Computational Power of Observed Quantum Turing Machines

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Quantum Computing Basics

State space

in a classical world of computation: countable A.
in a quantum world: Hilbert space $\mathbb{C}^A$

ket map

$$|.\rangle : A \to \mathbb{C}^A$$

s.t. $\{|x\rangle, x \in A\}$ is an orthonormal basis of $\mathbb{C}^A$

Arbitrary states

$$\mathbf{\Phi} = \sum_{x \in A} \alpha_x |x\rangle$$

s.t. $\sum_{x \in A} |\alpha_x|^2 = 1$

Quantum Computing Basics

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$$\langle . | \quad : \quad A \quad \rightarrow \quad \mathbf{L}(\mathbb{C}^A, \mathbb{C})$$

s.t. $\forall x, y \in A$,

$$\langle y| \, |x\rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad \text{“ Kronecker ”}$$

$\forall v, t \in A, |v\rangle\langle t| : \mathbb{C}^A \rightarrow \mathbb{C}^A$:

$$(|v\rangle\langle t|) \, |x\rangle = |v\rangle \, (\langle t| \, |x\rangle) = \begin{cases} |v\rangle & \text{if } t = x \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad \text{“ } |v\rangle\langle t| \approx t \mapsto v \text{ ”}$$

Evolution of isolated systems: Linear map $U \in \mathbf{L}(\mathbb{C}^A, \mathbb{C}^A)$

$$U = \sum_{x,y \in A} u_{x,y} \, |y\rangle\langle x|$$

which is an isometry ($U^\dagger U = I$).

Observation

$$\text{Let } \mathbf{\Phi} = \sum_{x \in A} \alpha_x |x\rangle$$

(Full) measurement in standard basis:

The probability to observe $a \in A$ is $|\alpha_a|^2$.

If $a \in A$ is observed, the state becomes $\mathbf{\Phi}_a = |a\rangle$.

Partial measurement in standard basis: Let $K = \{K_\lambda, \lambda \in \Lambda\}$ be a partition of A.

The probability to observe $\lambda \in \Lambda$ is $p_\lambda = \sum_{a \in K_\lambda} |\alpha_a|^2$

If $\lambda \in \Lambda$ is observed, the state becomes

$$\mathbf{\Phi}_\lambda = \frac{1}{\sqrt{p_\lambda}} P_\lambda \mathbf{\Phi} = \frac{1}{\sqrt{p_\lambda}} \sum_{a \in K_\lambda} \alpha_a |a\rangle$$

where $P_\lambda = \sum_{a \in K_\lambda} |a\rangle\langle a|$.

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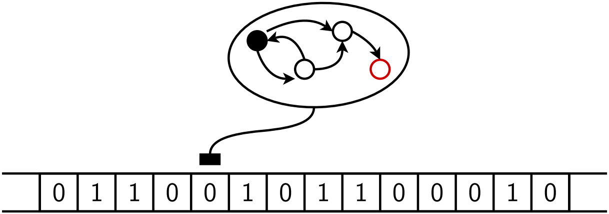
Deterministic Turing Machine (DTM)

Classical Turing machine (Q, Σ, δ):

$$\delta : Q \times \Sigma \to Q \times \Sigma \times \{-1, 0, 1\}$$



$(q, T, x) \in Q \times \Sigma^* \times \mathbb{Z}$ is a classical configuration.

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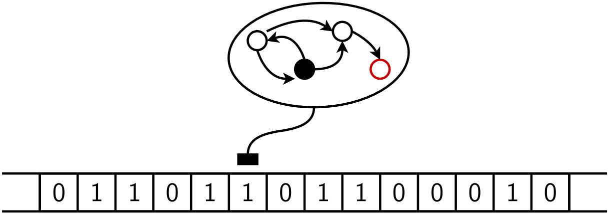

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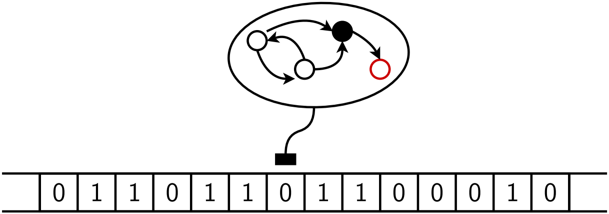

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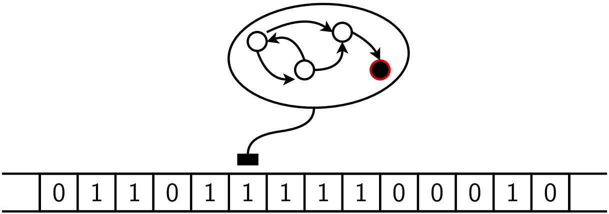

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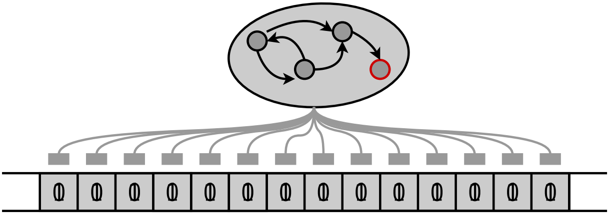
Quantum Turing Machine (QTM)

Quantum Turing machine $M = (Q, \Sigma, \delta)$:

$$\delta : Q \times \Sigma \times Q \times \Sigma \times \{-1, 0, 1\} \to \mathbb{C}$$



A quantum configuration is a superposition of classical configurations

$$\sum_{q \in Q, T \in \Sigma^*, x \in \mathbb{Z}} \alpha_{q,T,x} \, |q, T, x\rangle \in \mathbb{C}^{Q \times \Sigma^* \times \mathbb{Z}}$$

Evolution operator

$$U_M = \sum_{p,q\in Q,\sigma\in\Sigma,d\in\{-1,0,1\},T\in\Sigma^*,x\in\mathbb{Z}} \delta(p,T_x,q,\sigma,d)\,|q,T_x^\sigma,x+d\rangle\,\langle p,T,x|$$

A QTM (Q,Σ,δ) has to satisfy some well-formedness conditions...

Well-formedness conditions

Definition: A QTM M is well-formed iff U_M is an isometry, i.e. $U_M^\dagger U_M = I$

- The evolution of the machine does not violate the postulates of quantum mechanics.
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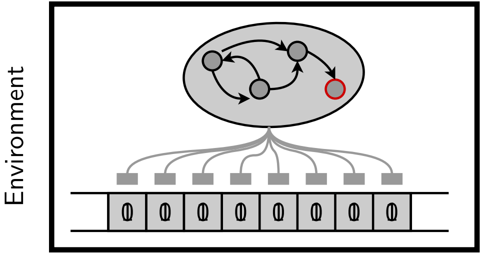
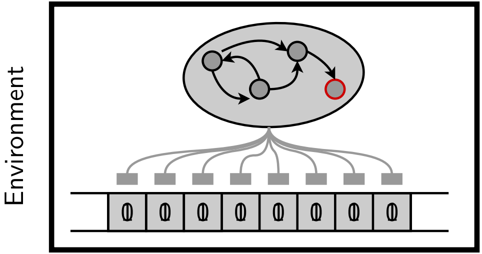


Halting of QTM



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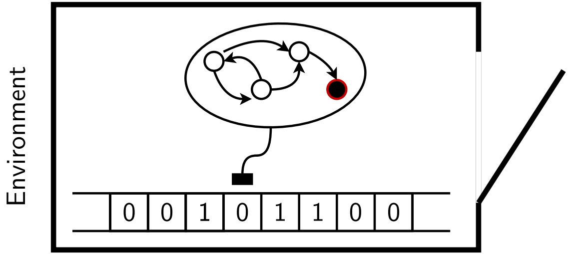

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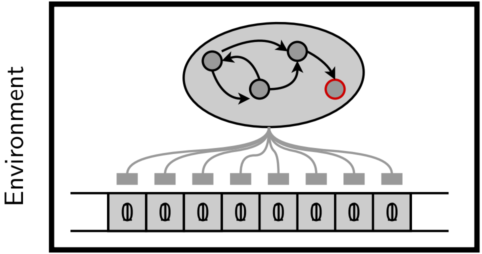

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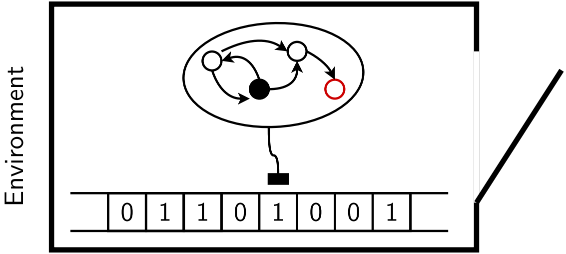
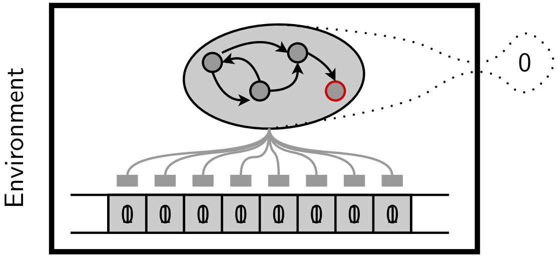


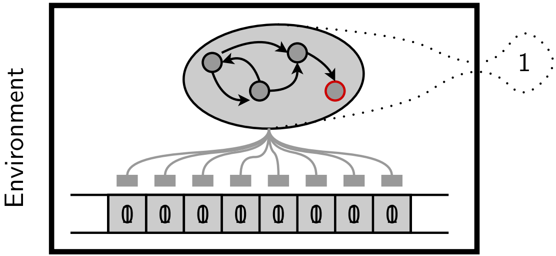
If the halting state is not reached, the computation is useless.

Halting of QTM



Halting qubit (Ad hoc)

Halting of QTM



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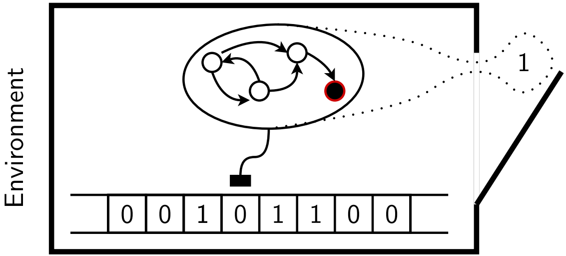

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‘Un-isolated’ QTM

Isolation assumption is probably too strong

- technical issues like the halting of QTM,
- models of QC (one-way model, measurement-only model) based on measurements.
- PTM and DTM are not well-formed QTM (reversible DTM does)
- quest of a universal QTM: a classical control is required.

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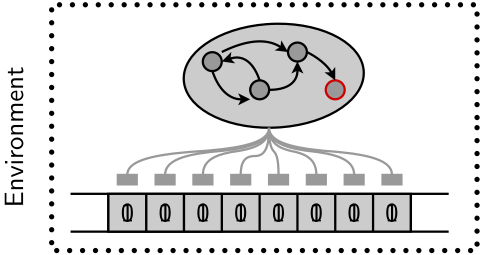

Modelling Environment: Observed QTM

Environment is modelled as a partial measurement of the configuration, characterised by a partition $K = \{K_\lambda\}_{\lambda \in \Lambda}$ of $Q \times \Sigma^* \times \mathbb{Z}$.

Definition: For a given QTM $M = (Q, \Sigma, \delta)$ and a given partition $K = \{K_\lambda\}_{\lambda \in \Lambda}$ of $Q \times \Sigma^* \times \mathbb{Z}$, $[M]_K$ is an Observed Quantum Turing Machine (OQTM).

Evolution of OQTM

One transition of $[M]_K$ is composed of:

1. partial measurement K of the quantum configuration;
2. transition of M;
3. partial measurement K of the quantum configuration.

Definition: An OQTM $[M]_K$ is well-observed iff

$$\sum_{\lambda \in \Lambda} P_\lambda U_M^\dagger U_M P_\lambda = I$$

where $P_\lambda = \sum_{(p,T,x) \in K_\lambda} |p, T, x\rangle \langle p, T, x|$.

Lemma: If a QTM M is well-formed then $[M]_K$ is a well-observed OQTM for any K.

Proof:

$$\begin{aligned}
\sum_{\lambda\in\Lambda} P_\lambda U_M^\dagger U_M P_\lambda &= \sum_{\lambda\in\Lambda} P_\lambda P_\lambda \\
&= \sum_{\lambda\in\Lambda} P_\lambda \\
&= \sum_{\lambda\in\Lambda} \sum_{p,T,x\in K_\lambda} |p,T,x\rangle\langle p,T,x| \\
&= \sum_{p,T,x\in Q\times\Sigma^*\times\mathbb{Z}} |p,T,x\rangle\langle p,T,x| \\
&= I
\end{aligned}$$

a weaker condition

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Proof:

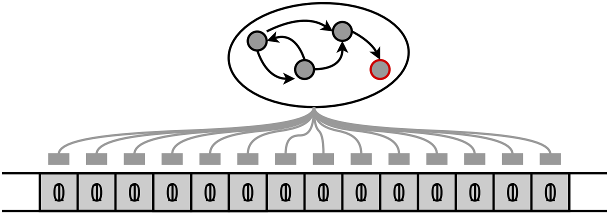
$$\begin{array}{rcl} \sum_{\lambda \in \Lambda} P_\lambda U_M^\dagger U_M P_\lambda & = & \sum_{\lambda \in \Lambda} P_\lambda P_\lambda \\ & = & \sum_{\lambda \in \Lambda} P_\lambda \\ & = & \sum_{\lambda \in \Lambda} \sum_{p,T,x \in K_\lambda} |p,T,x\rangle \langle p,T,x| \\ & = & \sum_{p,T,x \in Q \times \Sigma^* \times \mathbb{Z}} |p,T,x\rangle \langle p,T,x| \\ & = & I \end{array}$$

Example: halting of QTM

For a given QTM $M = (Q, \Sigma, \delta)$ s.t. $q_h \in Q$ is the unique halting state.

$$\begin{array}{lcl} K_h & = & \{q_h\} \times \Sigma^* \times \mathbb{Z} \\ K_{\bar{h}} & = & K \setminus K_h \end{array}$$

$[M]_{\{K_h, K_{\bar{h}}\}}$ evolves as follows:



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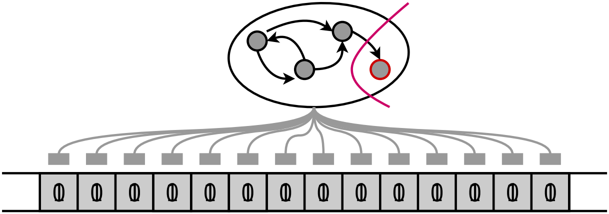

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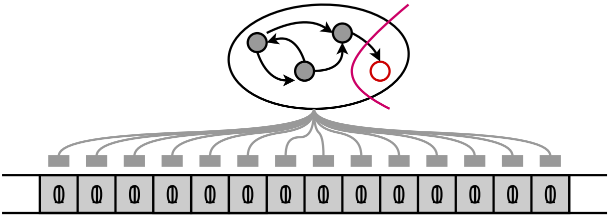

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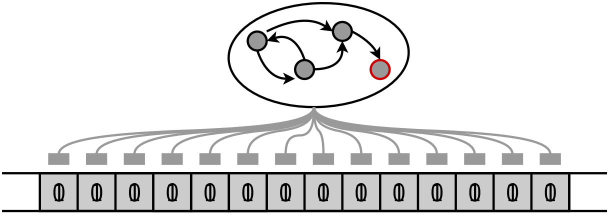

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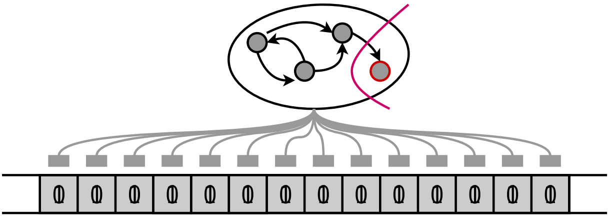

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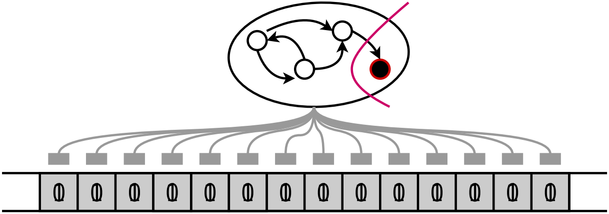

OQTM more expressive than QTM

Lemma: For any DTM $M = (Q, \Sigma, \delta)$, $[M]_{\{\{c\}, c \in Q \times \Sigma^* \times \mathbb{Z}\}}$ is a well-observed OQTM.

OQTM, a too powerful model ?!?

Theorem: There is a well-observed OQTM $[M_h]_{K_h}$ for deciding (with high probability), for any DTM M and any input u, whether M halts on input u.

(Proof) Hadamard QTM

Let $M_h = (\{q_0, q_1, q_2, q_h, q_{\bar{h}}\}, \Sigma, \delta_h)$ be a well-formed QTM, s.t. q_h and $q_{\bar{h}}$ are the halting states and for $\sigma \in \Sigma$

$$\begin{array}{rcr} \delta_h(q_0, \sigma, q_1, \sigma, 0) & = & 1/\sqrt{2} \\ \delta_h(q_0, \sigma, q_2, \sigma, 0) & = & 1/\sqrt{2} \\ \delta_h(q_1, \sigma, q_h, \sigma, 0) & = & 1/\sqrt{2} \\ \delta_h(q_1, \sigma, q_{\bar{h}}, \sigma, 0) & = & 1/\sqrt{2} \\ \delta_h(q_2, \sigma, q_h, \sigma, 0) & = & 1/\sqrt{2} \\ \delta_h(q_2, \sigma, q_{\bar{h}}, \sigma, 0) & = & -1/\sqrt{2} \end{array}$$

$\forall w \in \Sigma^*$,

$$\begin{array}{rcl} U_{M_h}^2 |q_0, w\rangle & = & U_{M_h}(\frac{1}{\sqrt{2}}(|q_1, w\rangle + |q_2, w\rangle)) \\ & = & \frac{1}{2}(|q_h, w\rangle + |q_{\bar{h}}, w\rangle + |q_h, w\rangle - |q_{\bar{h}}, w\rangle) \\ & = & |q_h, w\rangle \end{array}$$

For any DTM M and any input u, let $w_{M,u} \in \Sigma^*$ be an 'encoding' of M and u.

$$\begin{array}{rcl} K_0 & = & \{(q_1, w_{M,u}) \text{ s.t. } M(u) \text{ does not halt}\} \cup \{(q_{\bar{h}}, w)\} \\ K_1 & = & \{(q, w) \text{ s.t. } (q, w) \notin K_1\} \end{array}$$

What is the evolution of $[M_h]_{\{K_0,K_1\}}$ if the initial configuration is $(q_0, w_{M,u})$?

Evolution of M_h: $|q_0, w_{M,u}\rangle \to \frac{1}{\sqrt{2}}(|q_1, w_{M,u}\rangle + |q_2, w_{M,u}\rangle) \to |q_h, w_{M,u}\rangle$

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Towards a new definition of OQTM

- Initial proposition: K is a partition of $Q \times \Sigma^* \times \mathbb{Z}$.

- Focus on the (classical) control: K is a partition of $Q \times \mathbb{Z}$.

  Theorem: There is a QTM M_h' and a partition K of $Q \times \mathbb{Z}$, s.t. $[M_h]_K$ is well observed and decides (with high probability), for any DTM M and any input u, whether M halts on input u.

- Finite partition: K is a partition of $Q \times \Sigma$. (Q: internal states; Σ: symbol pointed out by the head.)

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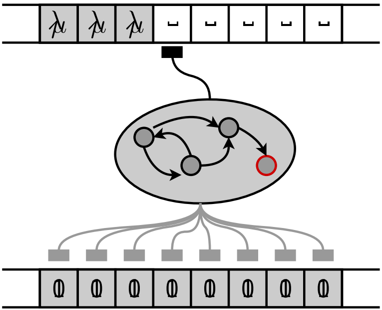
Simulation

Theorem: For any well-observed OQTM $[M]_K$ there exists a well-formed QTM M' which simulates $[M]_K$ within a quadratic slowdown.

Step one

If $M = (Q, \Sigma, \delta)$ and $K = \{K_\lambda\}_{\lambda \in \Lambda}$, let $\tilde{M} = (Q, \Sigma, \Lambda, \tilde{\delta})$ be a 2-tape QTM s.t.

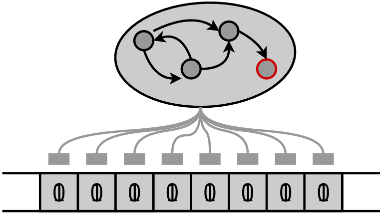
$$\tilde{\delta}(p, \tau, \llcorner\!\lrcorner, q, \sigma, \lambda, d, +1) = \begin{cases} \delta(p, \tau, q, \sigma, d) & \text{if } (p, \tau) \in K_\lambda \\ 0 & \text{otherwise} \end{cases}$$

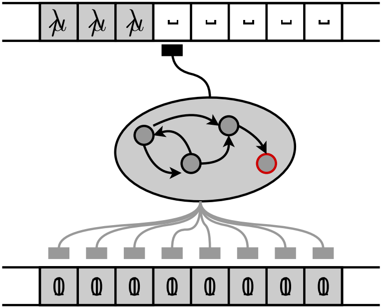


Lemma: $\tilde{M}$ is well formed.

Step two

Lemma: $[\tilde{M}]_{\tilde{K}}$ simulates $[M]_K$, where $\tilde{K} = \{Q \times \Sigma \times \{\lambda\}\}_{\lambda \in \Lambda}$





Step three

Since they act on distinct systems (the second head always moves to the right), the measurements can be postponed to the end of the computation:

Lemma: $\tilde{M}$ simulates $[\tilde{M}]_{\tilde{K}}$.

Lemma: There exists a well-formed 1-tape QTM M' which simulates $\tilde{M}$ within a quadratic slowdown.

Conclusion

- OQTM: extension of QTM with measurements;
- a more expressive (but not overpowerfull) machine: QTM, DTM, halting QTM.

Perspectives:

- Universal quantum Turing machine;
- what is the minimal k for which any OQTM $[M]_K$ can be efficiently simulated by an OQTM $[M']_{K'}$ where all regions of K' have a size less than k ?