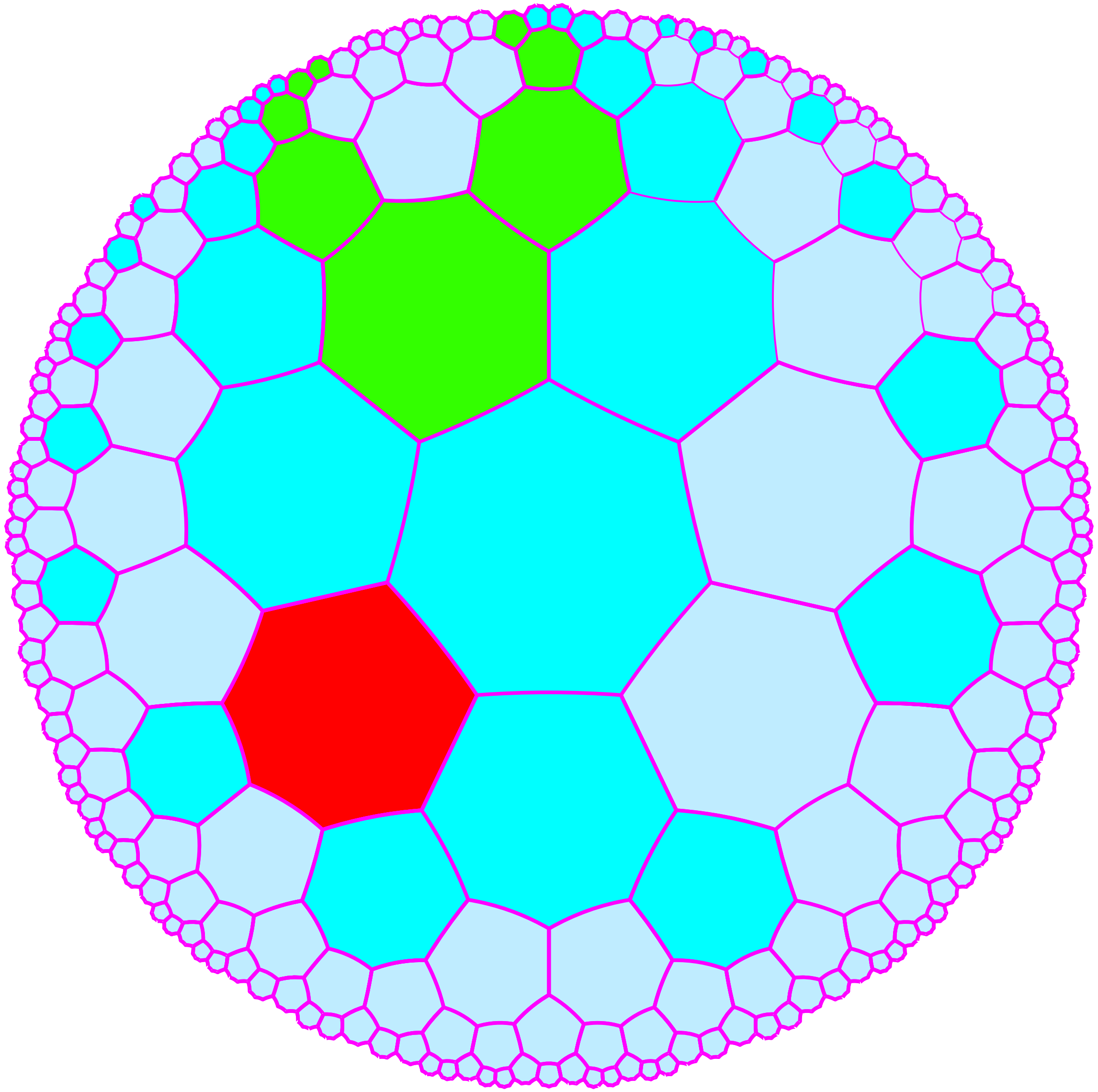
the locomotive on a track along an isocline

and it goes
along
the path



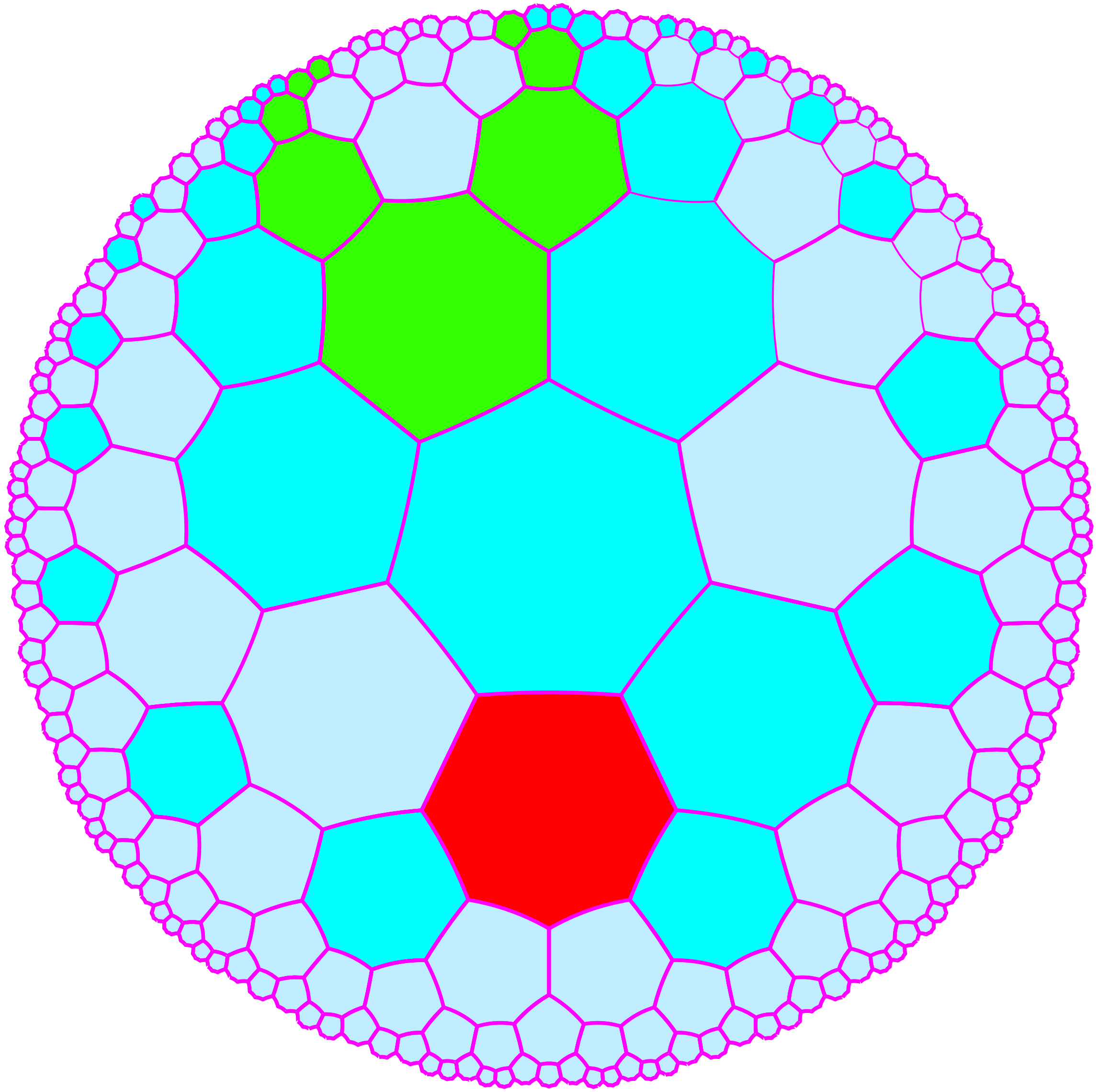
the locomotive on a track along an isocline

and it goes
along
the path



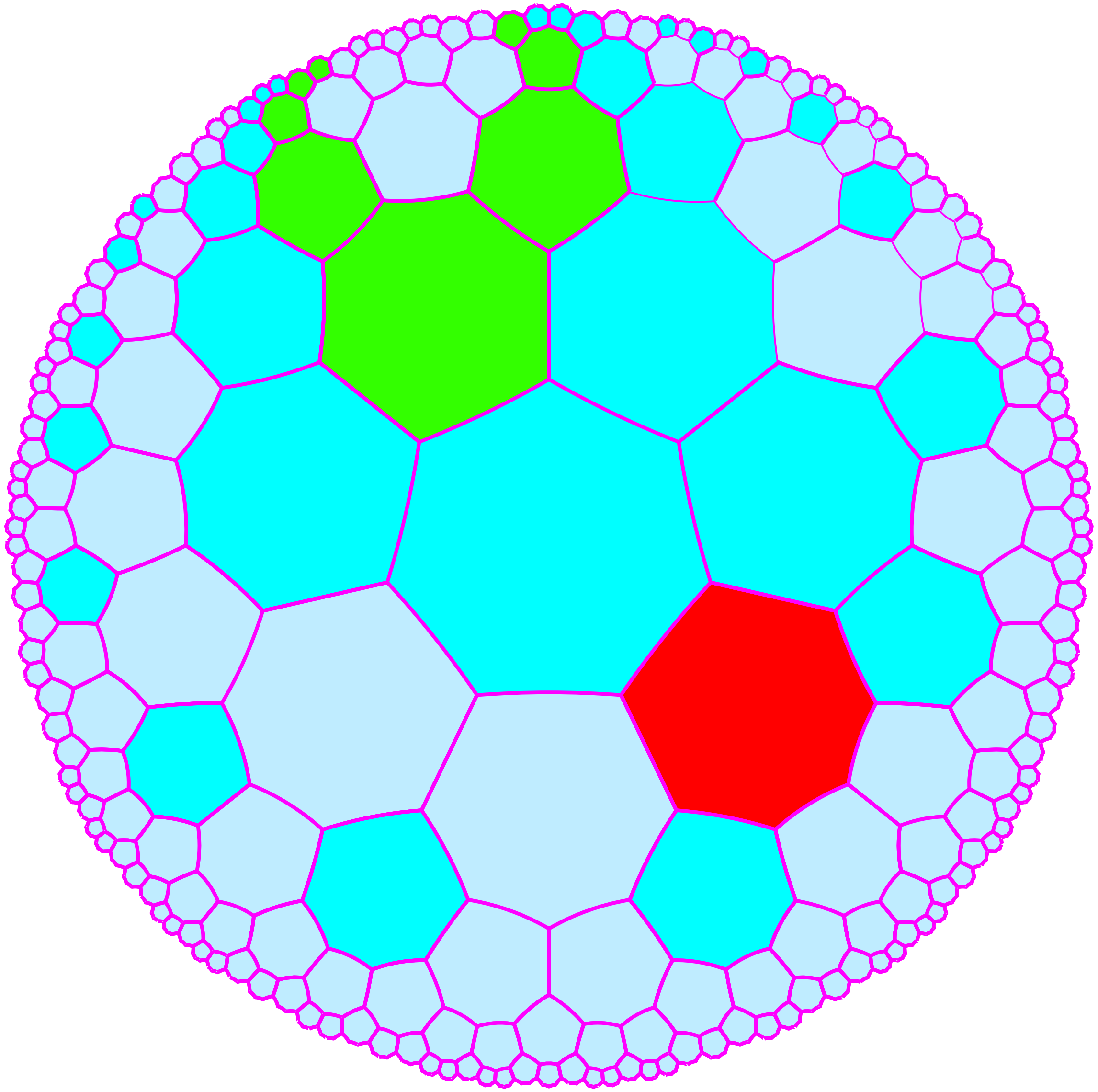
the locomotive on a track along an isocline

until
it reaches
the other
side



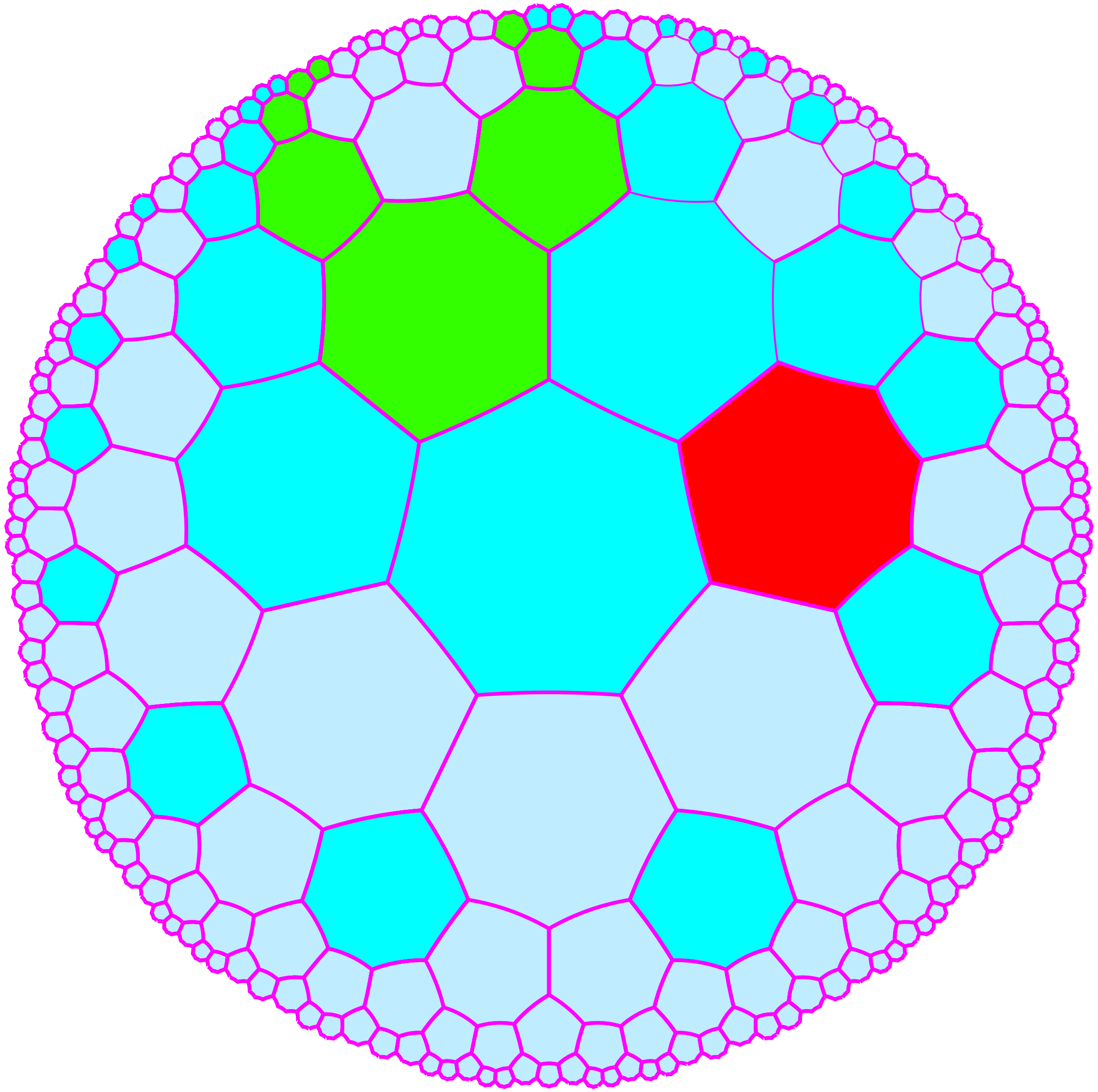
the locomotive on a track along an isocline

until
it reaches
the other
side



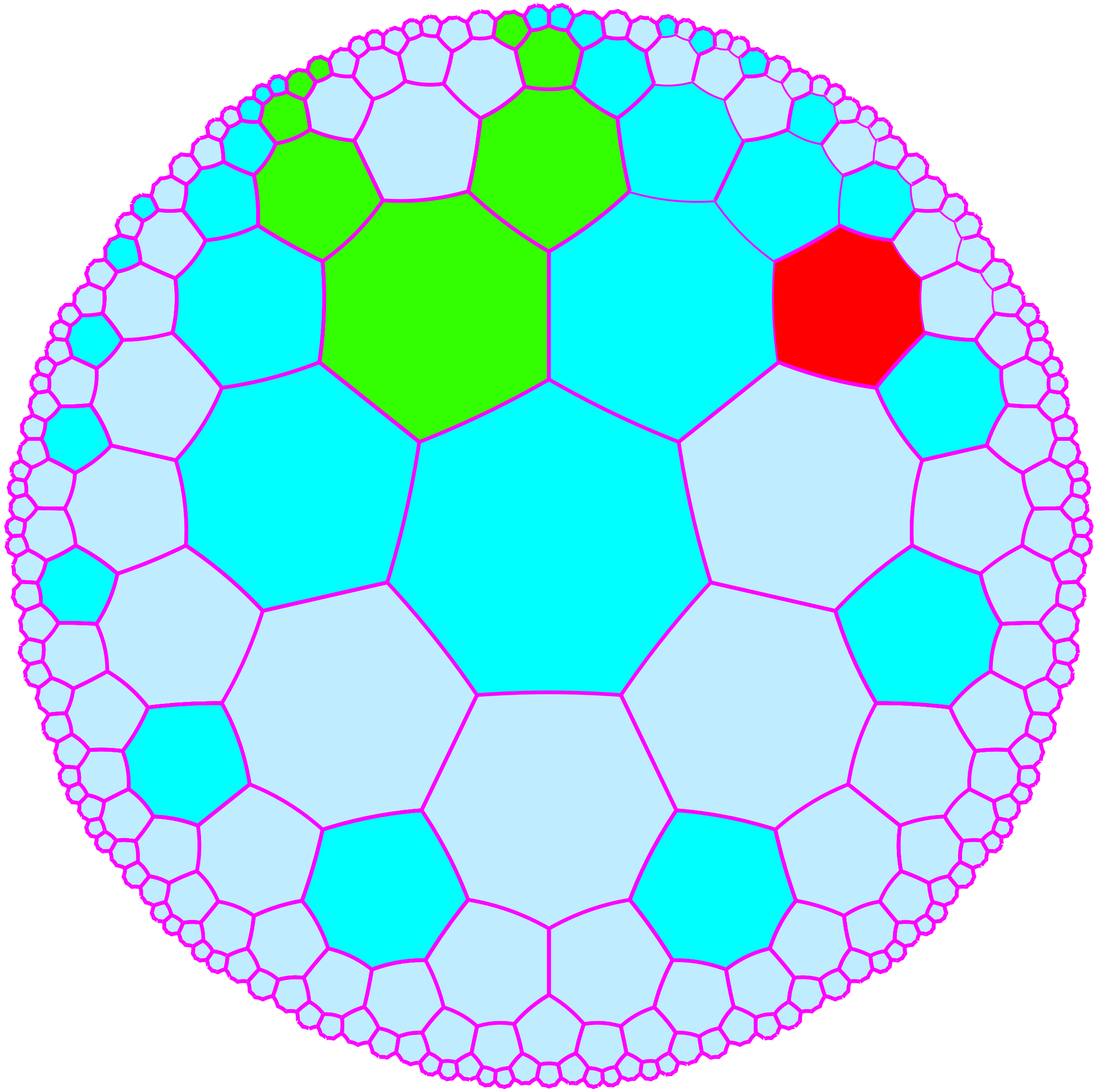
the locomotive on a track along an isocline

from where
it will be
leaving



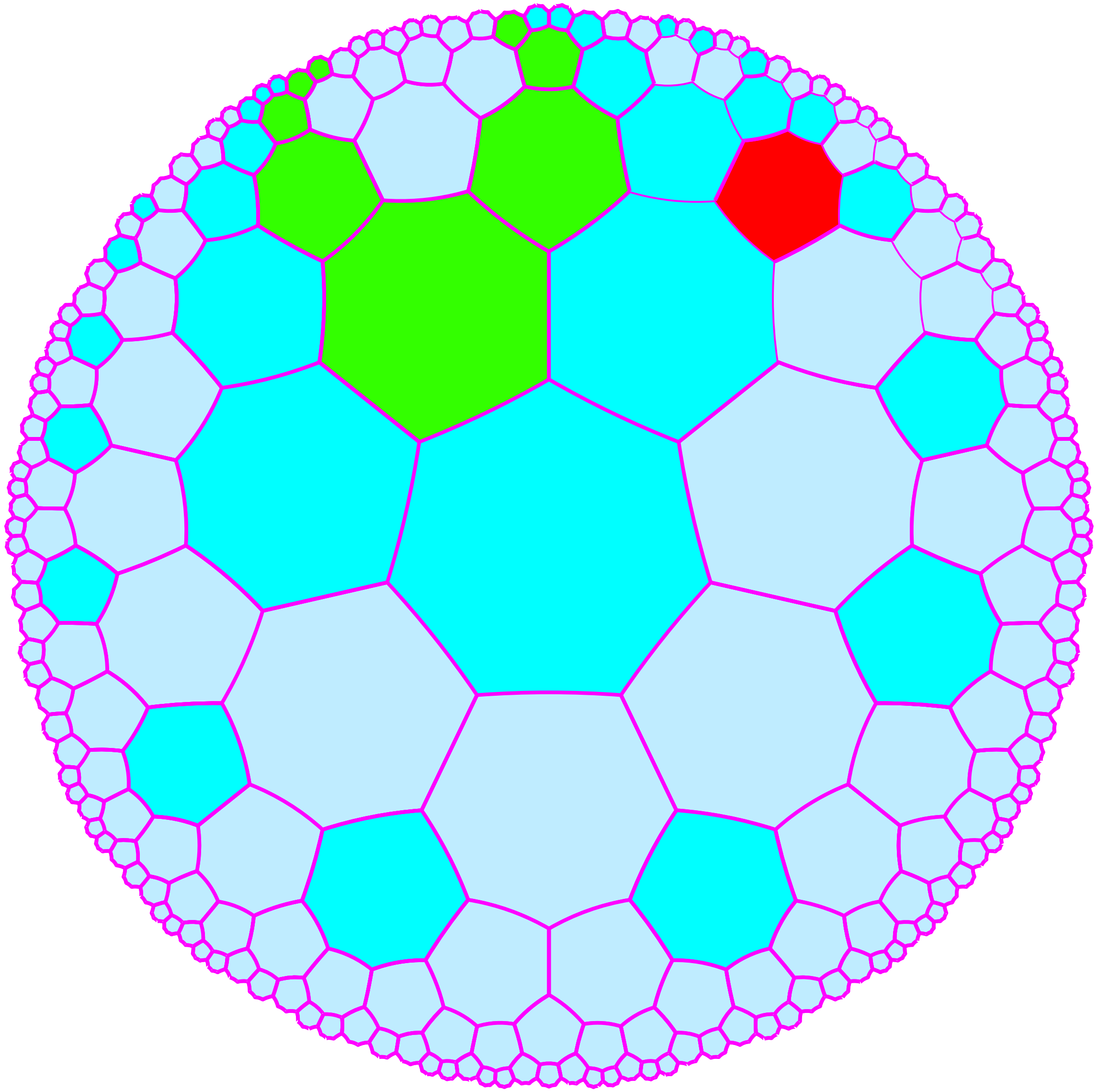
the locomotive on a track along an isocline

it will be
leaving



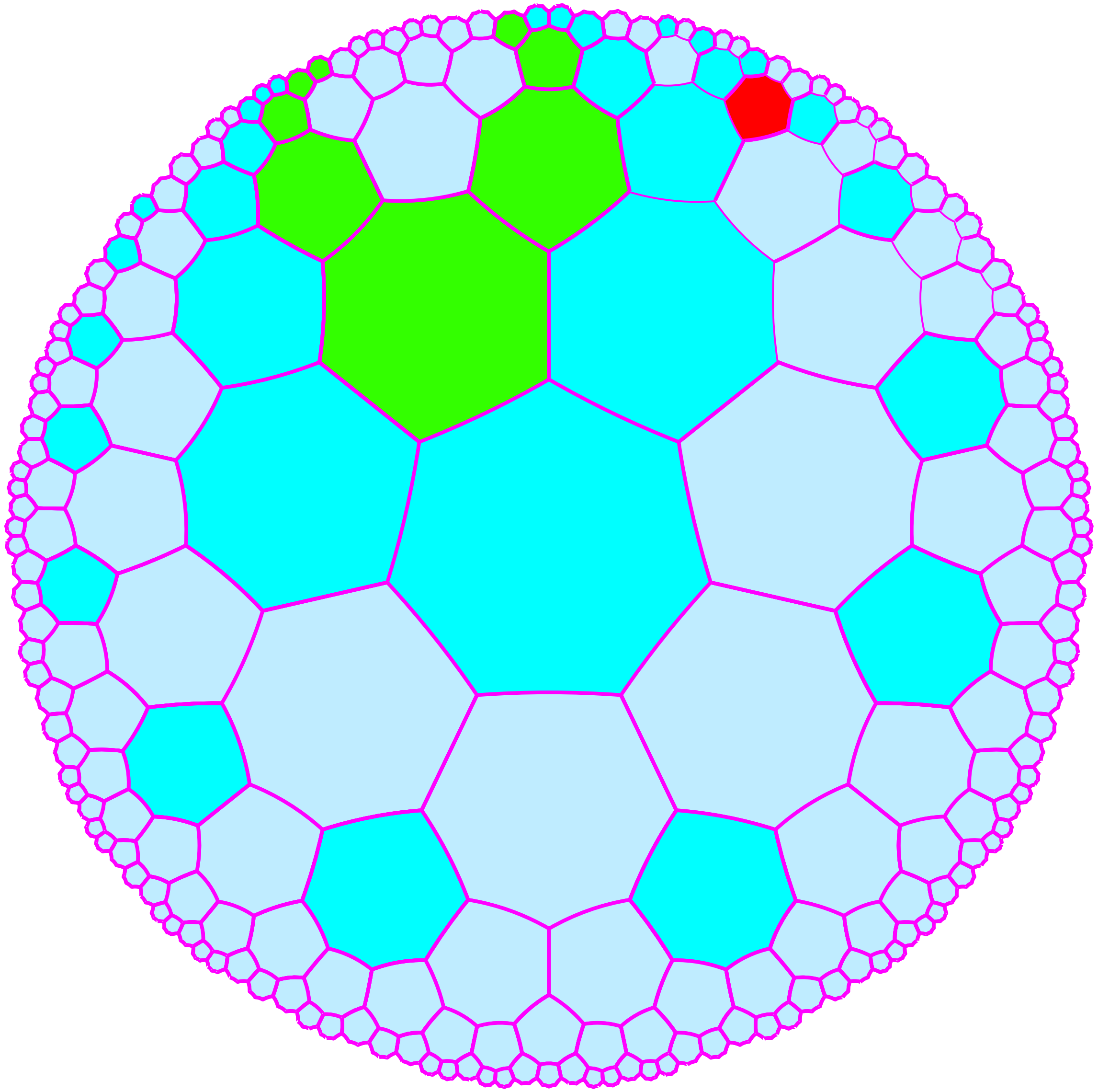
the locomotive on a track along an isocline

it will be
leaving



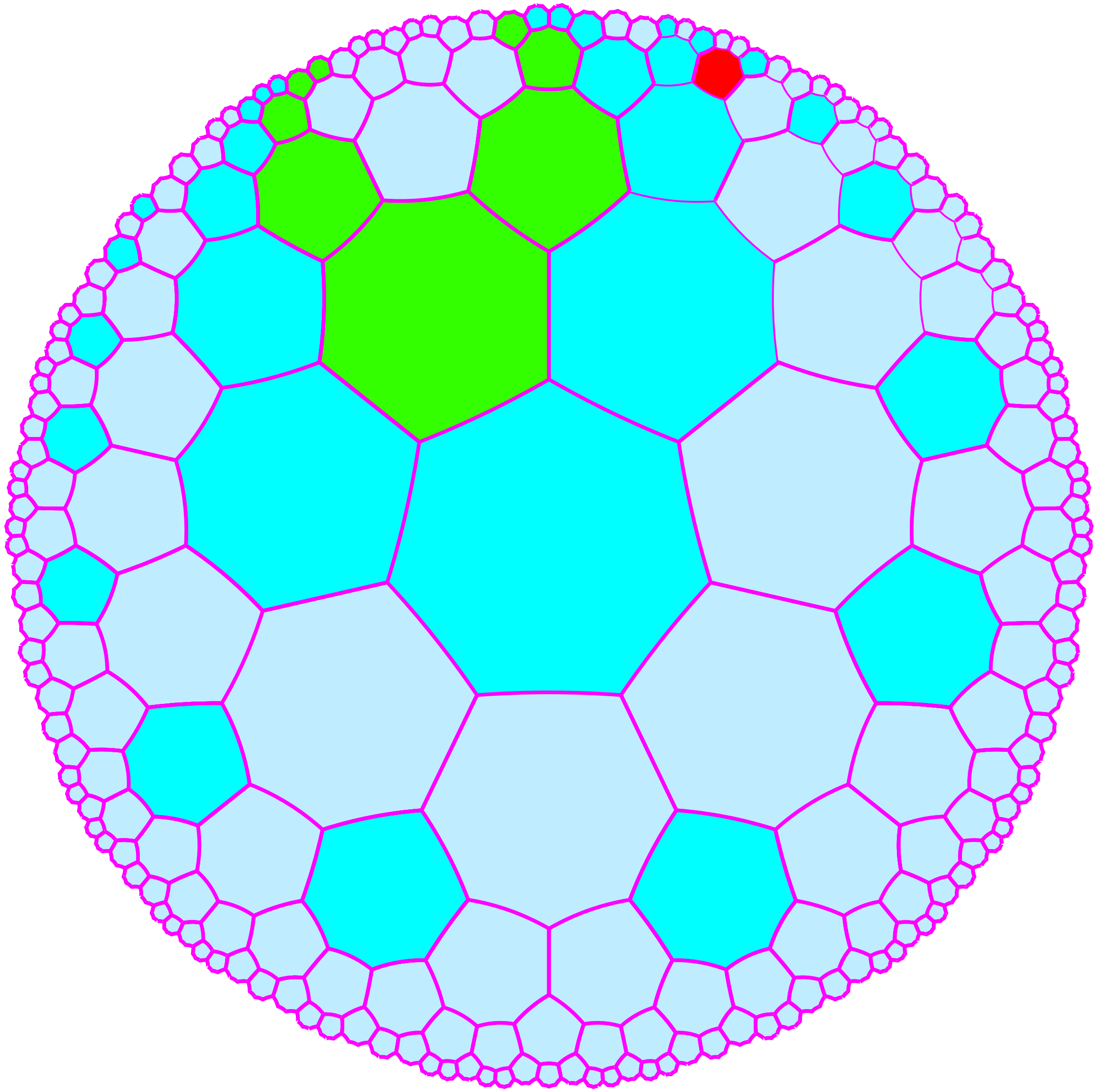
the locomotive on a track along an isocline

it will be leaving



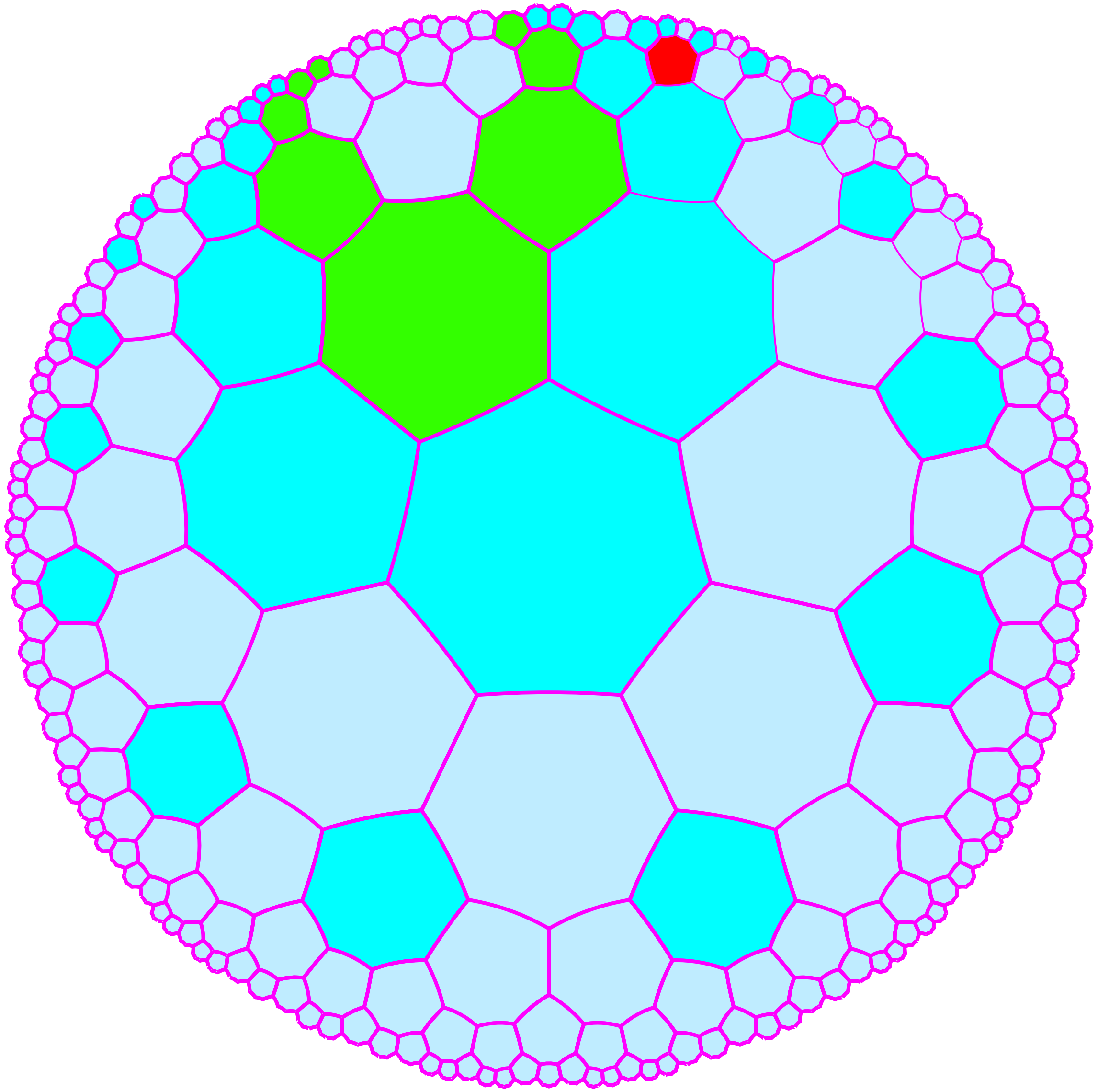
the locomotive on a track along an isocline

and now,
it is
leaving



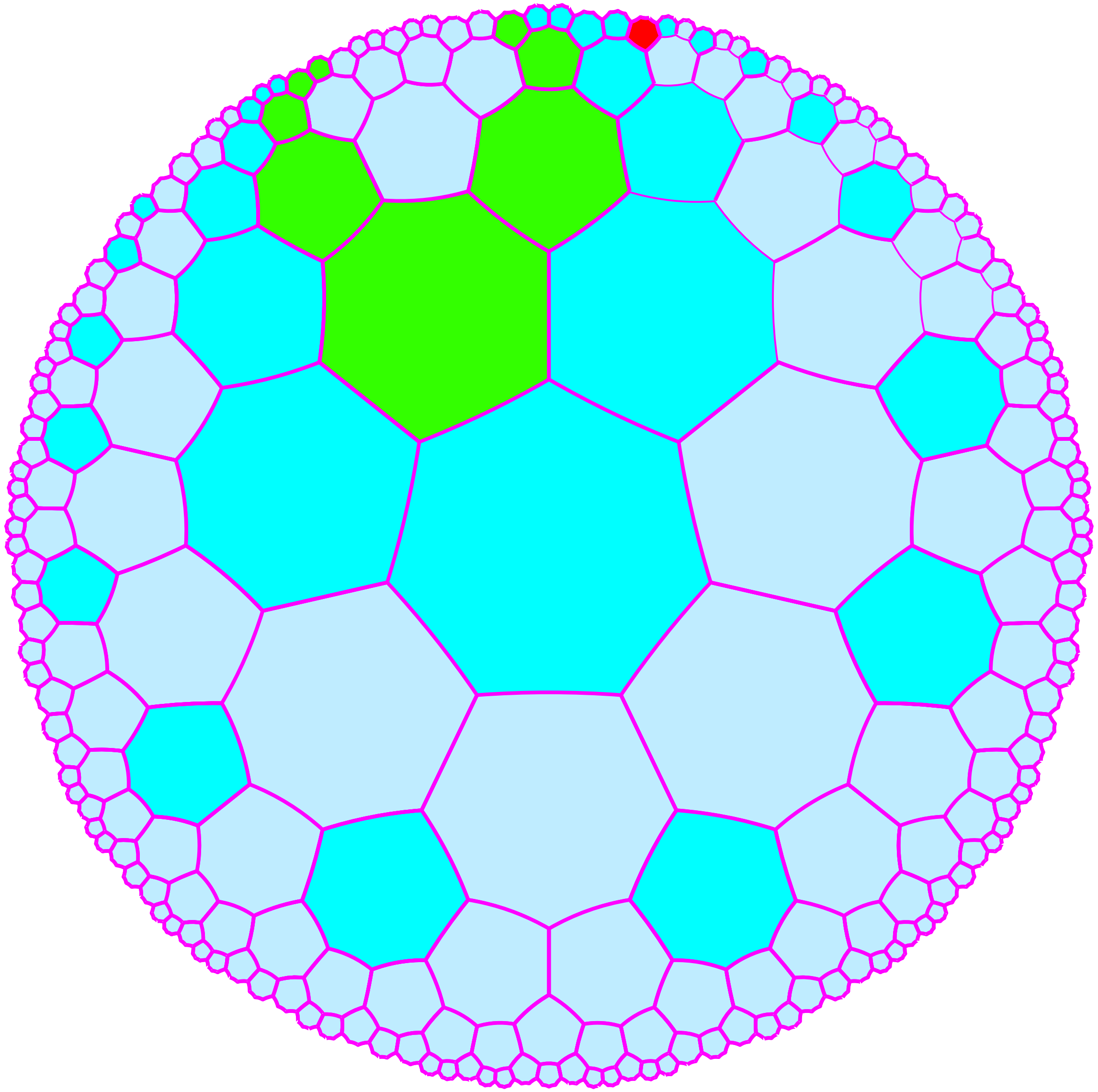
the locomotive on a track along an isocline

and now,
it is
leaving



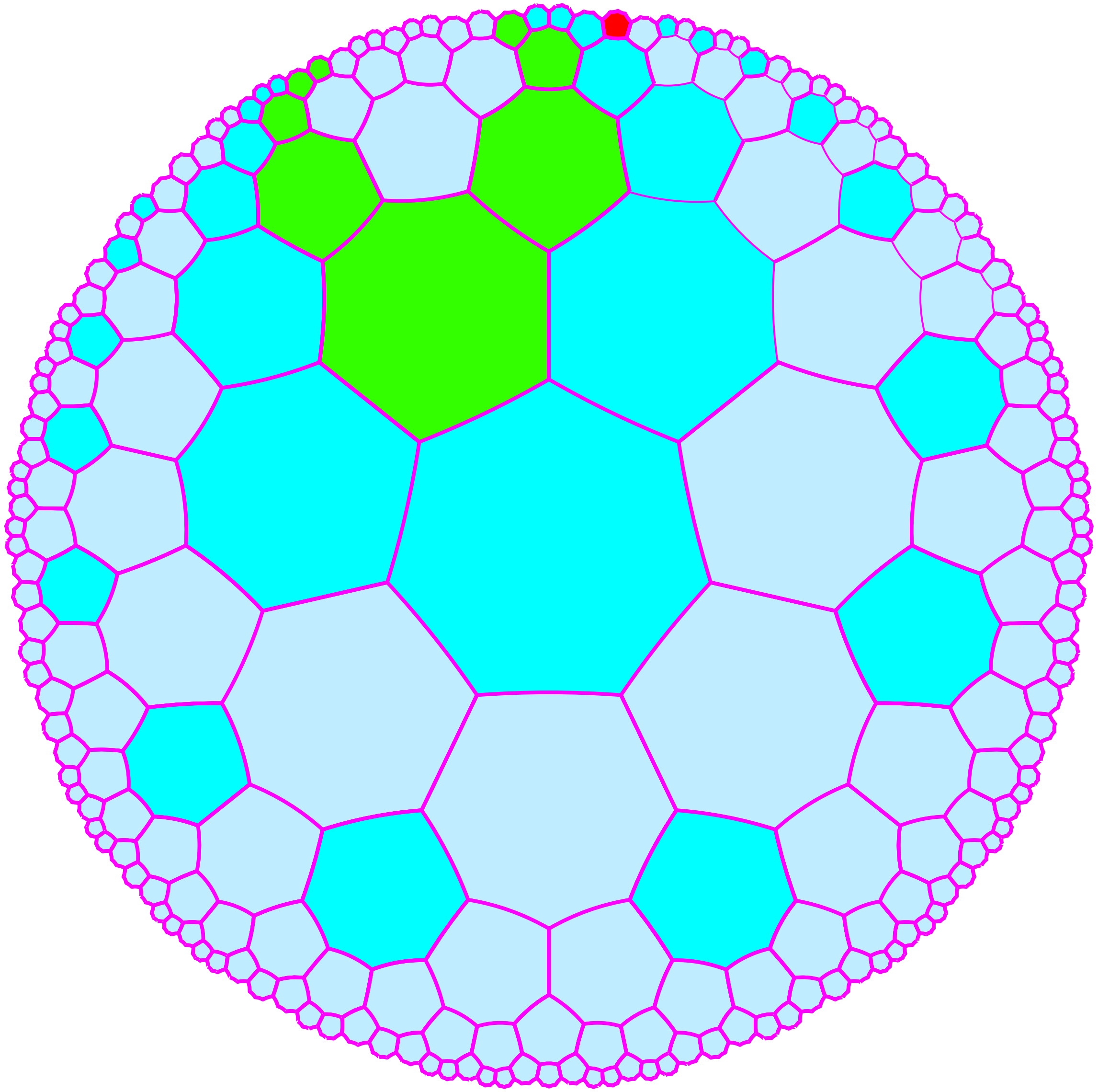
the locomotive on a track along an isocline

and now,
it is
leaving



the locomotive on a track along an isocline

and now,
it is
leaving



the locomotive on a track along an isocline

here,
idle again

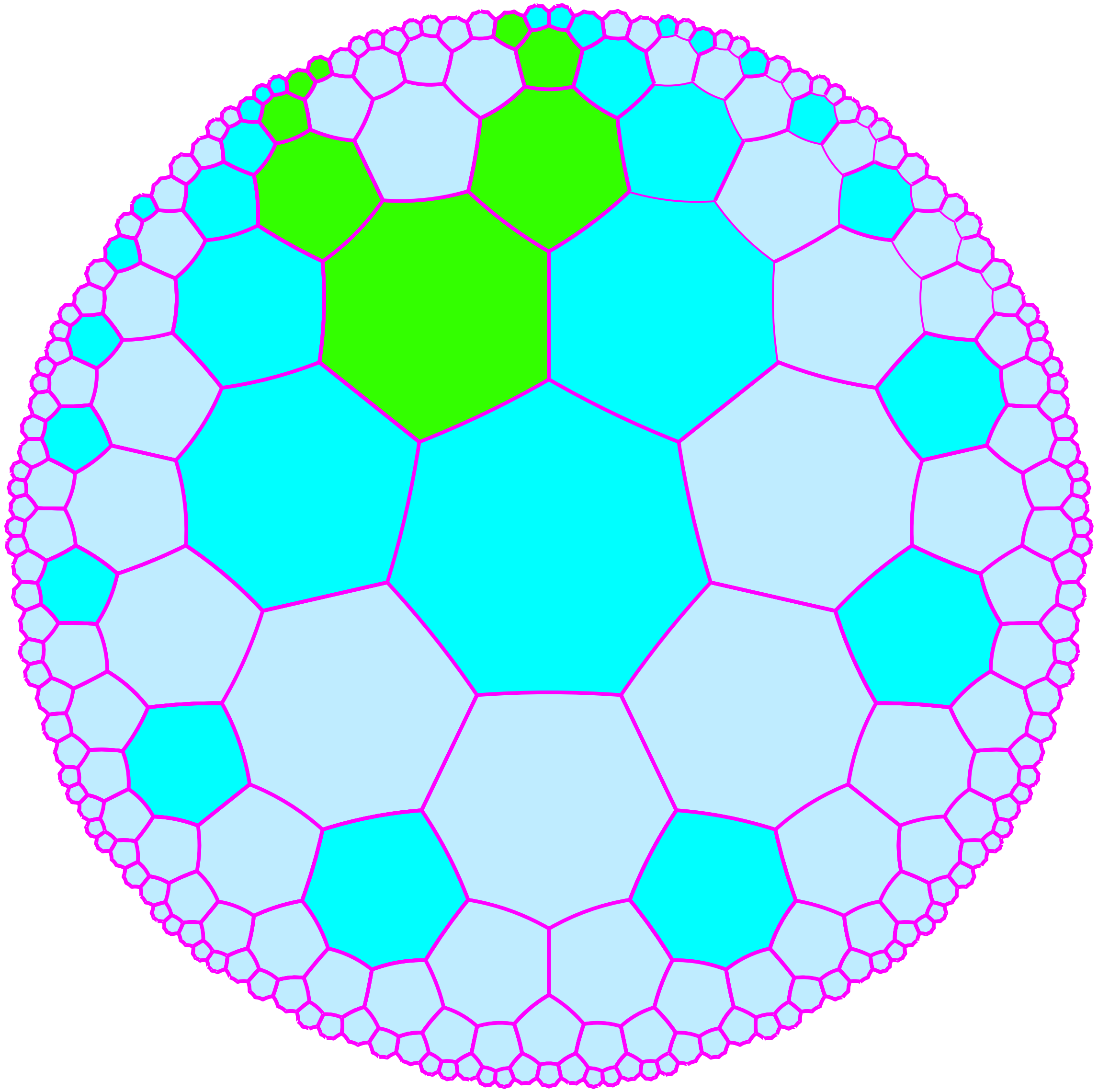


illustration of the crossing of a memory switch:

here, the passive passage through

the non-selected track, in sector 7

if the locomotive arrives through sector 7

it is sent to sector 4

and sector 7 becomes the **selected** track

left-hand side memory switch:

next, when the locomotive arrives from sector 7,

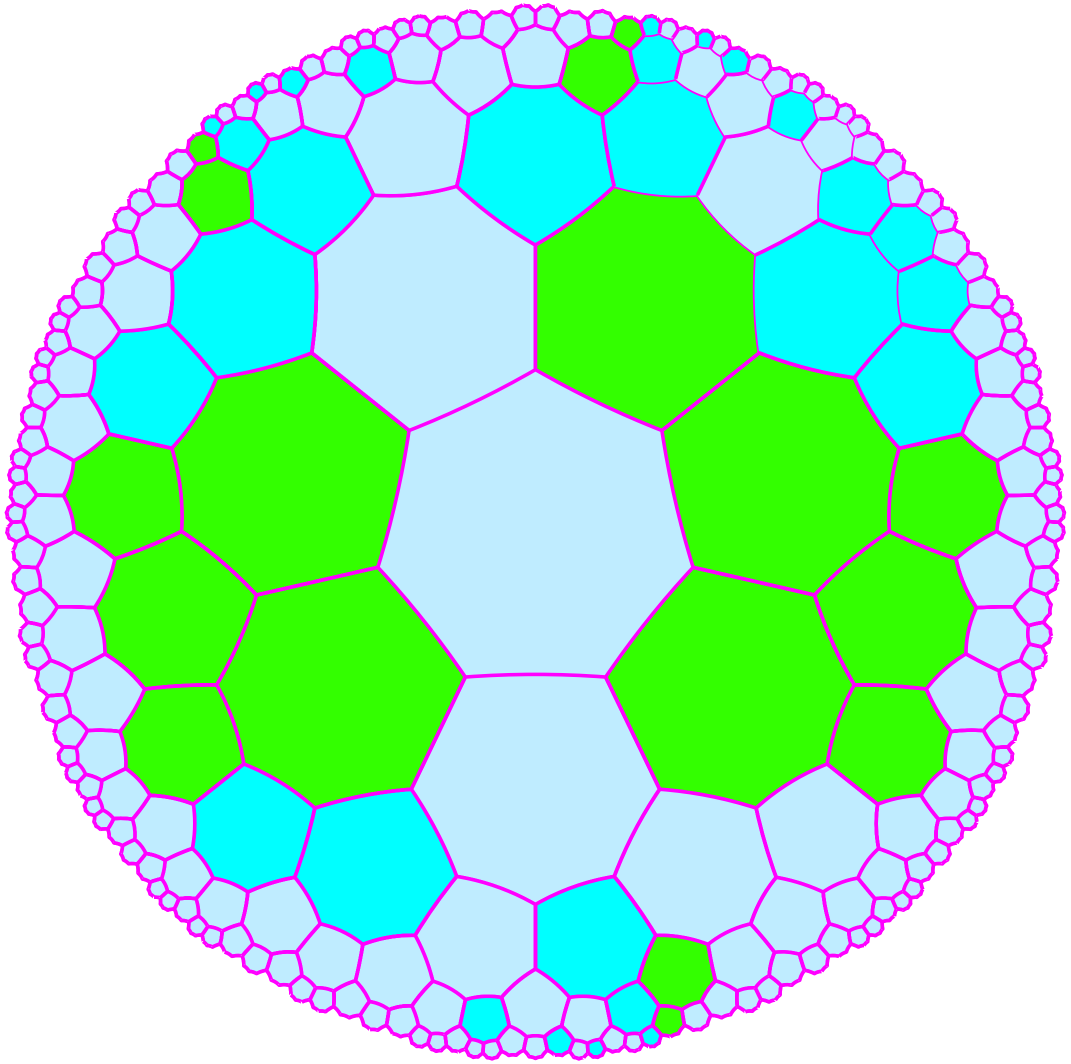
the **non-selected** track:

it is sent to sector 4

and sector 7 becomes the **selected** track

left-hand side memory switch:

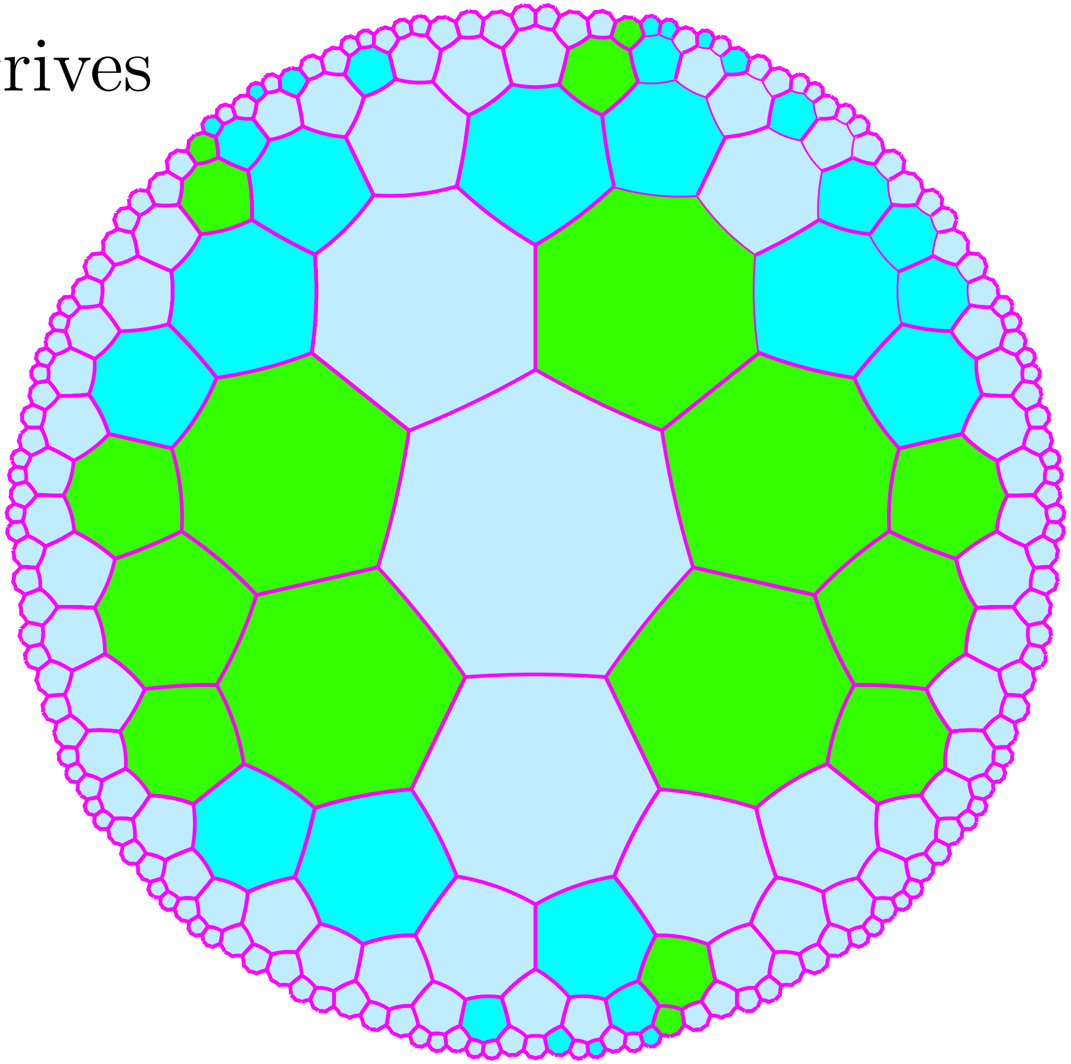
the locomotive
will arrive
through sector 7



here, idle
configuration

left-hand side memory switch:

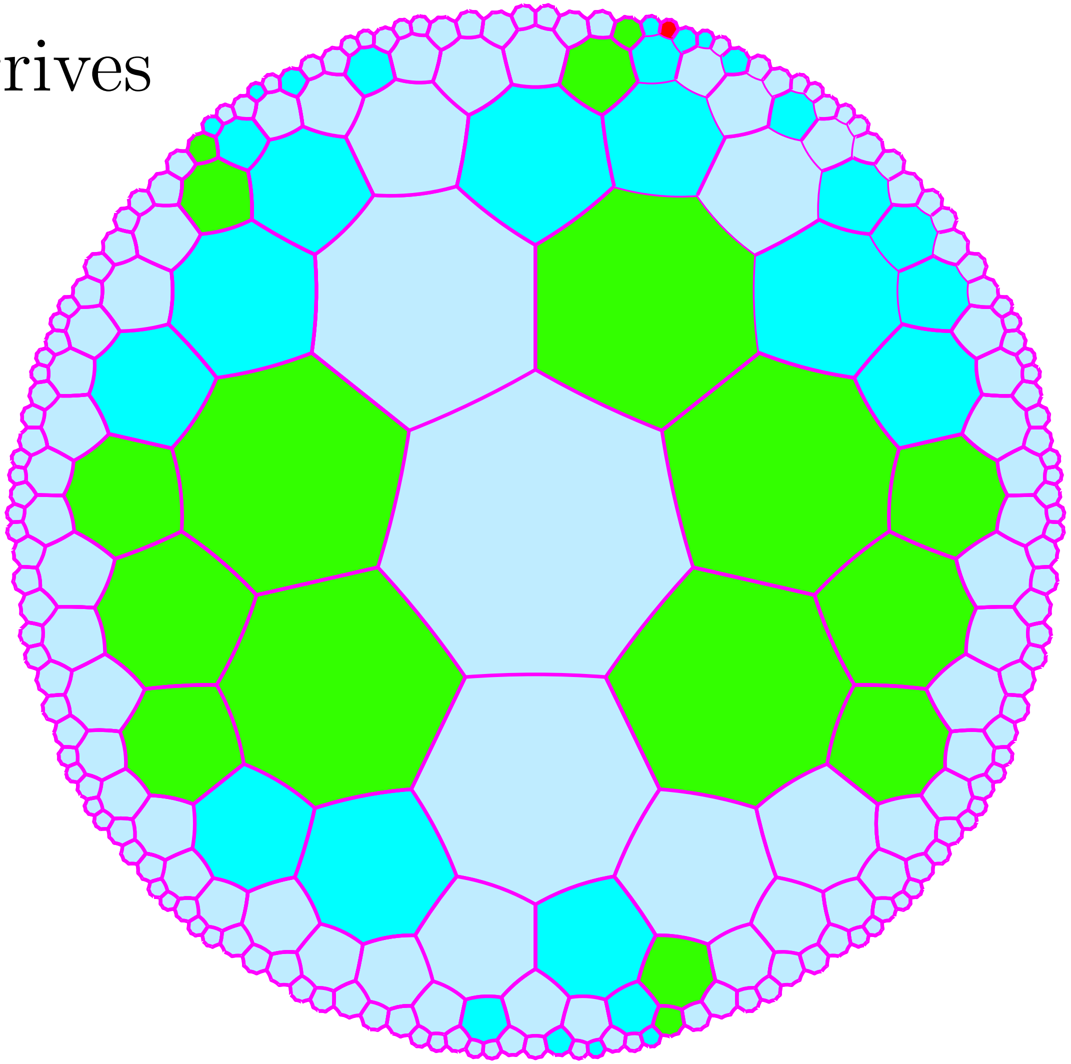
the locomotive arrives through sector 7



front in `32(1)`

left-hand side memory switch:

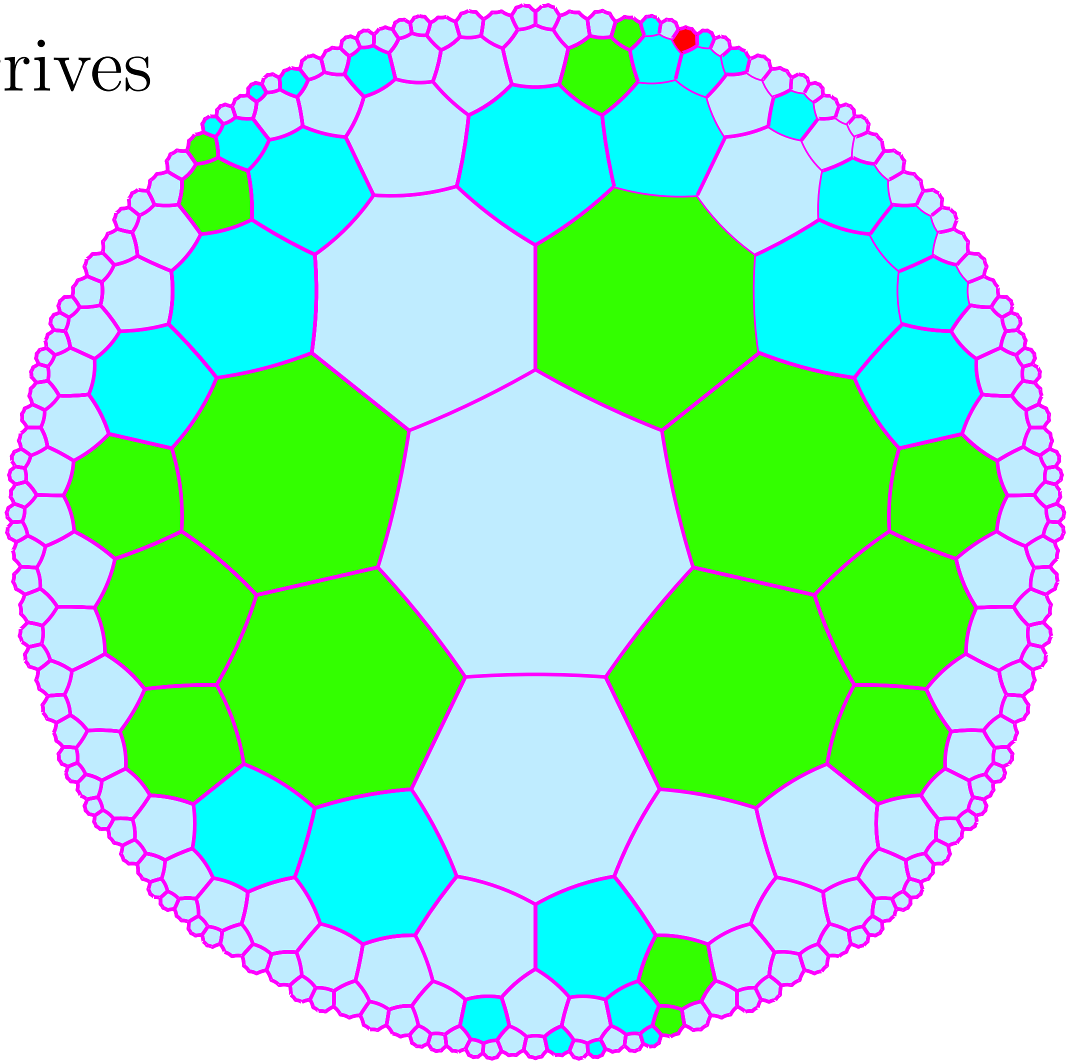
the locomotive arrives through sector 7



front in `31(1)`

left-hand side memory switch:

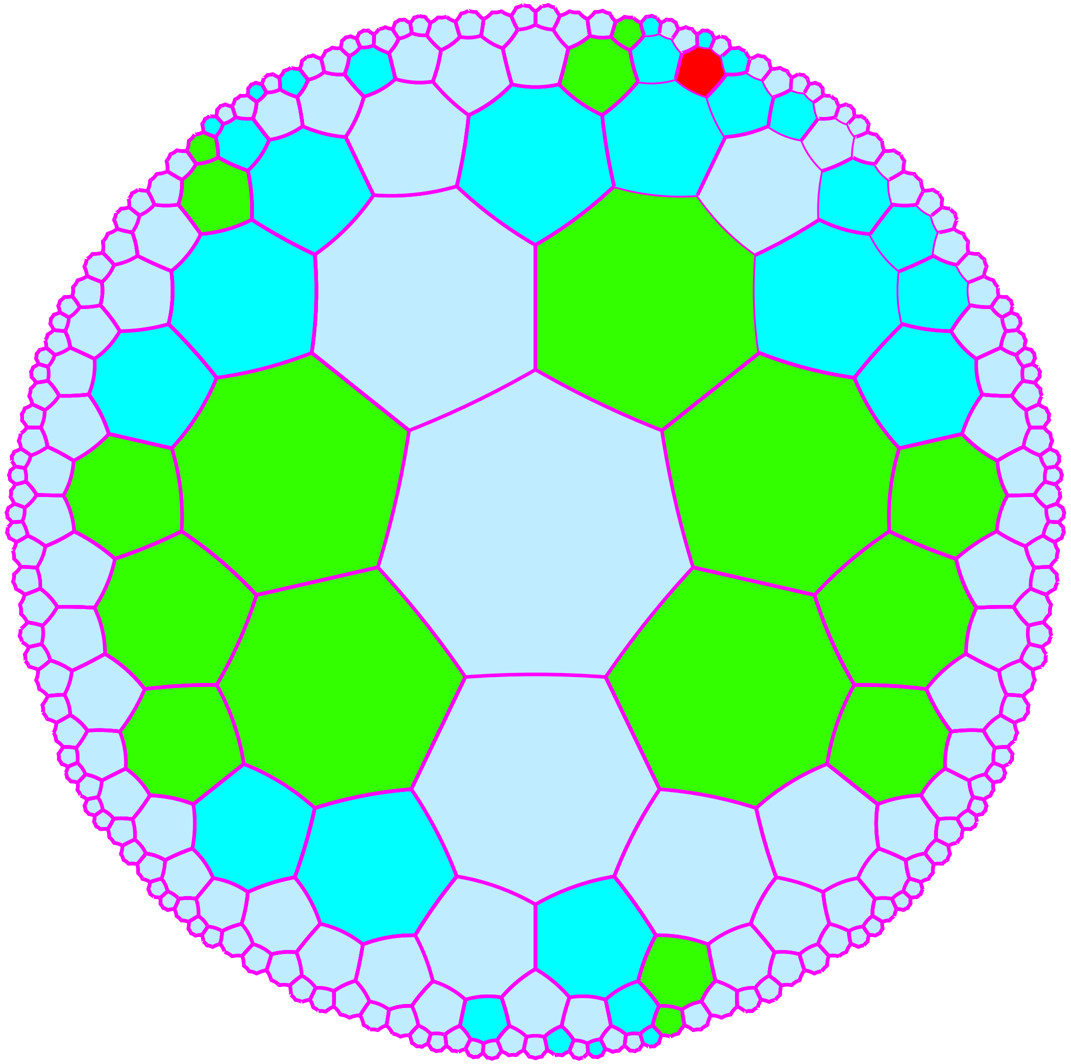
the locomotive arrives through sector 7



front in `11(1)`

left-hand side memory switch:

and it goes
to sector 4



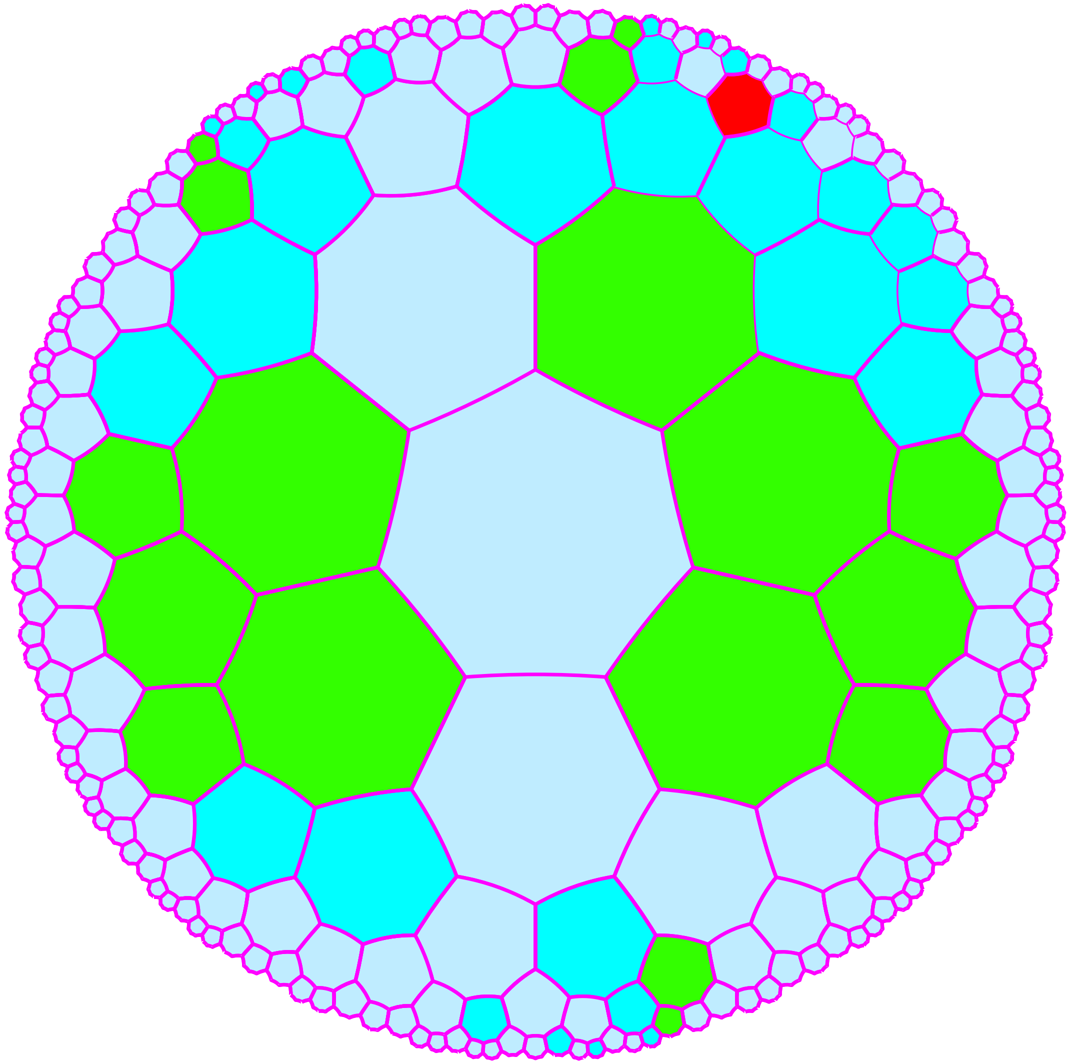
front in `10(1)`

left-hand side memory switch:

and it goes
to sector 4

but here, it
meets with
an obstacle

front in `3(1)`

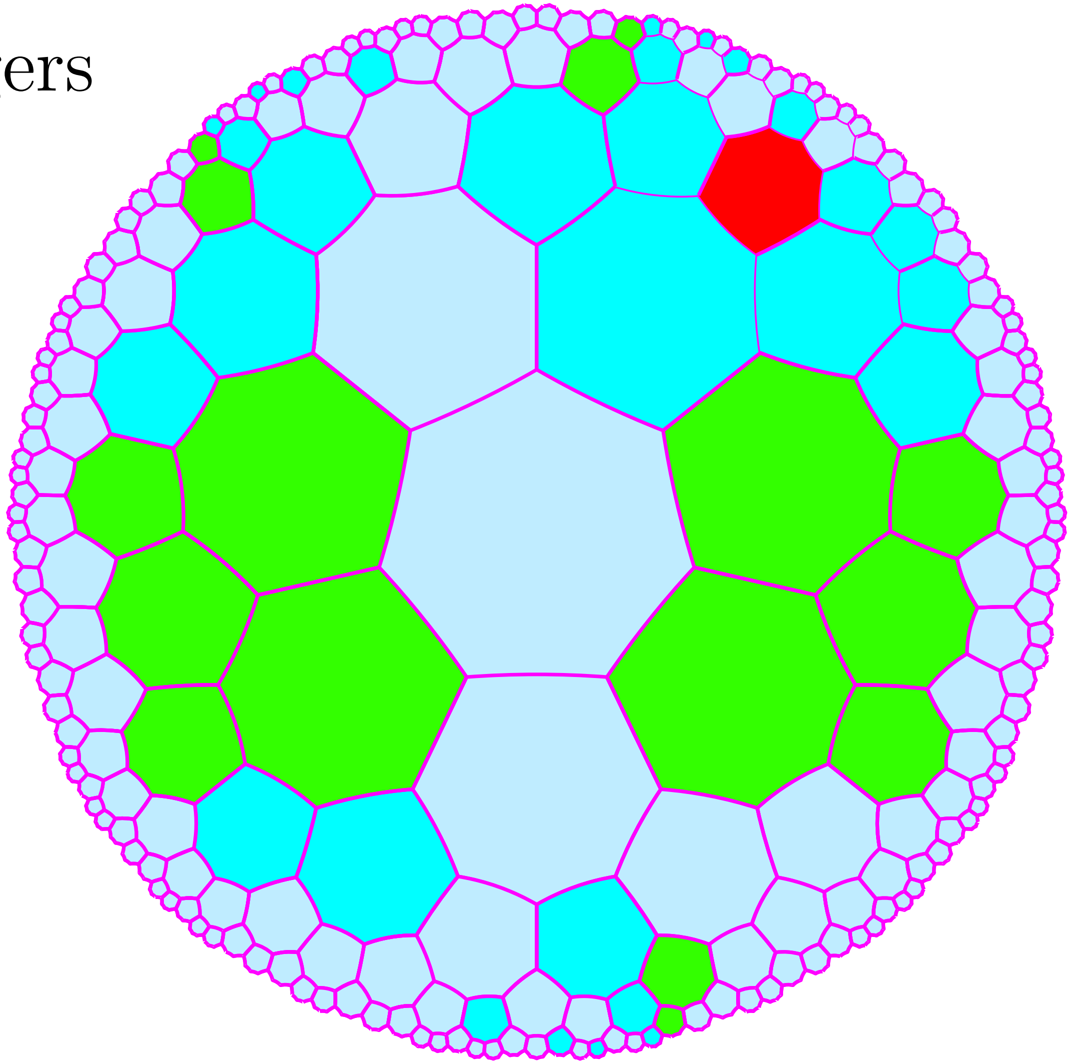


left-hand side memory switch:

the obstacle triggers
the change of the
selected track

here, first step:
the obstacle
is removed

front in `1(1)`

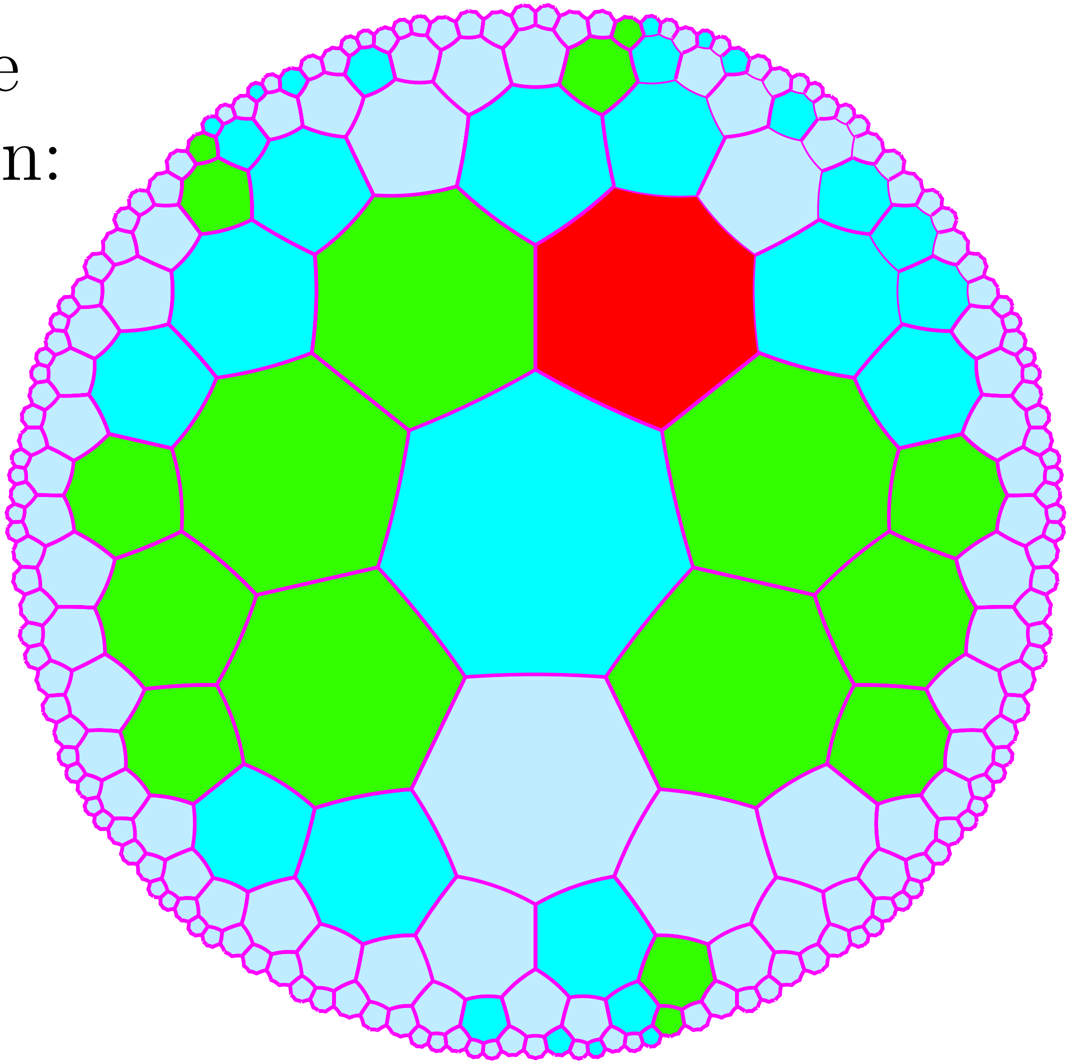


left-hand side memory switch:

second step of the
change of selection:

the obstacle is
put on track 1

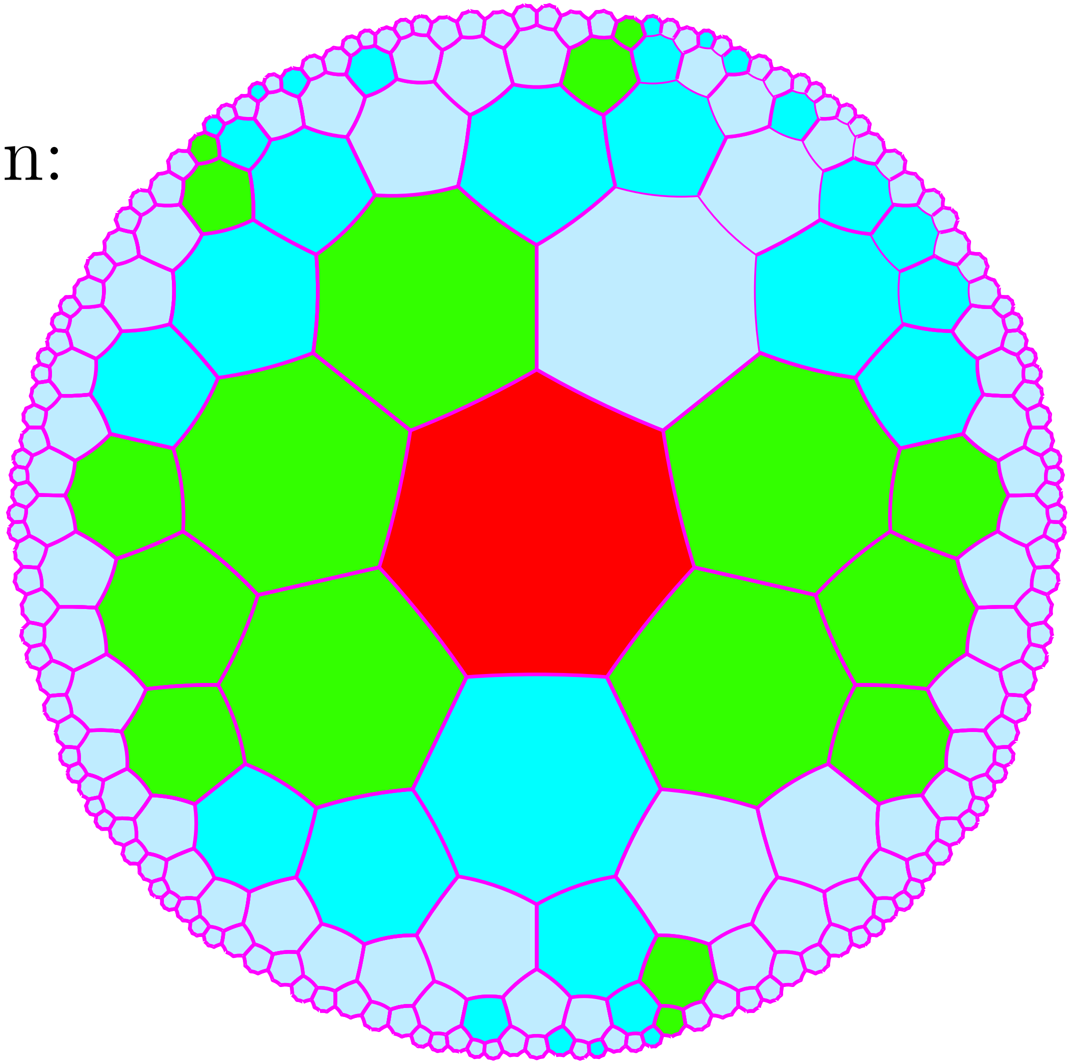
front in 0



left-hand side memory switch:

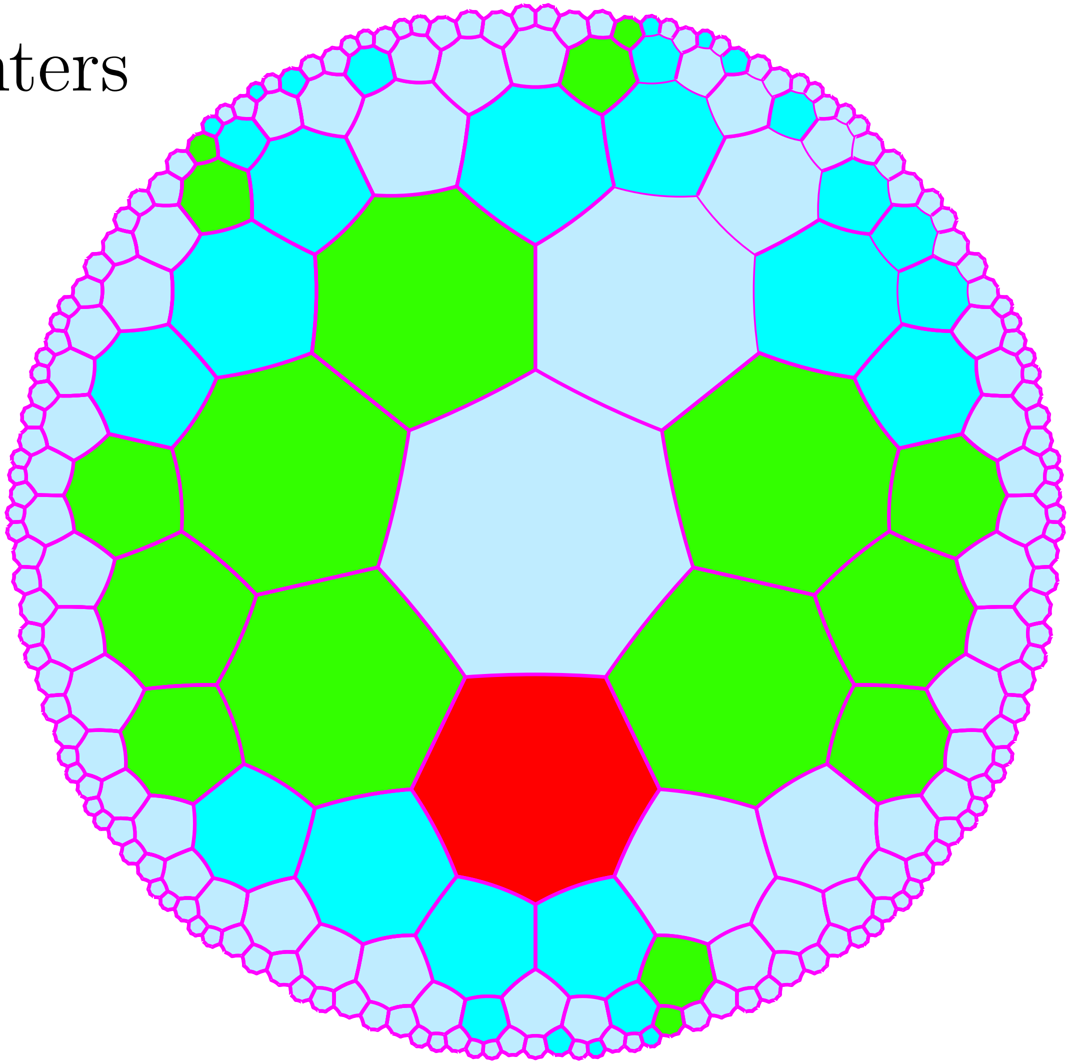
last step of the change of selection:

the track 7 is now free, and the locomotive enters sector 4 front in `1(4)`



left-hand side memory switch:

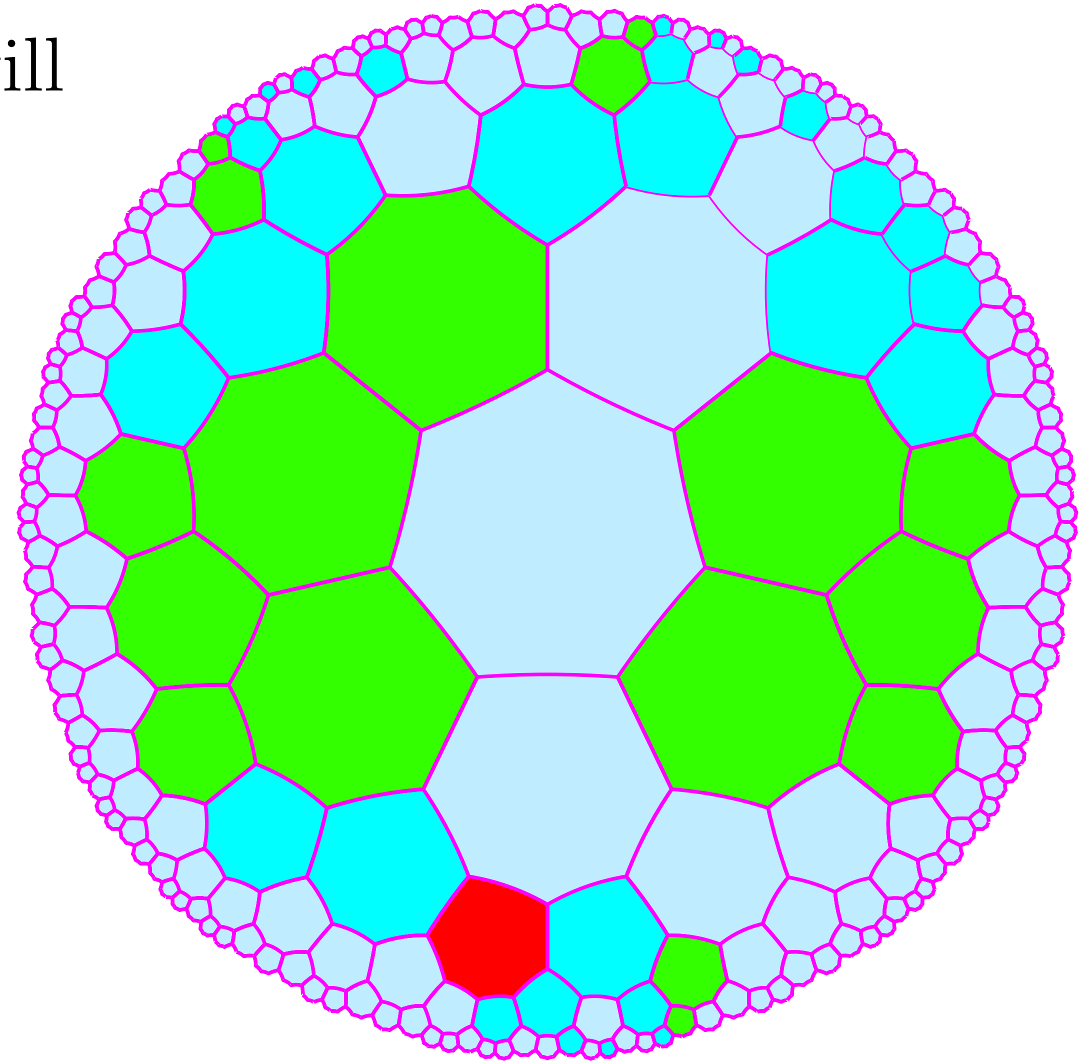
the locomotive enters sector 4



front in `3(4)`

left-hand side memory switch:

the locomotive will leave through sector 4

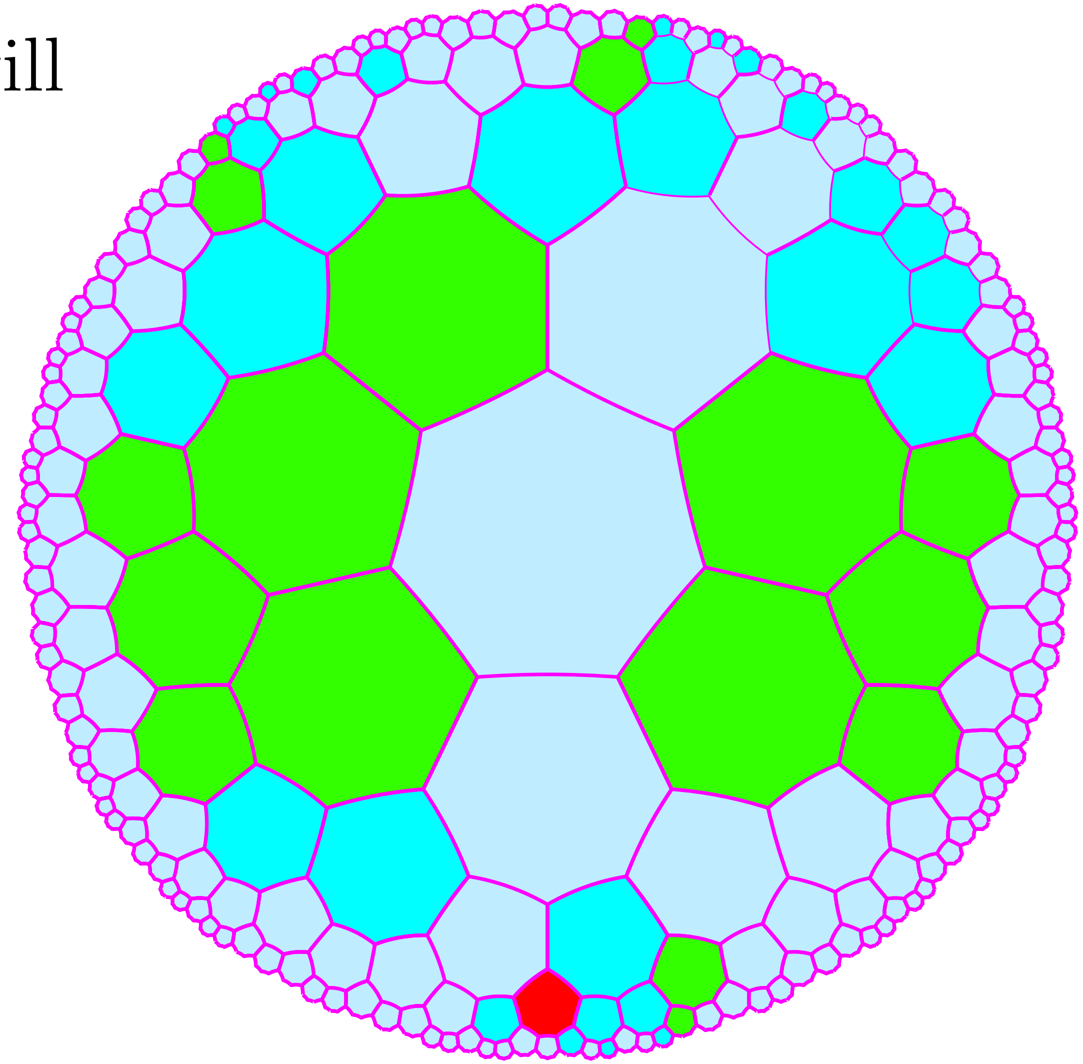


front in `10(4)`

left-hand side memory switch:

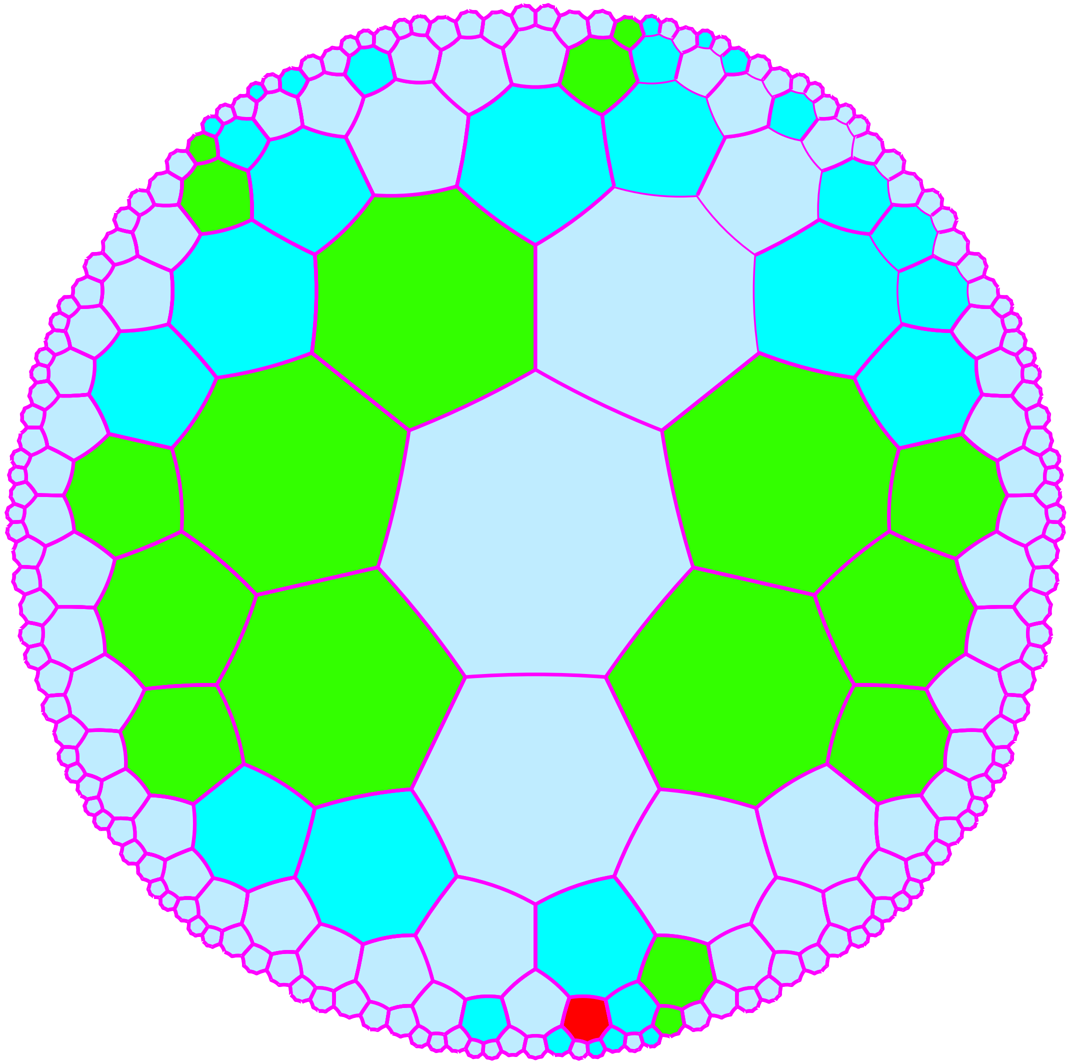
the locomotive will leave through sector 4

front in `11(4)`



left-hand side memory switch:

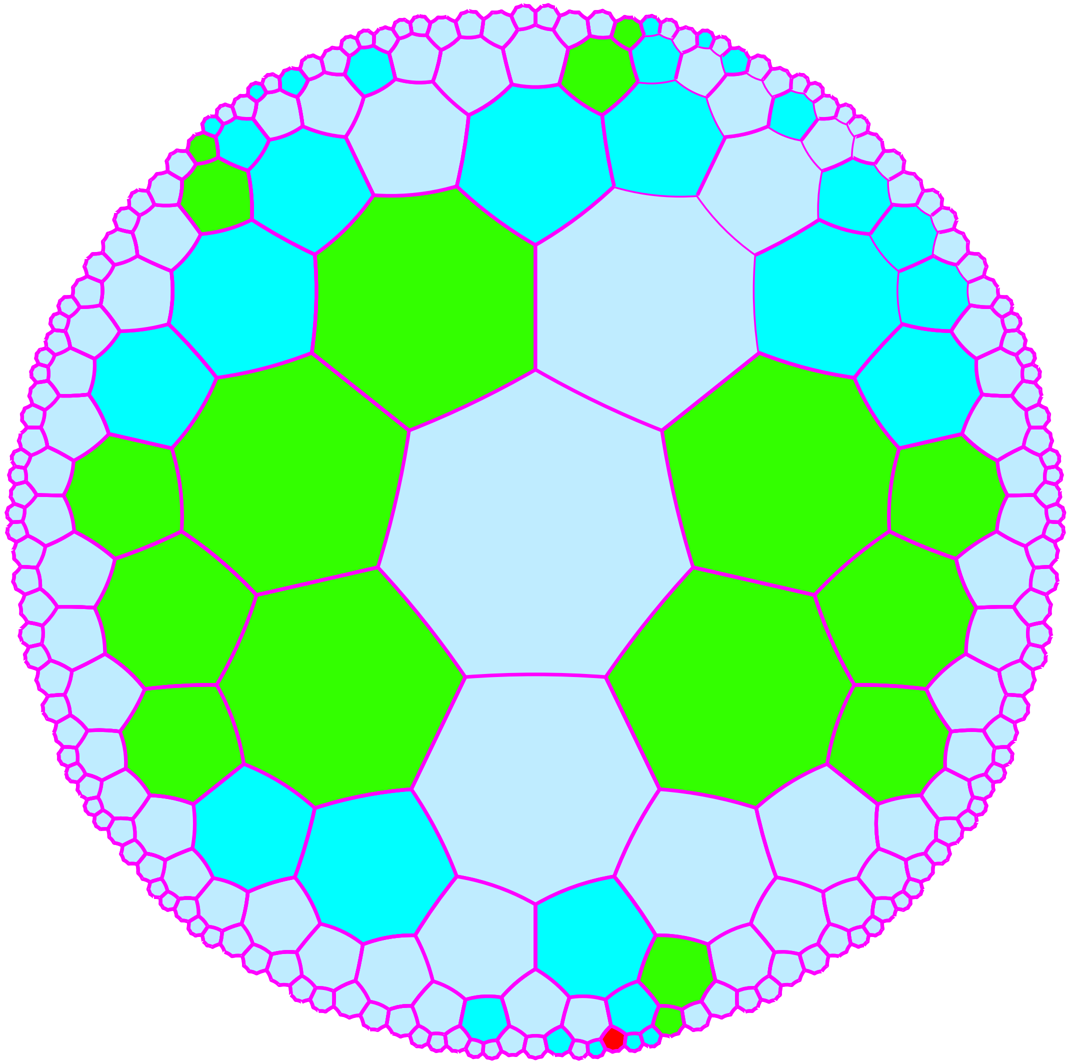
the locomotive is leaving sector 4



front in `31(4)`

left-hand side memory switch:

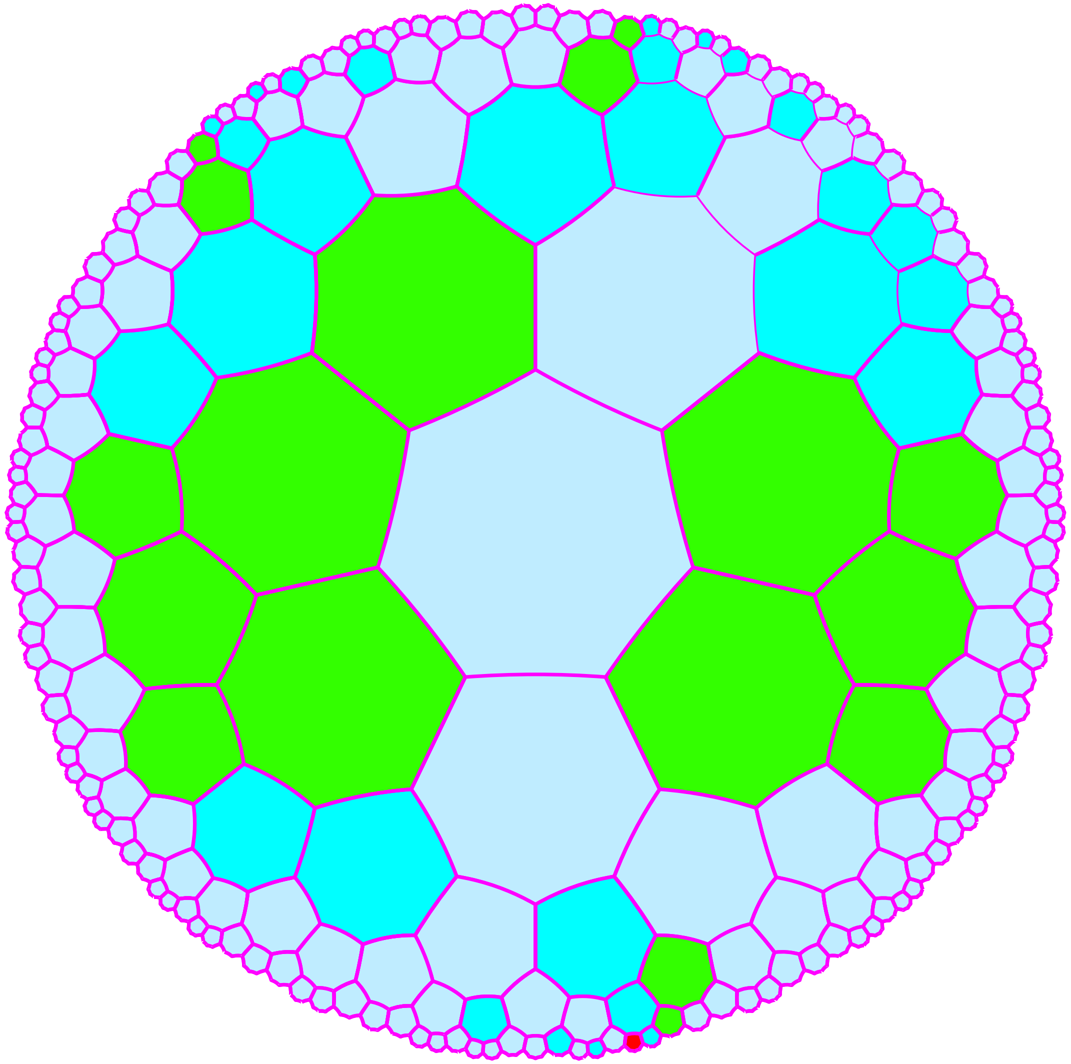
the locomotive is leaving sector 4



front in `32(4)`

left-hand side memory switch:

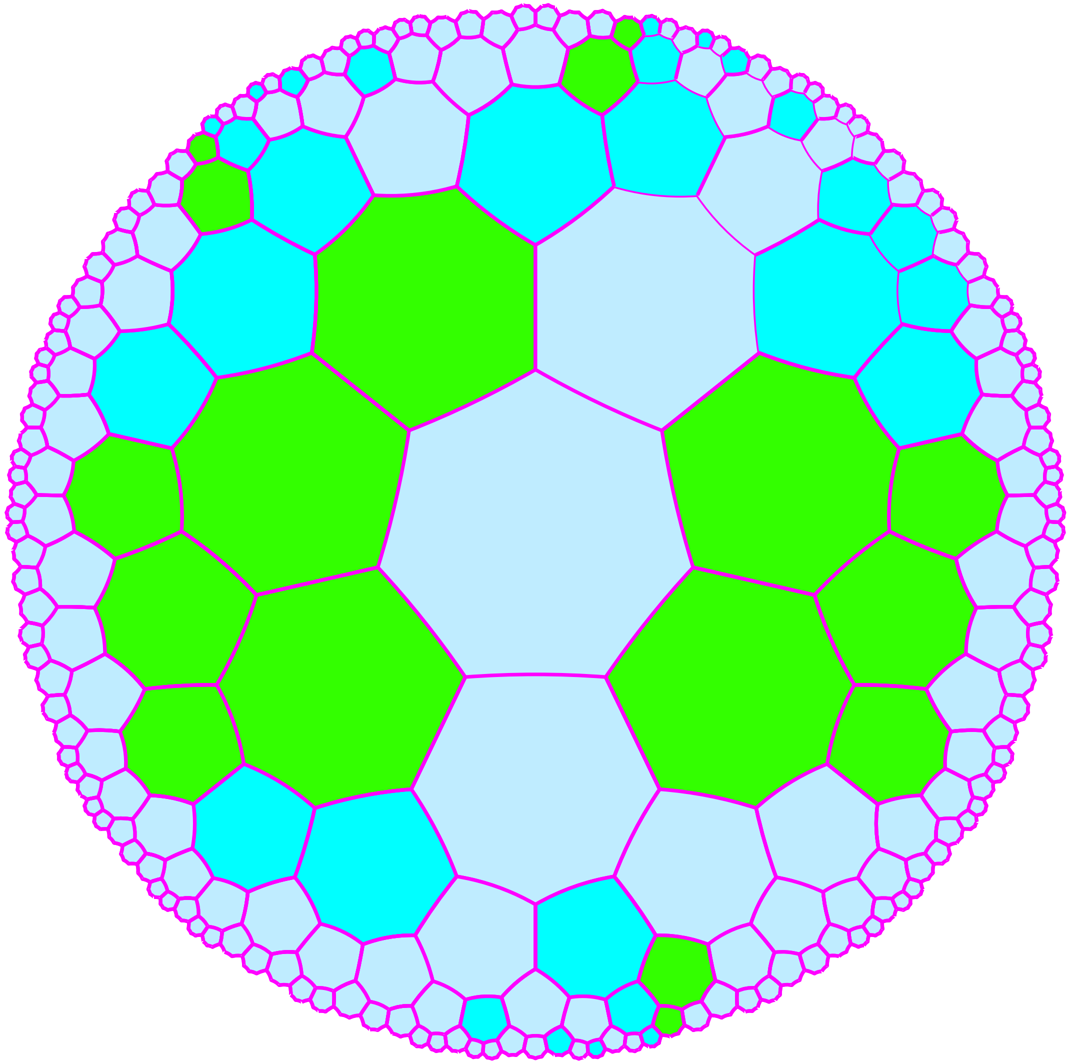
the locomotive is
leaving sector 4



front in `88(4)`

left-hand side memory switch:

the locomotive
left sector 4



idle again,
but as a
right-hand side
memory switch

4.3.c in the hyperbolic $3D$ space

possible to implement the same model

important differences:

take advantage of the $3D$ space
to replace crossings by bridges

also: more neighbours for each cell:
12 of them instead of 7 in the heptagrid

the result:

5 states, dodecagrid, MM, 2004

again, weakly universal rotation invariant CA

another property:

let $\{A, B, C, D, E\}$ be the states
consider a rule: $\eta_0\eta_1..\eta_{12} \rightarrow \eta_0'$

define its **reduced pattern** as the word
$A^{a_1}B^{a_2}C^{a_3}D^{a_4}E^{a_5}; \eta_0\eta_0', \sum a_i = 12$

then: the mapping from the rules to their reduced pattern is injective

4.4 beyond the Turing barrier

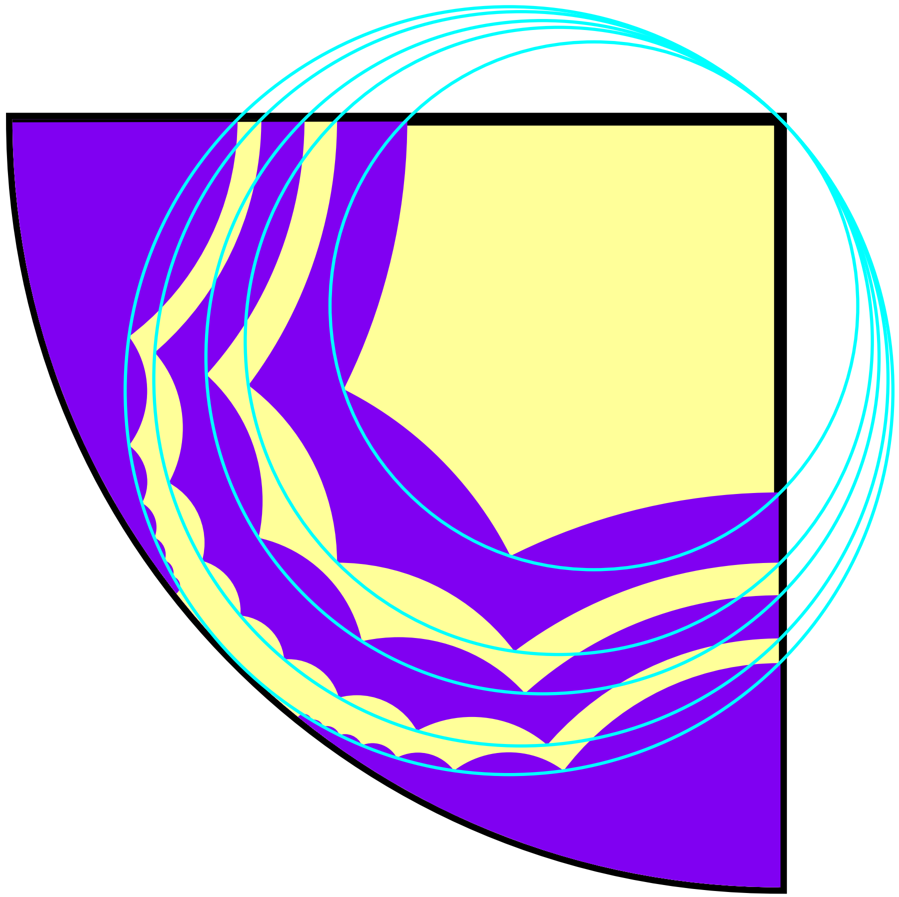
4.4.a infinigons and infinigrids

4.4.b register CA's on an infinigrid

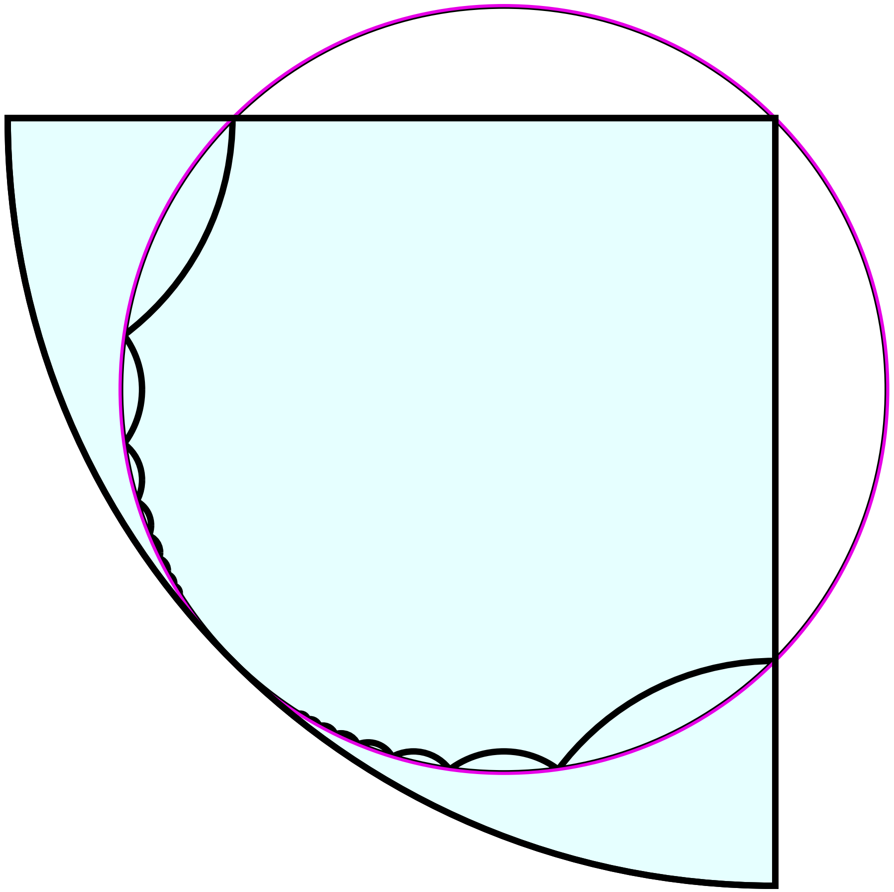
4.4.a infinigons and infinigrids

plane again:
viewing the regular rectangular polygons at once,

their limit
$\Rightarrow$ infinigon:



the visual limit:



the basic construction

define a sequence of segments, $x_n x_{n+1}$, $n \in \mathbb{Z}$, such that:

- $\forall\, n : x_n x_{n-1}, x_n x_{n+1} = x_{n+1} x_n, x_{n+1} x_{n+2}$
- $\forall\, n : ||x_n x_{n+1}||_h = ||x_{n+1} x_{n+2}||_h$

claim:

the x_n's belong to an e-circle Γ if $x_0 = 0$ and $||x_n x_{n+1}||_e = x$, $x \in]0, 1[$

then, diameter of $\Gamma = \dfrac{x}{\cos(\frac{\alpha}{2})}$

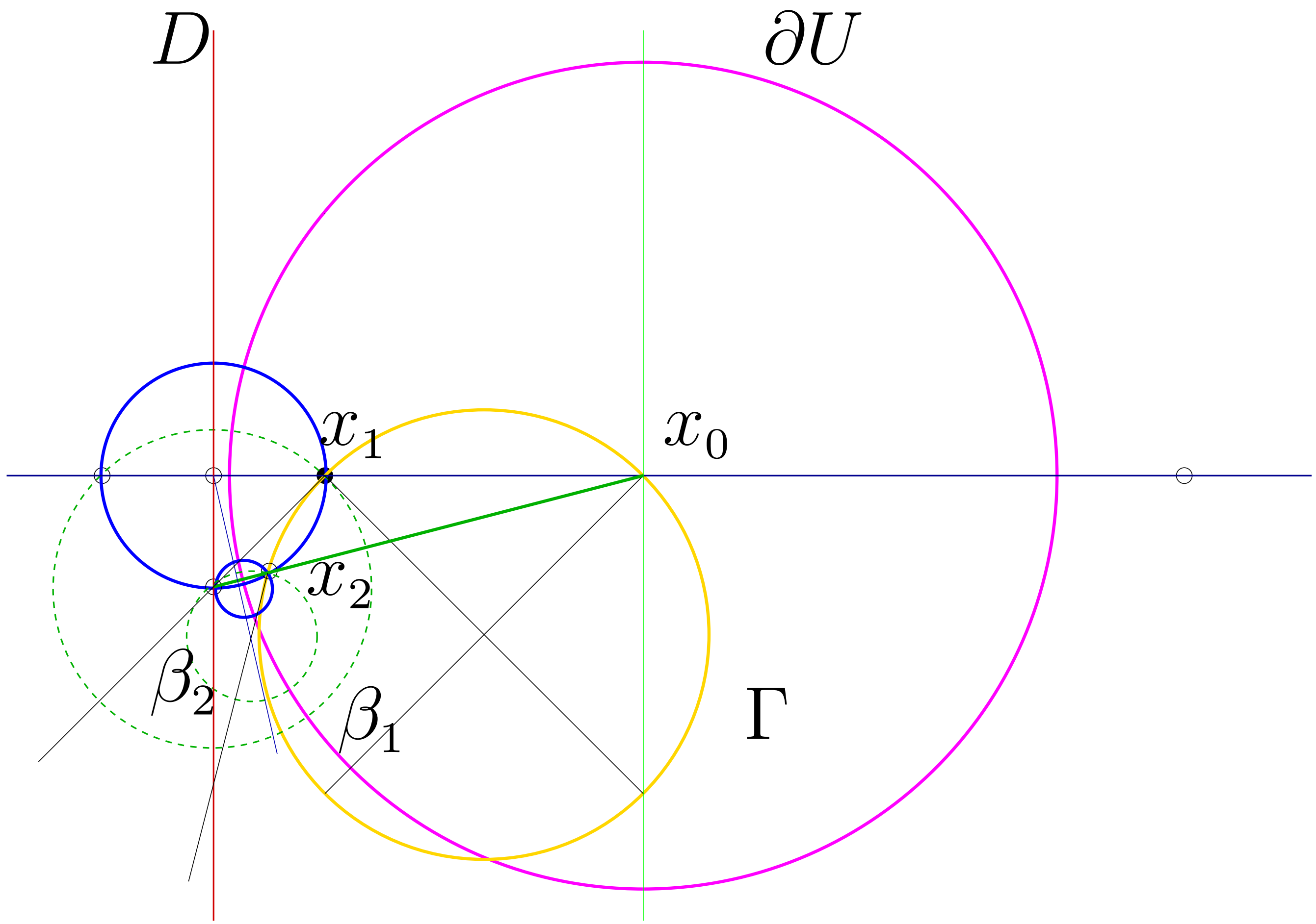
let U denote the open unit disc ;

- if $\Gamma \subset U$, x_n's either a regular polygon or a dense subset of an annulus
- if $\Gamma \subset \overline{U}$ and $\Gamma \not\subset U$, then Γ *horocycle* and x_n's *basic infinigon*
- if $\Gamma \not\subset \overline{U}$, then Γ *equidistant curve* and x_n's *open infinigon*

points at infinity of an infinigon:

- basic infinigon: a single point
- open infinigon: a closed interval of ∂U

the basic construction in the disc model:



infinigrids:

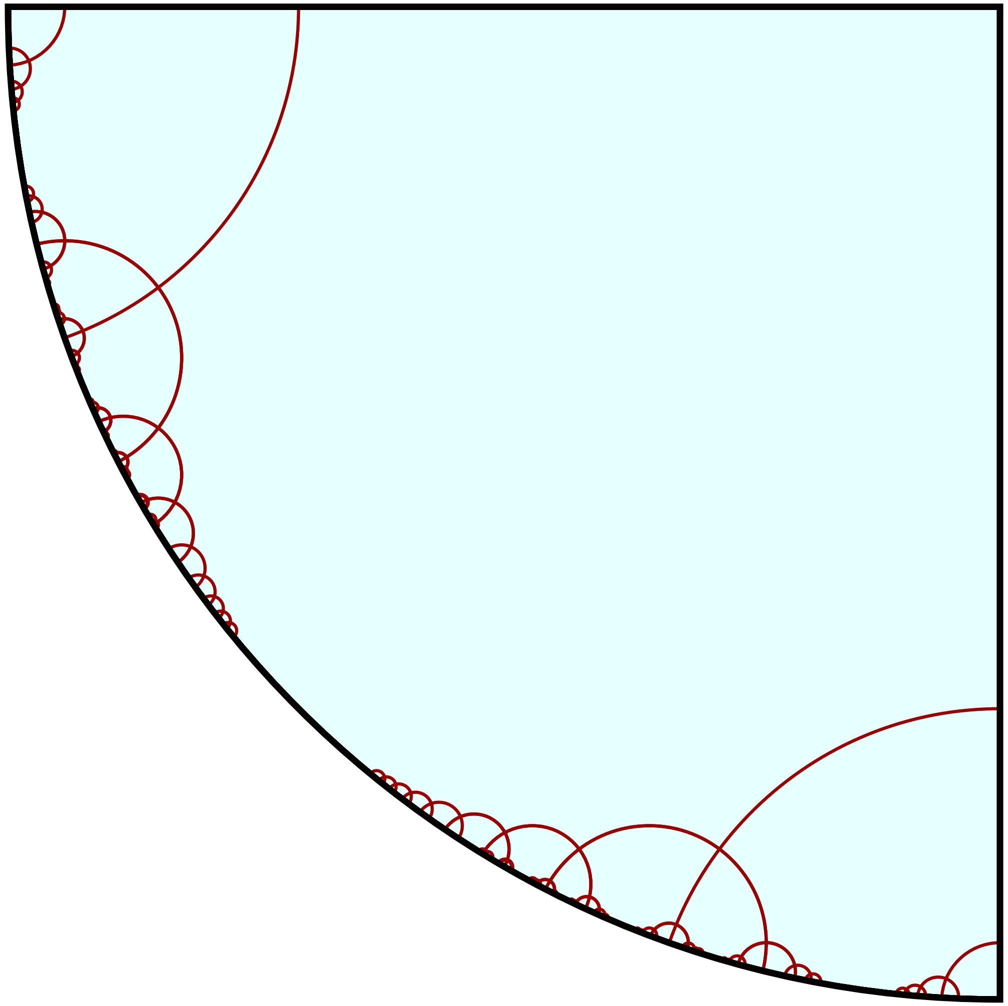
tessellation:

fix an infinigon ; replicate it by reflections in its sides and repeat the process with the images, recursively

theorem 1 (Coxeter/Rozenfeld/Margenstern) – *an infinigon generates a tiling by tessellation iff its interior angle* $= \dfrac{2\pi}{k}$, $k \geq 3$

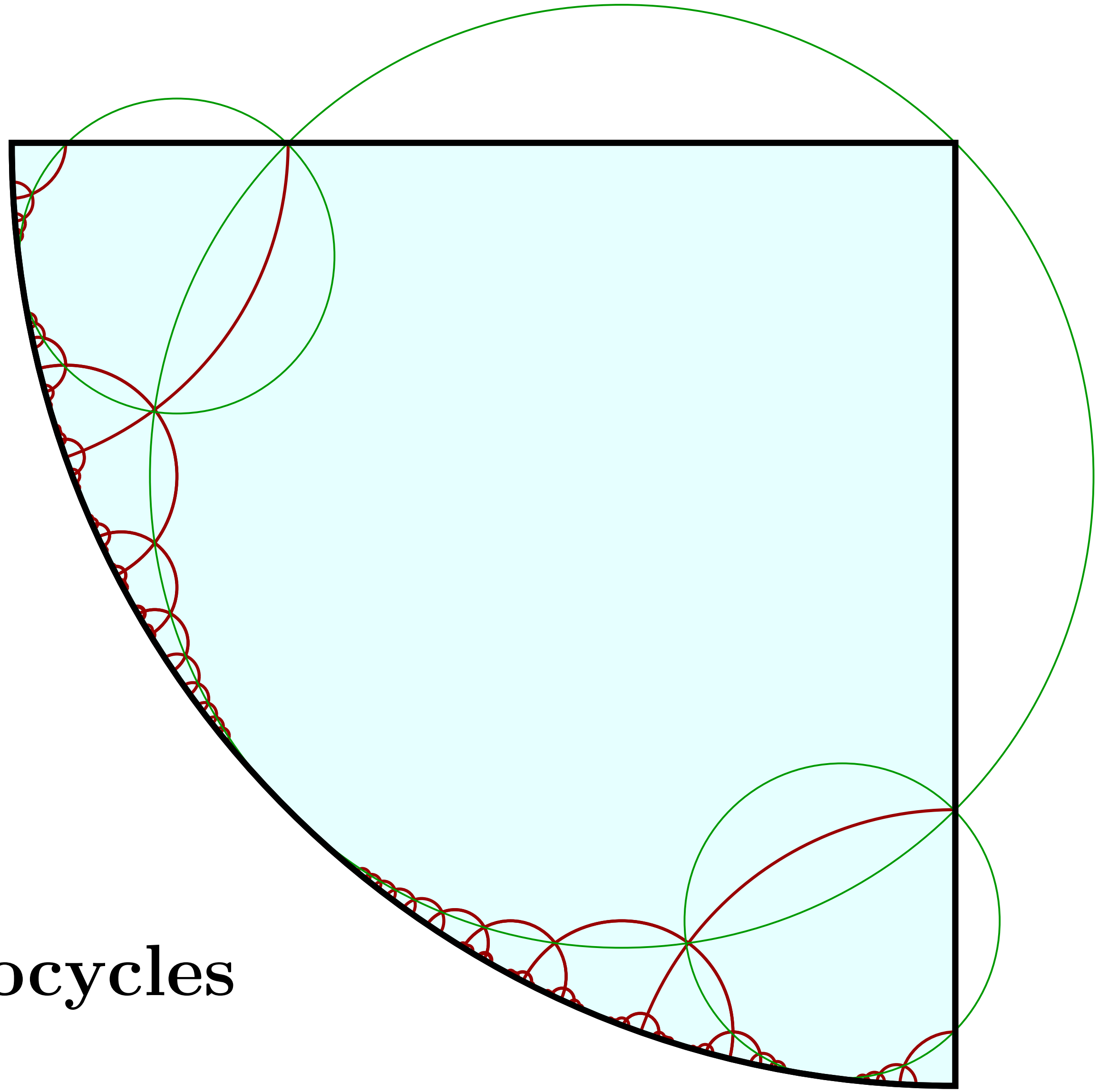
infinigon: basic or open

disc model: the rectangular infinigrid



the same

with horocycles



important property:

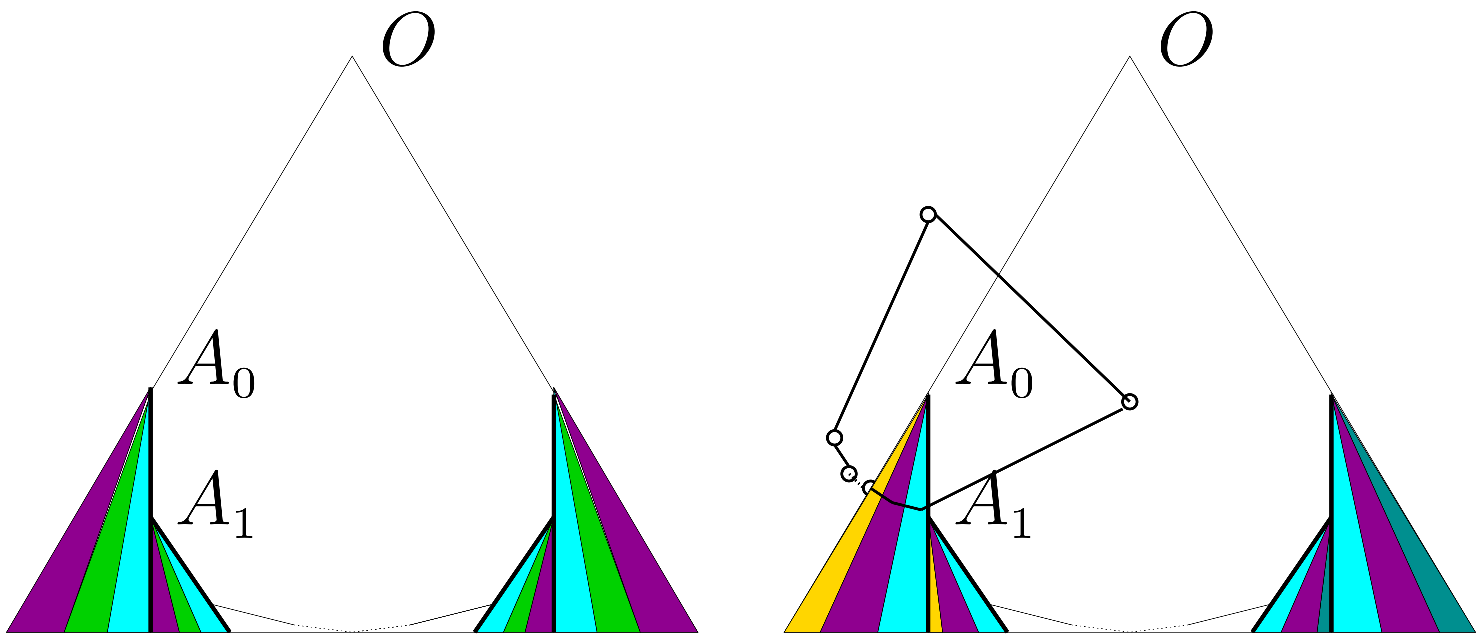
the splitting method can be extended to the infinigrids

theorem 2 (Margenstern) – *the tiling generated by a an infinigon is in a one-to-one correspondance with an infinite tree with an infinite branching in each node*

proof based on a recursive splitting

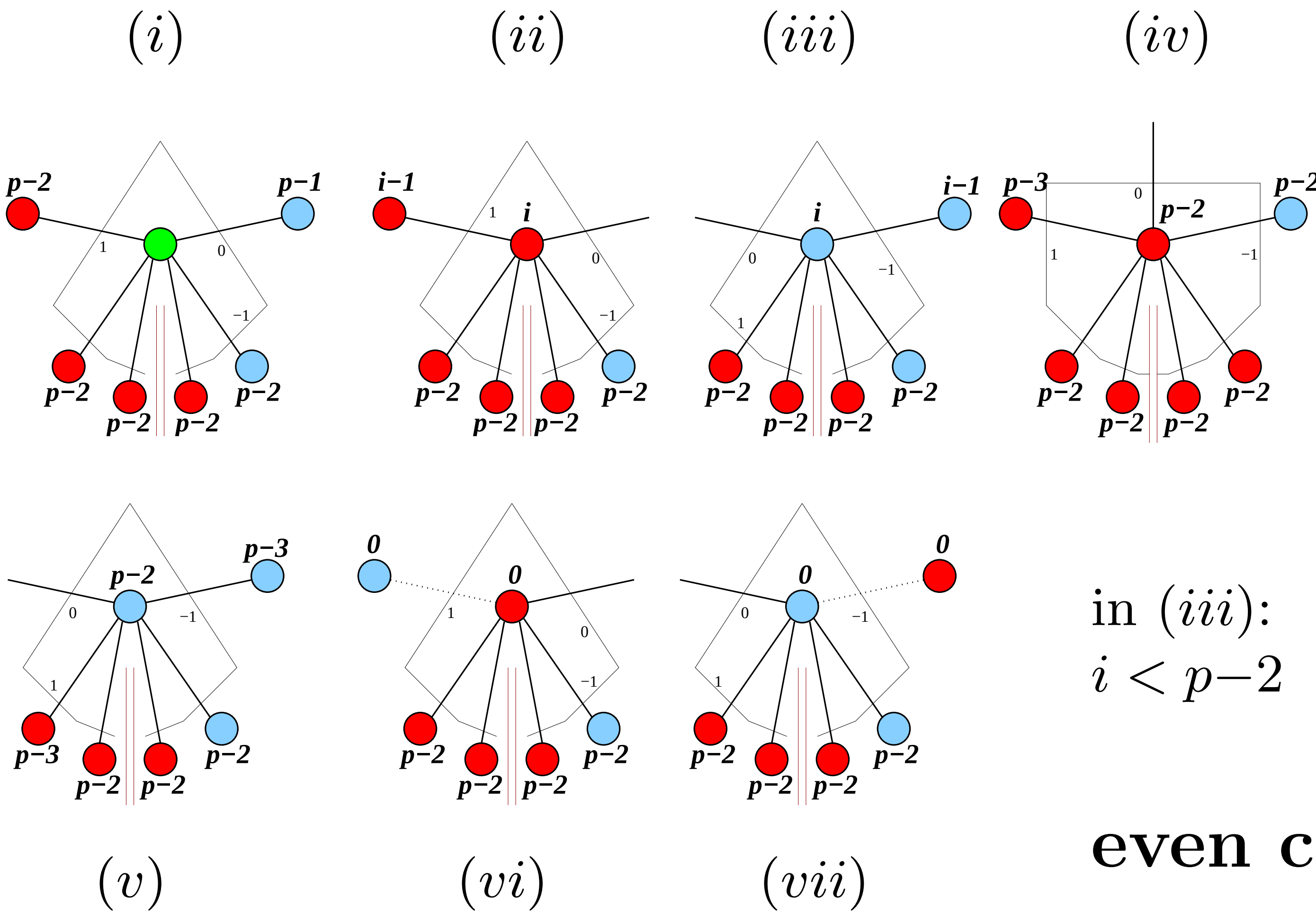
by recursion, generate a spanning tree of the dual graph

the splitting of $I\!H^2$:



even case odd case

the construction algorithms:

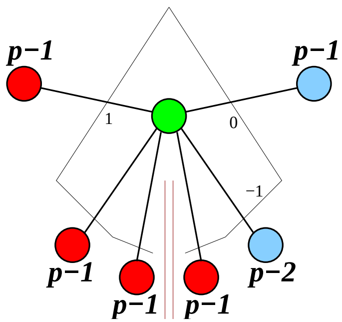


in (iii):
$i < p{-}2$

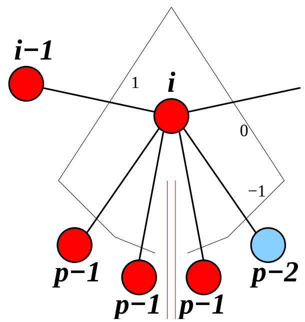
even case

the construction algorithms (continued):

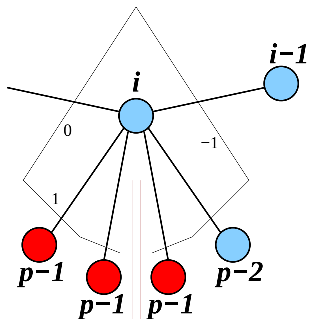
(i)



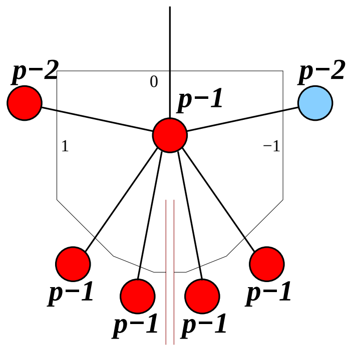
(ii)

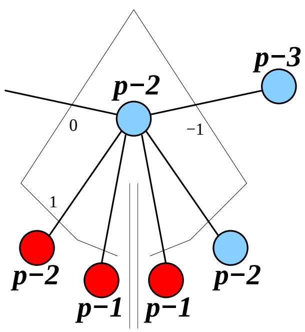


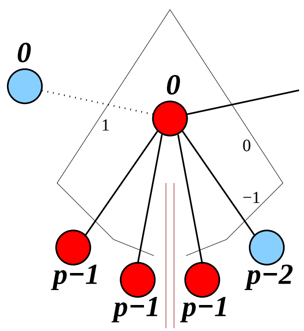
(iii)

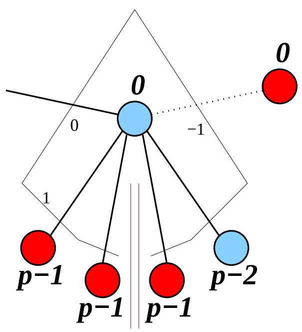


(iv)









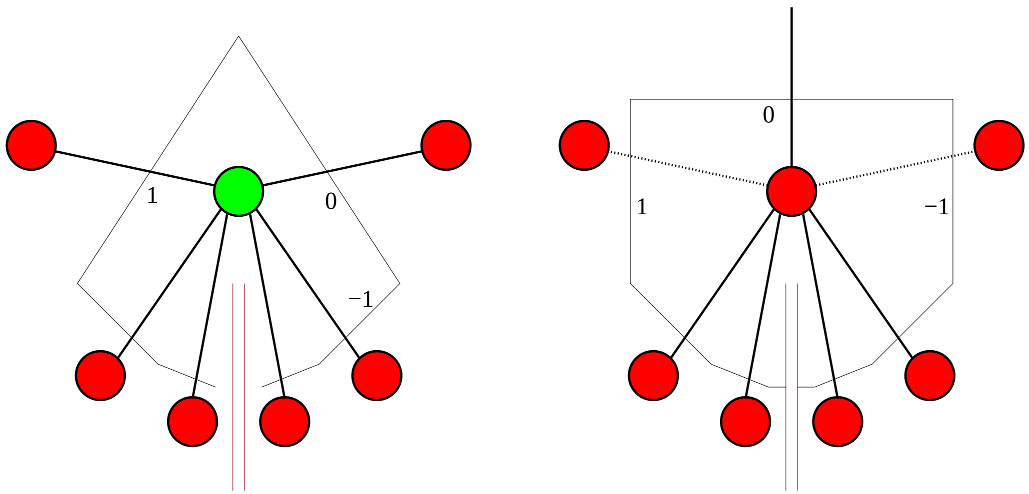
(v) (vi) (vii)

in (iii):
$i < p{-}2$

odd case

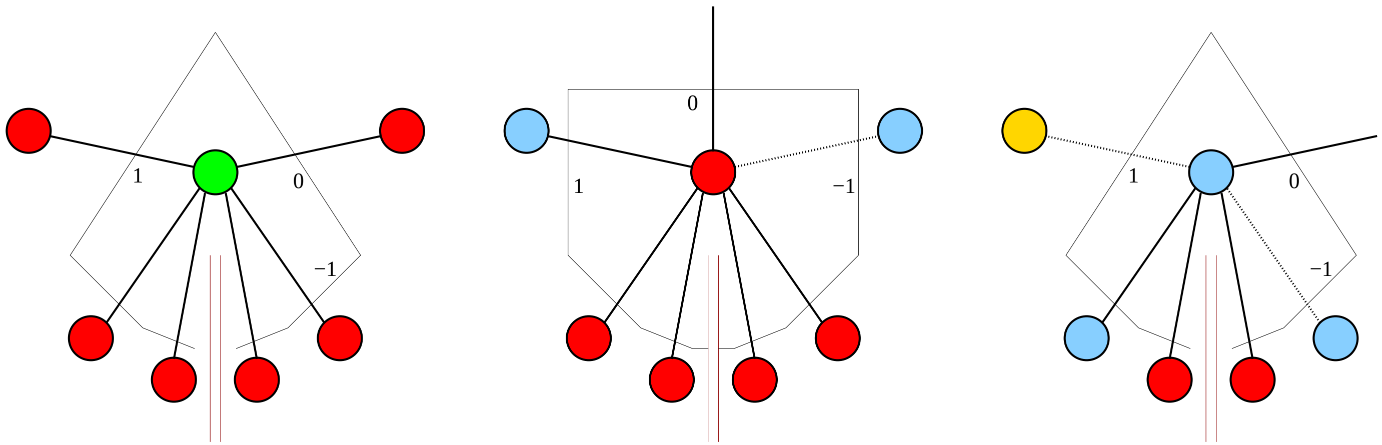
special cases:

$$k = 3$$



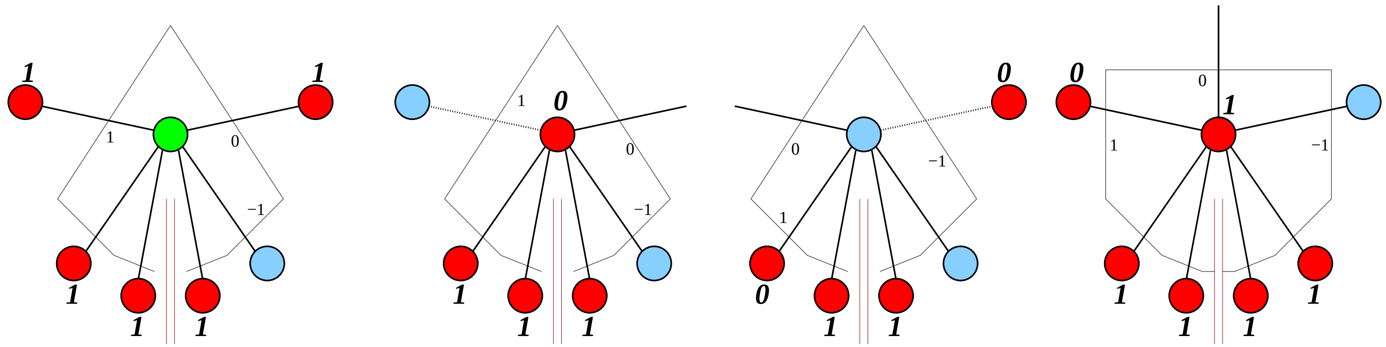
special cases:

$$k = 4$$



special cases:

$k = 5$



4.4.b register CA's on the infinigrid

first, it is an **adapted** CA to the infinigrid:

its transition function δ *is of the following form:*

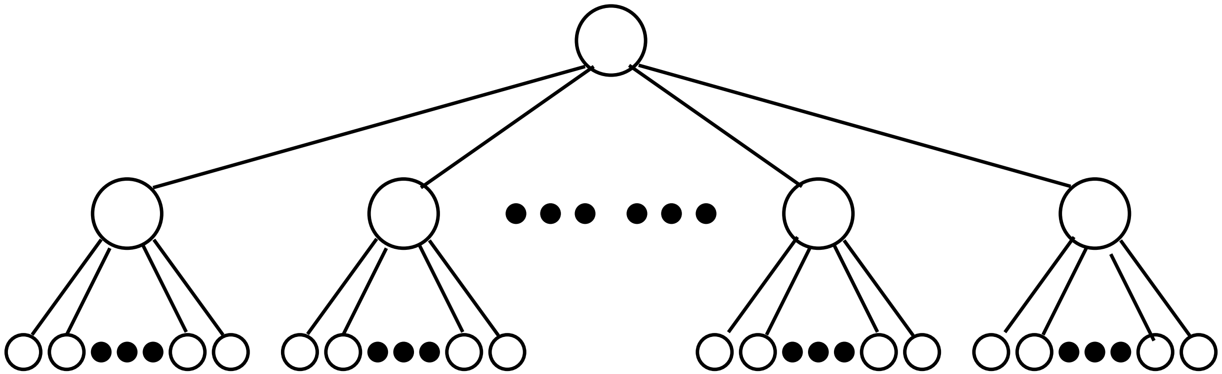
$$\delta : Q \times \{0,1\}^{|Q|} \mapsto Q$$

with $<s,t{+}1> \; = \delta(<s,t>, z_1(s,t), \ldots, z_{|Q|}(s,t))$
where the states of the CA are $1, \ldots, k$ *and*

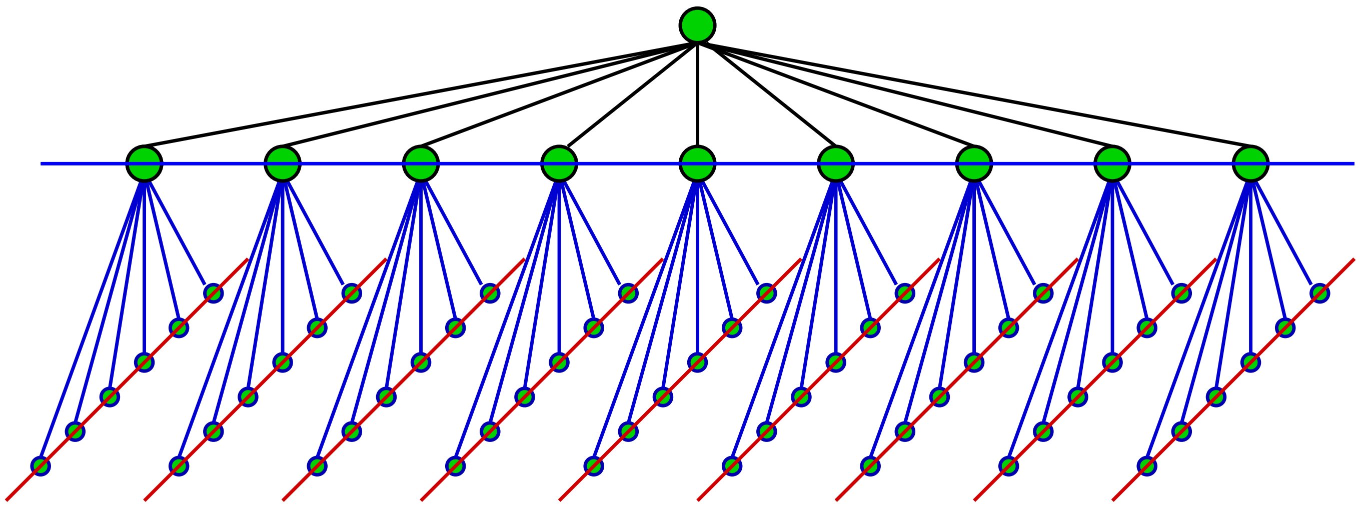
$$z_i(s,t) = \begin{cases} 1 & \textit{if there is a neighbour of the} \\ & \textit{cell in state } i \textit{ at time } t \\ 0 & \textit{otherwise} \end{cases}$$

addresses of cells: $(a_1, \ldots, a_n)$, $a_i \in \mathbb{N}$

infinigrids: the tree representation



infinigrids: another representation



theorem 1 – (SG-MM, 2002)
there is a CA U which is adapted on the infini-grid and such that for any arithmetical formula F in Σ^0_n or in Π^0_n, U recognises whether F is true or not

proof

we may assume F being closed
let $F = \exists x_1 \forall x_2 \ldots \xi x_n\, G(x_1, \ldots, x_n)$
where G prim. rec. with values in $\{0, 1\}$
initialization of F:
put $G(a_1, \ldots, a_n)$ in the cell $(a_1, \ldots, a_n)$
$(a_1, \ldots, a_n)$ oversees $(a_1, \ldots, a_n, z)$ for all z's
$\blacksquare$

second, a register CA has the following facilities:

- *states contain* **accept** *and* **reject**
- *in each cell, two* **registers***:* **a** *and* **x**

  a *read-only, holds the address of the cell*

  x *read-write, to compute integers*

permitted operations: copy **a**, $+$, $-$, $/$, $*$,

mod, sg, $\overline{sg}$, $\{(n)_i\}_{i=1}^{|n|}$, *any in* 1 *step*

data in unary via the root, initially not in 1

halting: root in **accept** or **reject**

theorem 2 – (SG-MM, 2002)
register CA's on the infinigrid are able to decide the truth of any Σ^0_n ; they can do that in time linear in the length of the formula

proof

same basic idea as in theorem 1,
+ Matiyasevich's theorem on the existence of a polynomial representing any partial recursive function by the associated diophantine equation

constant time for reporting the result to the root: indeed 12 levels ■

THANK YOU FOR YOUR ATTENTION