Ideal Decompositions For Vector Addition Systems

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Vector Addition Systems with States (VASS)

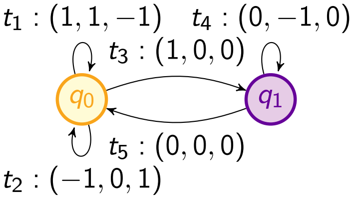

Figure: A 3-dimensional VASS.

$$q_0, 1, 0, 1 \xrightarrow{t_1} q_0, 2, 1, 0 \xrightarrow{t_2} q_0, 1, 1, 1 \xrightarrow{t_3} q_1, 2, 1, 1$$

Central Problems

Many problems are decidable on VASS, notably

INPUT: V a VASS, c, c' two configurations.

Reachability: $c \xrightarrow{*} c'$?

Coverability: $c \xrightarrow{*} c''$ for some configuration $c'' \sqsupseteq c'$?

Well Structure Transition Systems (WSTS)

WSTS [Abdulla & Čerans & Jonsson & Tsay 2000][Finkel & Schnoebelen 2001]:

- Many problems are decidable, including coverability.
- Based on a well quasi-order (wqo) on configurations.
- VASS are WSTS.

$\Longrightarrow$ The VASS coverability problem is decidable.

Coverability Set

V a VASS, c a configuration.

$$\mathit{Cover}(c) \stackrel{\text{def}}{=} \{c' \mid \exists c'' \sqsupseteq c' \ c \xrightarrow{*} c''\}$$

Computable thanks to a coverability tree [Karp & Miller 1969]:

- Forward exploration of a reachability tree.
- A finite description of $\mathit{Cover}(c)$ is obtained from nodes' labels.

Complete WSTS

Ingredient for defining a coverability tree algorithm [Finkel & Goubault-Larrecq 2009,2012]:

- An acceleration procedure.
- A way to represent downward-closed sets of configurations.

$\Longrightarrow$ wqo ideals are the right objects.

VASS Reachability Problem

Decidable:

- Several attempts and partial solutions, notably by Sacerdote & Tenney in 1977.
- First proved by Mayr in 1981.
- Clarified by Kosaraju in 1982 and Lambert in 1992.

We call the resulting algorithm, the KLMST:

- Refinement of a finite set of structures following some conditions.
- At first sight little to do with WSTS.

$\Longrightarrow$ wqo ideals are the right objects [Leroux & Schmitz 2015].

Overview of the Talk

Ideals provide the data structures involved:

- Karp & Miller's coverability tree algorithm which computes the ideal decomposition of the coverability set using configuration ideals.
- The KLMST algorithm, which computes the ideal decomposition of the downward-closure of the set of runs using run ideals.

This talk:

- Present wqo ideals.
- Overview algorithmic applications through two algorithms.

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Well Quasi-Orders

A relation $\leq$ on a set X is a wqo if:

- $\leq$ is a quasi-order : $\begin{cases} \text{reflexive: } x \leq x \\ \text{transitive: } x \leq y \wedge y \leq z \Rightarrow x \leq z \end{cases}$
- Infinite sequences $x_1, x_2, \ldots$ are good: $x_i \leq x_j$ for some $i < j$.

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Example

$(Q, =)$ is wqo if Q is finite.
$(\mathbb{N}, \leq)$ is wqo.

$(\mathbb{Z}, \leq)$ is not a wqo since $0, -1, -2, \ldots$ is a bad sequence.

Downward-Closed Sets

Let $(X, \leq)$ be a quasi-ordered set.

The downward-closure of $S \subseteq X$:

$$\downarrow S = \{x \in X \mid \exists s \in S \; x \leq s\}$$

$D \subseteq X$ is downward-closed if $\downarrow D = D$.

Lemma

A quasi-ordered set is wqo if, and only if, it satisfies the descending chain property: chains $D_0 \supsetneq D_1 \supsetneq \cdots$ of downward-closed sets are finite.

Ideals

$(X, \leq)$ a wqo.

A set $S \subseteq X$ is directed if $\forall x, y \in S \; \exists s \in S$ such that $x, y \leq s$.

An ideal is a non-empty directed downward-closed set.

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Example

$(Q, =)$ with Q finite. Ideals are $\{q\}$ with $q \in Q$.

$(\mathbb{N}, \leq)$: Ideals are $\mathbb{N}$ and $\{0, \ldots, n\}$ with $n \in \mathbb{N}$.

Ideal Decomposition

Theorem ([Kabil & Pouzet : 1992],[Finkel & Goubault-Larrecq : 2009], [Goubault-Larrecq & Karandikar & Narayan Kumar & Schnoebelen : In preparation])

Every downward-closed set is the union of a unique finite family of incomparable for the inclusion ideals.

Application:

- Effective way for representing downward-closed sets.

Dickson's Lemma

The Cartesian product $(X_1, \leq_1) \times (X_2, \leq_2)$ of two quasi-ordered sets is the quasi-ordered set $(X, \leq)$ defined by:

$$(x_1, x_2) \leq (y_1, y_2) \iff x_1 \leq_1 y_1 \ \wedge \ x_2 \leq_2 y_2$$

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Lemma (Dickson's Lemma)

$(X_1, \leq_1)$ and $(X_2, \leq_2)$ wqo $\Longrightarrow (X_1, \leq_1) \times (X_2, \leq_2)$ wqo.

$$\mathit{Ideals}(X_1, \leq_1) \times (X_2, \leq_2)$$

$$=$$

$$\left\{ I_1 \times I_2 \;\middle|\; \begin{array}{l} I_1 \in \mathit{Ideals}(X_1, \leq_1) \wedge \\ I_2 \in \mathit{Ideals}(X_2, \leq_2) \end{array} \right\}$$

Higman's Lemma

Given a quasi-ordered set $(X, \leq)$, we define $(X, \leq)^*$ as the set X^* of words over X quasi-ordered by $\leq_*$ defined by:

$$x_1 \dots x_n \leq_* y_1 \dots y_m$$
$$\iff$$
$$\exists i_1 < \dots < i_n \;\mid\; x_1 \leq y_{i_1} \wedge \dots \wedge x_n \leq y_{i_n}$$

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Lemma (Higman's Lemma)

$(X, \leq)$ *wqo* $\Longrightarrow (X, \leq)^*$ *wqo.*

$(X, \leq)$ a wqo.

An atom of $(X, \leq)$ is a language of the form:

- $\{\varepsilon\} \cup I$ where I is an ideal of $(X, \leq)$, or
- $(I_1 \cup \ldots \cup I_n)^*$ where $I_1, \ldots, I_n$ are ideals of $(X, \leq)$.

Theorem ([Jullien 1969], [Kabil & Pouzet : 1992], [Finkel & Goubault-Larrecq : 2009])

Ideals of $(X, \leq)^*$ are the finite product of atoms of $(X, \leq)$.

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Ordering Configurations

V a VASS with:

- Q finite set of control states.
- d counters.

The set of configurations is equipped with $\sqsubseteq$ defined by:

$$(\mathrm{Confs}, \sqsubseteq) \stackrel{\mathrm{def}}{=} (Q, =) \times (\mathbb{N}, \leq)^d$$

Coverability Set As Downward-Closed Sets

The coverability set is downward-closed:

$$\mathit{Cover}(c) \stackrel{\text{def}}{=} \{c' \mid \exists c'' \sqsupseteq c' \; c \xrightarrow{*} c''\}$$
$$= \downarrow \{c'' \mid c \xrightarrow{*} c''\}$$

$\Longrightarrow$ $\mathit{Cover}(c)$ is a finite union of configuration ideals.

We introduce $\omega \notin \mathbb{N}$ and $\mathbb{N}_\omega \stackrel{\text{def}}{=} \mathbb{N} \cup \{\omega\}$ ordered by:

$$0 \leq 1 \leq 2 \leq \cdots \leq \omega$$

Ideals of $(\mathbb{N}, \leq)$ are:

$$\{n \in \mathbb{N} \mid n \leq x\}$$

Where $x \in \mathbb{N}_\omega$.

Representing Configuration Ideals

$$(\mathrm{Confs}, \sqsubseteq) \stackrel{\text{def}}{=} (Q, =) \times (\mathbb{N}, \leq)^d$$

Ideals of $(\mathrm{Confs}, \sqsubseteq)$ are the sets:

$$[\![q, \mathsf{x}]\!]_{Confs} \stackrel{\text{def}}{=} \{q\} \times \{\mathsf{v} \in \mathbb{N}^d \mid \mathsf{v} \leq \mathsf{x}\}$$

where (q, x) is an extended configuration in $Q \times \mathbb{N}_\omega^d$.

Extending the Step Relation

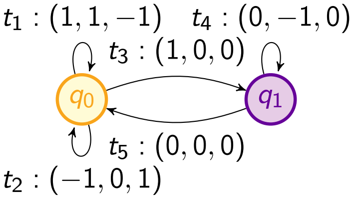
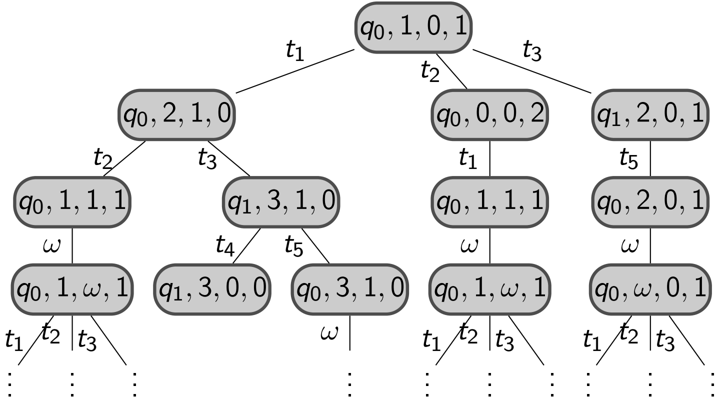


Figure: A 3-dimensional VASS.

$$q_0, 1, 0, \omega \xrightarrow{t_1} q_0, 2, 1, \omega \xrightarrow{t_2} q_0, 1, 1, \omega \xrightarrow{t_3} q_1, 2, 1, \omega$$

The Coverability Tree Construction



A prefix of the tree computed by the Karp and Miller algorithm.

$$Cover(q_0, 1, 0, 1) \ = \ [\![q_0, \omega, \omega, \omega]\!]_{Confs} \cup [\![q_1, \omega, \omega, \omega]\!]_{Confs}$$

Applications

Once the decomposition of the coverability set into ideals is computed:

- The coverability problem reduces to find an ideal that contains a configuration.
- The place boundedness problem reduces to check that every ideal satisfies the place boundedness condition.

Complexity View Point

The size of the coverability set $\stackrel{\text{def}}{=}$ size of the decomposition into maximal ideals (numbers encoded in binary).

- There exists a family of initialized VASS with finite but Ackermannian-sized reachability sets [Cardoza & Lipton & Meyer 1976].
- Lower-bound tight since the Karp and Miller algorithm is terminating in at most an Ackermannian number of steps [Figueira & Figueira & Schmitz & Schnoebelen 2011].

$\Longrightarrow$ The Karp and Miller algorithm is optimal.

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Ordering Runs

V is a VASS with a set T of transitions.

$$(\mathit{PreSteps}, \preceq) \stackrel{\text{def}}{=} (\mathit{Confs}, \sqsubseteq) \times (T, =) \times (\mathit{Confs}, \sqsubseteq)$$
$$(\mathit{PreRuns}, \trianglelefteq) \stackrel{\text{def}}{=} (\mathit{Confs}, \sqsubseteq) \times (\mathit{PreSteps}, \preceq)^* \times (\mathit{Confs}, \sqsubseteq)$$

A run is a prerun of the following form:

$$(c, (c_1, t_1, c'_1) \ldots (c_k, t_k, c'_k), c')$$

with:

$$c = c_1 \xrightarrow{t_1} c'_1 = c_2 \cdots c_k \xrightarrow{t_1} c'_k = c'$$

$\mathit{Runs}(c, c')$ is the set of runs from c to c'.

Reachability Problem

Reduces to the emptyness of:

$$\downarrow Runs(c, c')$$

This set can be uniquely decomposed as maximal prerun ideals.

Prestep Ideals

Ideals of $(PreSteps, \preceq)$ have the following form, where $e = (c, t, c')$ is an extended prestep, i.e. c, c' are extended configurations, and $t \in T$:

$$[\![e]\!]_{PreSteps} = [\![c]\!]_{Confs} \times \{t\} \times [\![c']\!]_{Confs} \ .$$

Prerun Ideals

Ideals of $(\mathit{PreRuns}, \trianglelefteq)$ have the following form, where p is a regular expression denoting a product over extended steps and c, c' are extended configurations:

$$[\![c, p, c']\!]_{\mathit{PreRuns}} = [\![c]\!]_{\mathit{Confs}} \times [\![p]\!]_{\mathit{PreSteps}^*} \times [\![c']\!]_{\mathit{Confs}} \ .$$

with:

$$p ::= a_1 \cdots a_n \ , \qquad\qquad a ::= e + \varepsilon \mid E^*$$

where e ranges over extended presteps and E over finite sets of extended presteps, with semantics:

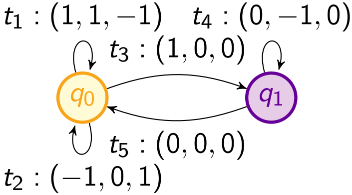
$$[\![a_1 \cdots a_n]\!]_{\mathit{PreStep}^*} \stackrel{\text{def}}{=} [\![a_1]\!]_{\mathit{PreStep}^*} \cdots [\![a_n]\!]_{\mathit{PreStep}^*}$$

$$[\![e + \varepsilon]\!]_{\mathit{PreStep}^*} \stackrel{\text{def}}{=} [\![e]\!]_{\mathit{PreSteps}} \cup \{\varepsilon\}$$

$$[\![E^*]\!]_{\mathit{PreStep}^*} \stackrel{\text{def}}{=} \Big(\bigcup_{e \in E} [\![e]\!]_{\mathit{PreSteps}} \Big)^*$$

Example

$c = (q_0, 1, 0, 1)$



$c' = (q_1, 2, 2, 1)$

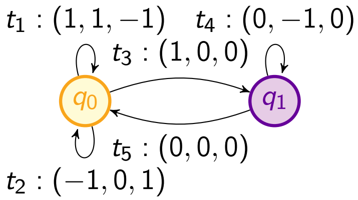
Any sequence of transitions in

$$\{t_1 t_2, t_2 t_1\}^{n+2} t_3 t_4^n$$

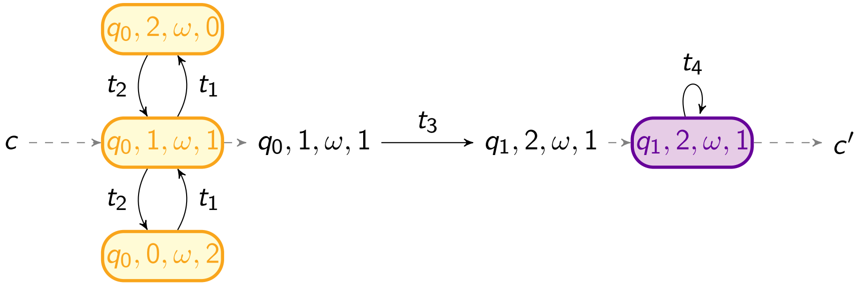
for $n \geq 0$ provides runs in $Runs(c, c')$.

Example

$c = (q_0, 1, 0, 1)$



$c' = (q_1, 2, 2, 1)$



$$\downarrow Runs(c, c') \;=\; [\![c, E_0^* \cdot (e_1 + \varepsilon) \cdot E_1^*, c']\!]_{PreRuns}$$

The KLMST Algorithm

Theorem (Leroux & Schmitz 2015)

The KLMST algorithm computes an ideal decomposition of $\downarrow Runs(c, c')$.

$\Longrightarrow$ the decomposition of $\downarrow Runs(c, c')$ into maximal ideals is effectively computable.

Applications

Once the decomposition of $\downarrow Runs(c, c')$ into ideals is computed:

- The reachability problem reduces to the emptyness of the decomposition.
- Provide a way to compute the downward-closure of the set of words of transitions from c to c', first proved in [Habermehl & Meyer & Wimmel 2012][Zetzsche 2015].

Complexity

- The ideal decomposition of $\downarrow Runs(c, c')$ is at least Ackermannian. We exhibit in [Leroux & Schmitz 2015] a cubic-Ackermannian upper-bound.
- The reachability problem may have a better complexity. The best lower bound in exponential space [Cardoza & Lipton & Meyer 1976].

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Overview

Coverability tree algorithm Karp and Miller	The KLMST algorithm Mayr, Kosaraju, and Lambert
configuration ideals	run ideals
$\downarrow$ reachability set	$\downarrow$ runs

VASS Extensions

The ideal framework provides abstract fundations for generalizing classical algorithms to VASS extensions. The coverability tree construction has been recently extended to:

- Unordered data Petri nets [Hofman & Lasota & Lazić & Leroux & Schmitz & Totzke 2016]
- Branching VASS [Verma & Goubault-Larrecq 2005],[Jacobé de Naurois 2014].
- Pushdown VASS [Leroux & Praveen & Sutre 2014].

Other Applications

Other recent applications of wqo ideals:

- Lazić and Schmitz in 2015 revisited the backward coverability algorithm for VASS.
- Use of ideal decompositions for computing the downward-closure of formal languages by Zetzsche in 2015.
- Decidability of separation by piecewise testable languages by Czerwiński, Martens, van Rooijen, Zeitoun, and Zetzsche in 2015.