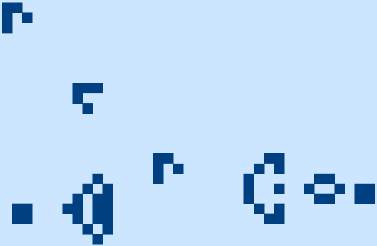
Tutorial on Cellular Automata

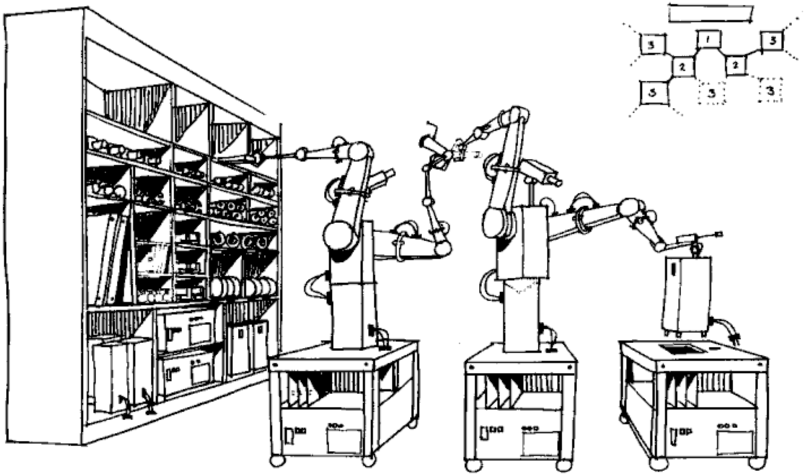
Nicolas Ollinger

LIF, Aix-Marseille Université, CNRS

UC 2011, Turku — June 2011



Theory of self-reproducing automata



Cellular automata emerged in the late 40s from the work of Ulam and von Neumann.

Cellular Automata

A **cellular automaton** (**CA**) is a discrete dynamical model.

Space is discrete and consists of an infinite regular grid of cells. Each cell is described by a **state** among a common **finite set**.

Time is discrete. At each clock tick cells change their state **deterministically**, **synchronously** and **uniformly** according to a common **local update rule**.

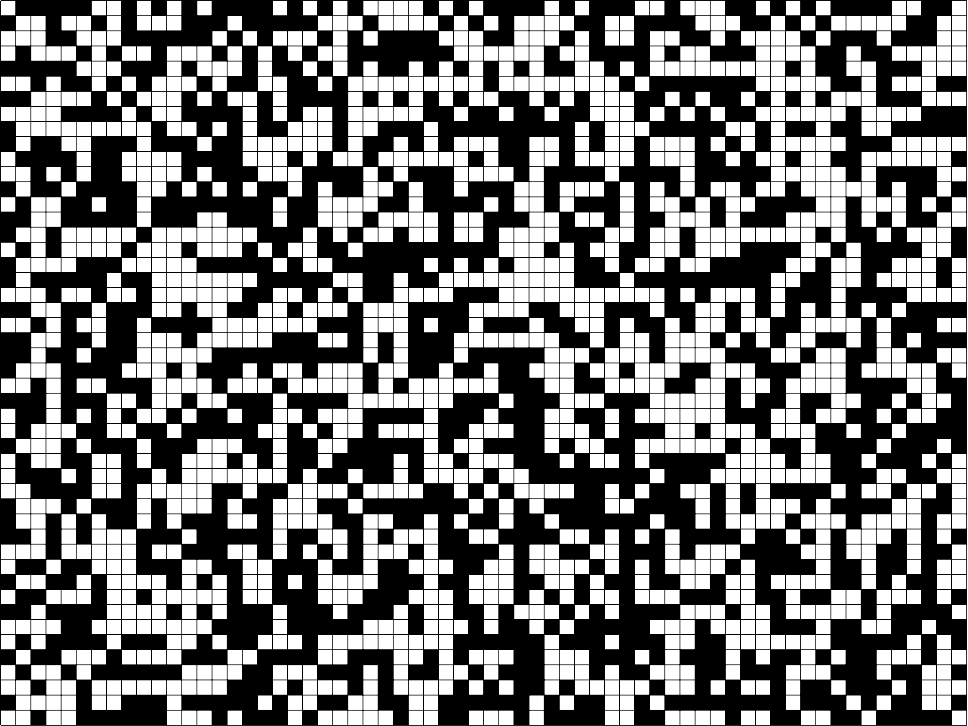
Conway's famous Game of Life

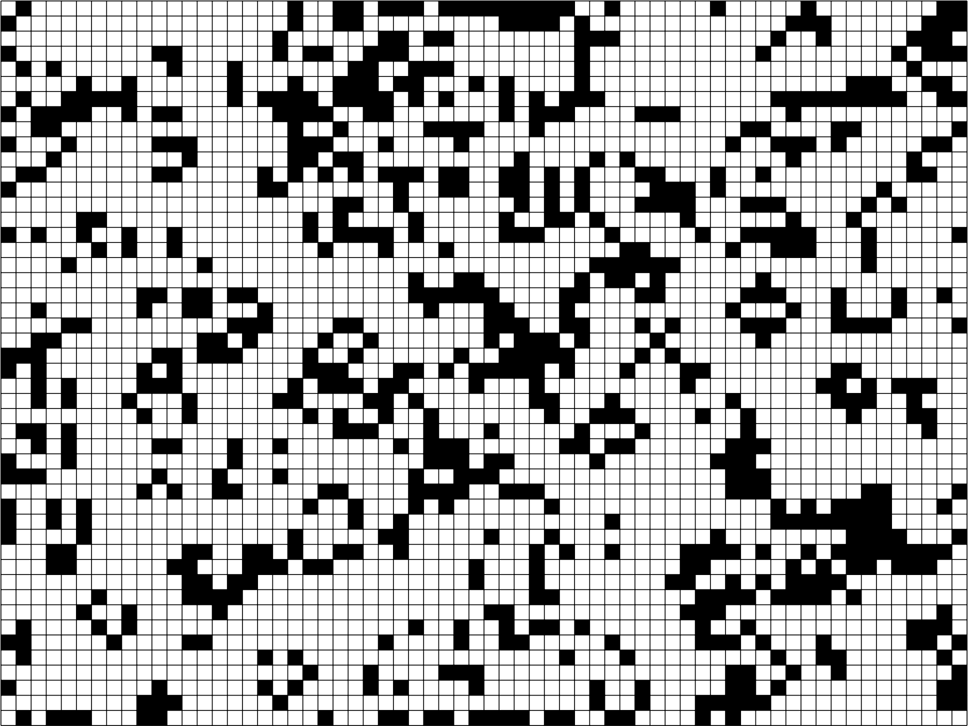
The **Game of Life** is a 2D CA invented by Conway in 1970.

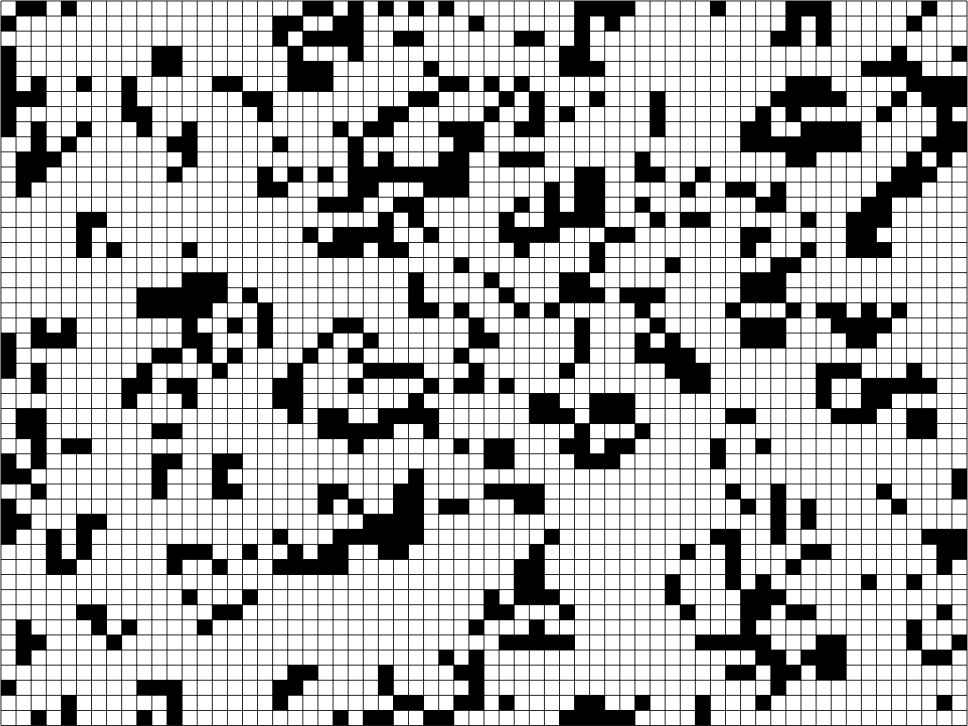
Space is an infinite chessboard of **alive** or **dead** cells.

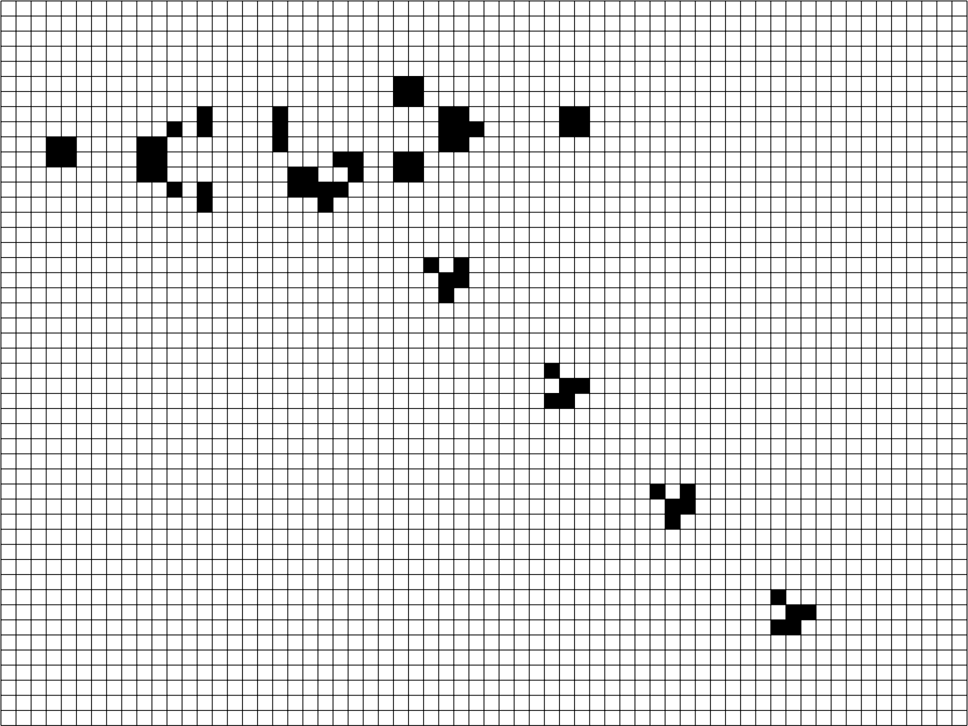
The **local update rule** counts the number of alive cells among the eight surrounding cells :

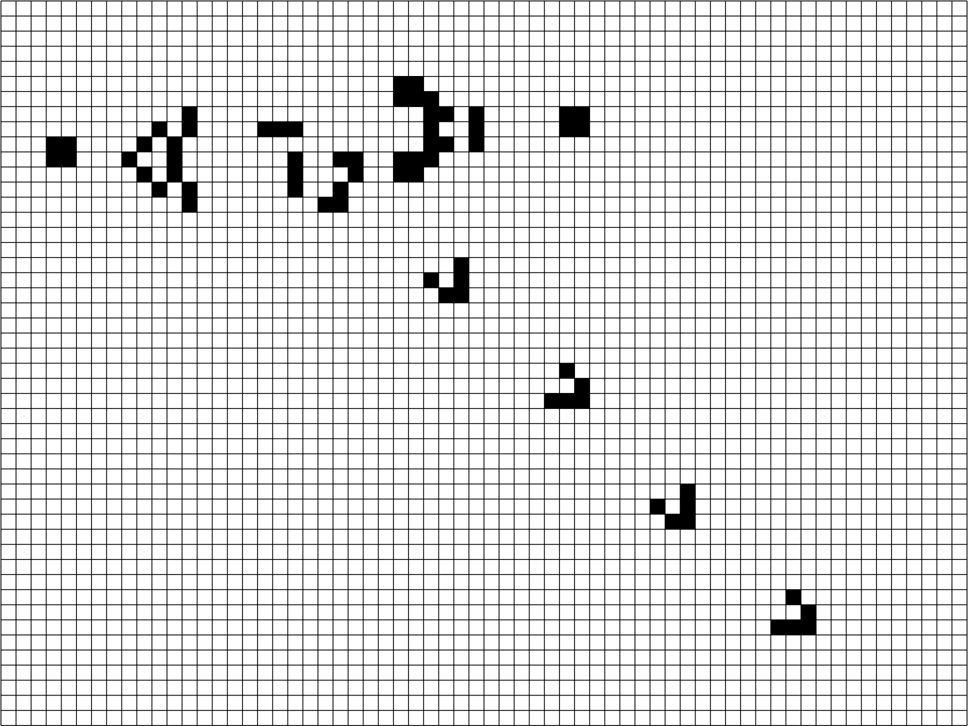
- **exactly three** alive cells give **birth** to dead cells ;
- **less than three** alive cells kill by **loneliness** ;
- **more than four** alive cells kill by **overcrowding** ;
- otherwise the cell remains in the same state.

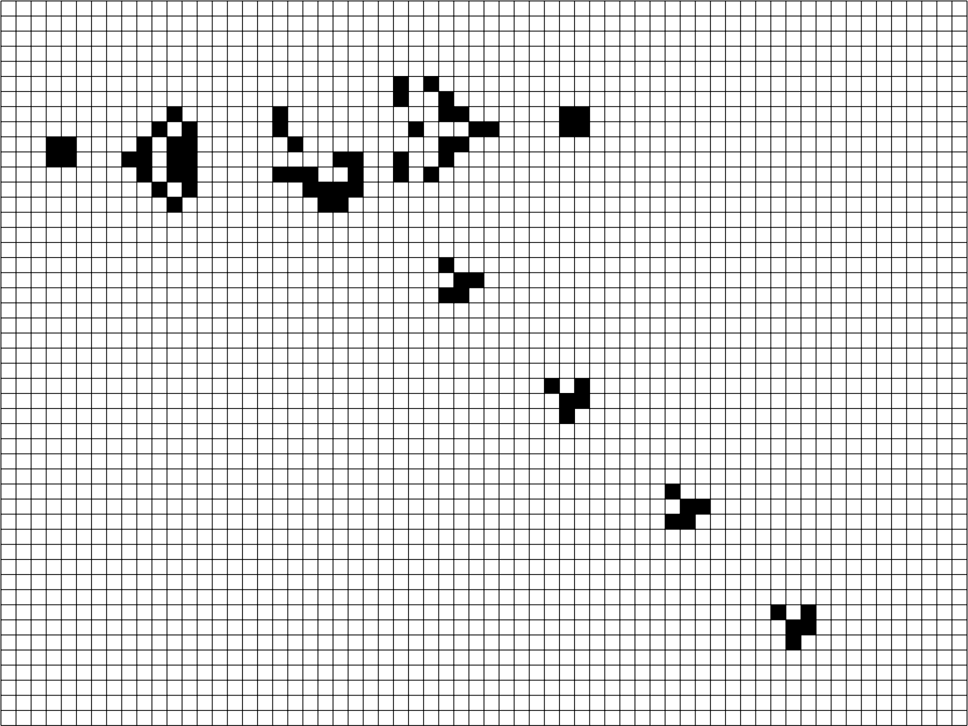


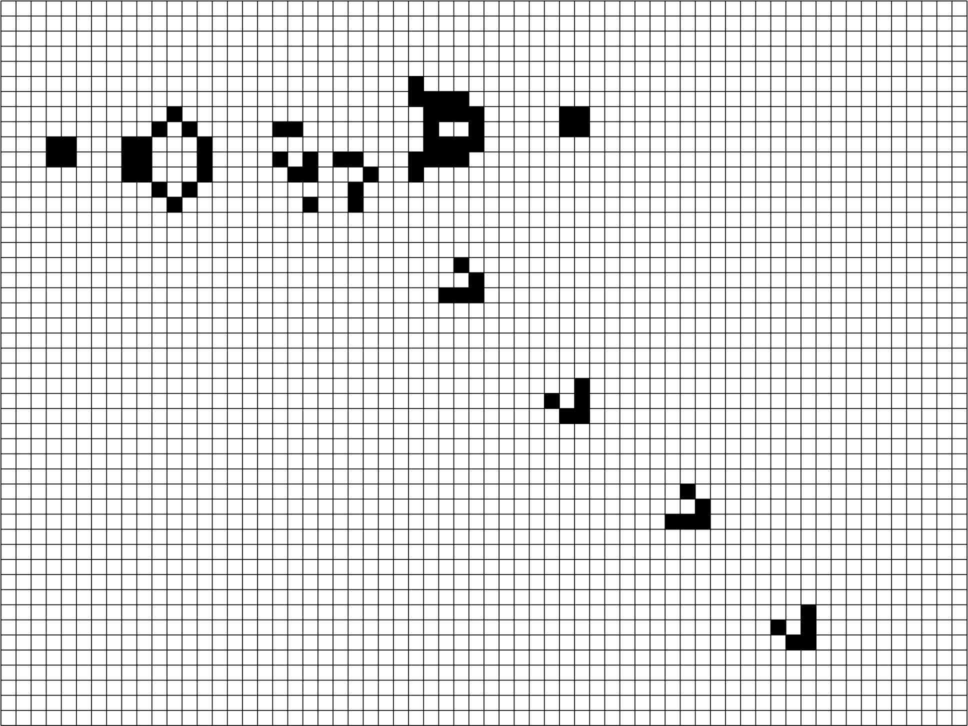


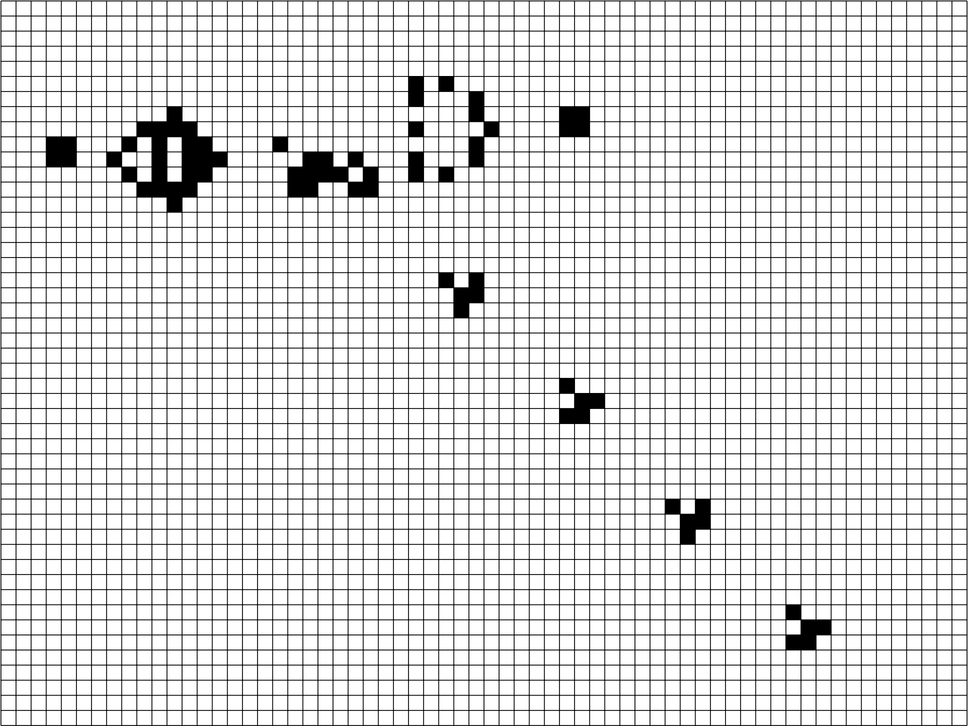


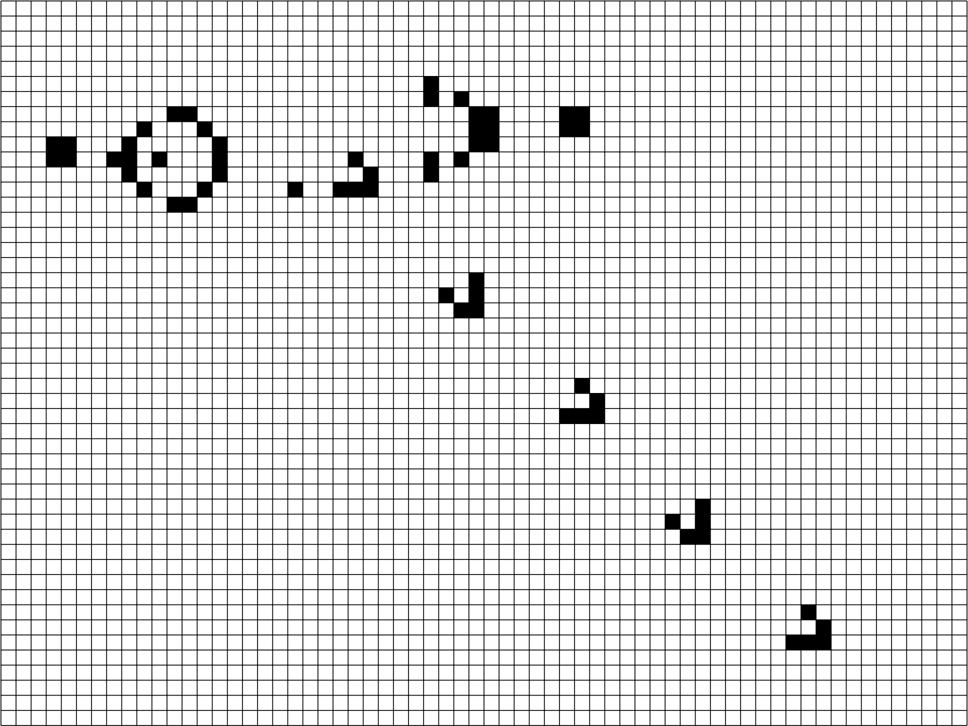


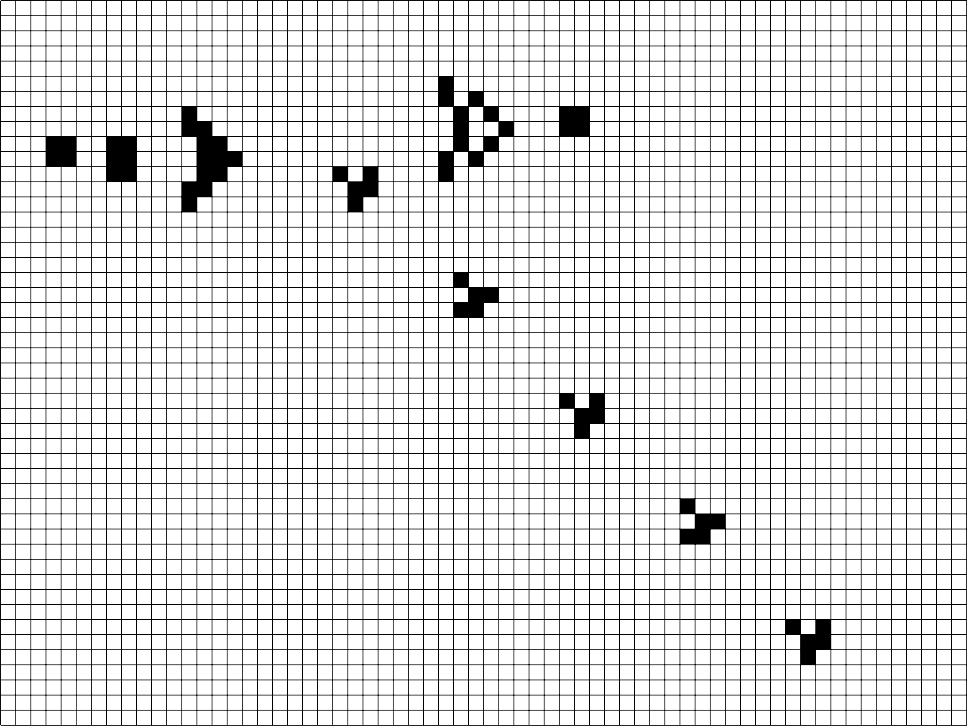


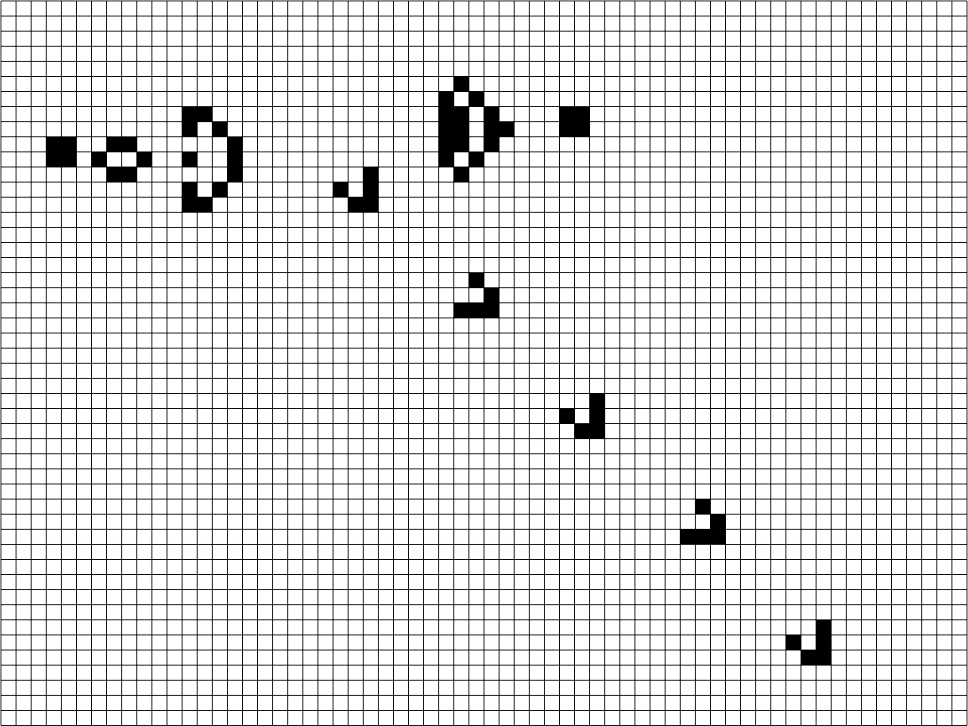








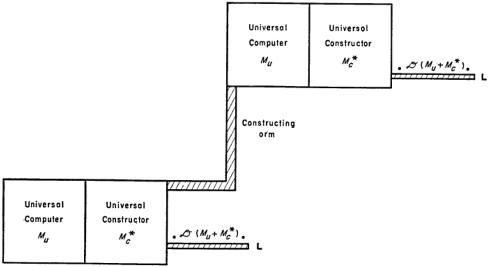




von Neumann self-reproducing CA

Class	Name	Symbol	Number	Summary of transition rule
Ordinary	Unexcitable	**U**	1	Direct process changes **U** into sensitized states and then into $T_{u\alpha 0}$ or C_{00}. Reverse process kills $T_{u\alpha\epsilon}$ or $C_{\epsilon\epsilon'}$ into **U**.
Ordinary	Confluent	$C_{\epsilon\epsilon'}$: $\begin{cases}\epsilon = 0 \text{ (quiescent)} \\ \epsilon = 1 \text{ (excited)}\end{cases}$ $\begin{cases}\epsilon' = 0 \text{ (next quiescent)} \\ \epsilon' = 1 \text{ (next excited)}\end{cases}$	4	Receives conjunctively from $T_{0\alpha\epsilon}$ directed toward it; emits with double delay to all $T_{u\alpha\epsilon}$ not directed toward it. Killed to **U** by $T_{1\alpha 1}$ directed toward it; killing dominates reception.
Ordinary	Transmission ($T_{u\alpha\epsilon}$)	$T_{0\alpha\epsilon}$: $\overset{0}{\rightarrow}$, $\uparrow 0$, $\overset{0}{\leftarrow}$, $\downarrow 0$. *Class:* $u = 0$ (ordinary), $u = 1$ (special). *Excitation:* $\epsilon = 0$ (quiescent), $\epsilon = 1$ (excited)	8	Receives disjunctively from $T_{0\alpha\epsilon}$ directed toward it and from $C_{\epsilon\epsilon'}$; emits in output direction with single delay (a) to $T_{0\alpha\epsilon}$ not directed toward it and to $C_{\epsilon\epsilon'}$ (b) to **U** or sensitized states by direct process (c) to kill $T_{1\alpha\epsilon}$ by reverse process. Killed to **U** by $T_{1\alpha 1}$ directed toward it; killing dominates reception.
Special	Transmission ($T_{u\alpha\epsilon}$)	$T_{1\alpha\epsilon}$: $\overset{1}{\rightarrow}$, $\uparrow 1$, $\overset{1}{\leftarrow}$, $\downarrow 1$. *Output direction:* $\alpha = 0$ (right), $\alpha = 1$ (up), $\alpha = 2$ (left), $\alpha = 3$ (down)	8	Receives disjunctively from $T_{1\alpha\epsilon}$ directed toward it and from $C_{\epsilon\epsilon'}$; emits in output direction with single delay (a) to $T_{1\alpha\epsilon}$ not directed toward it (b) to **U** or sensitized states by direct process (c) to kill $T_{0\alpha\epsilon}$ or $C_{\epsilon\epsilon'}$ by by reverse process. Killed to **U** by $T_{0\alpha 1}$ directed toward it; killing dominates reception.
	Sensitized	$S_\emptyset$ → S_0, S_1; S_0 → S_{00}, S_{01}; S_{00} → S_{000}; S_1 → S_{10}, S_{11}	8	These are intermediary states in the direct process. $T_{u\alpha 1}$ directed toward **U** converts it to $S_\emptyset$. Thereafter, S_Σ is followed by (a) $S_{\Sigma 1}$ if some $T_{u\alpha 1}$ is directed toward the cell (b) $S_{\Sigma 0}$ otherwise, until the direct process terminates in a $T_{u\alpha 0}$ or C_{00}. See Figure 10.

A **29 states** CA with **von Neumann** neighborhood with wires and construction/destruction abilities.

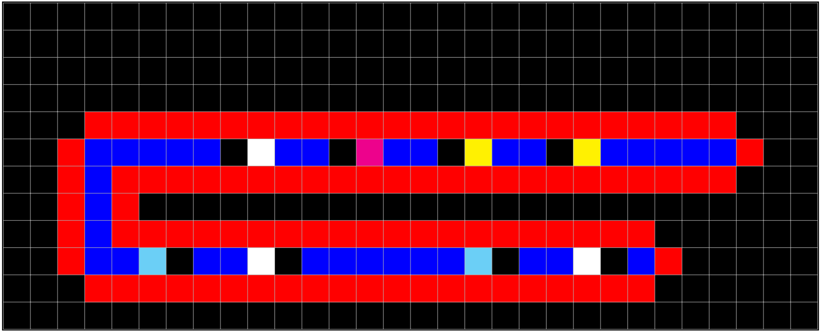


Self-reproduction using **Universal Computer** + **Universal Constructor**.

(Theory of Self-Reproducing Automata, edited by Burks, 1966)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

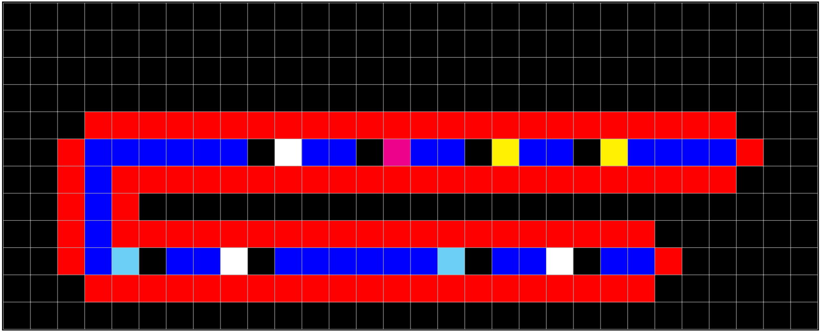


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

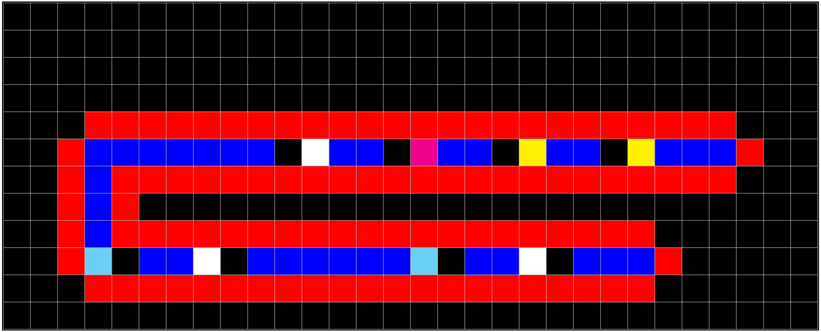


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

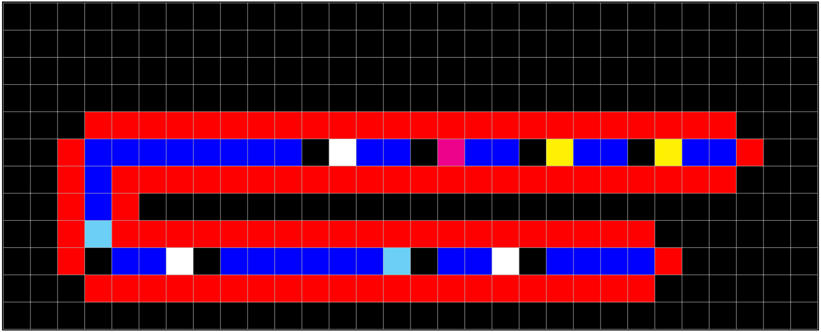


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

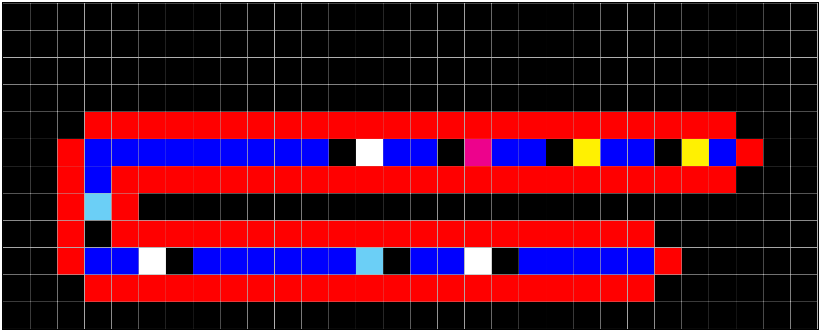


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

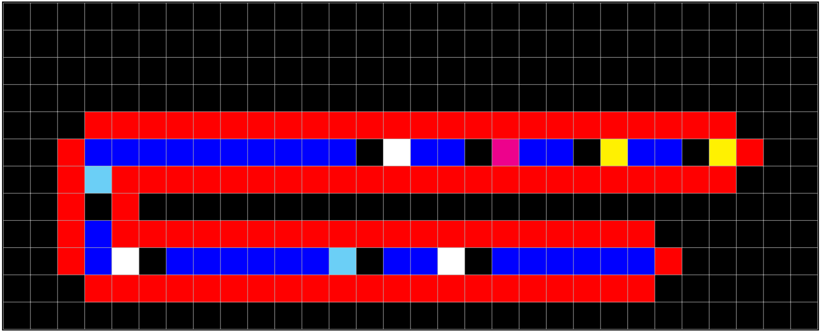


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

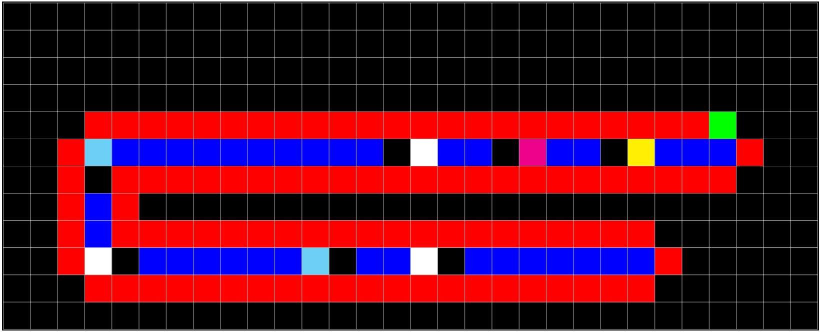


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

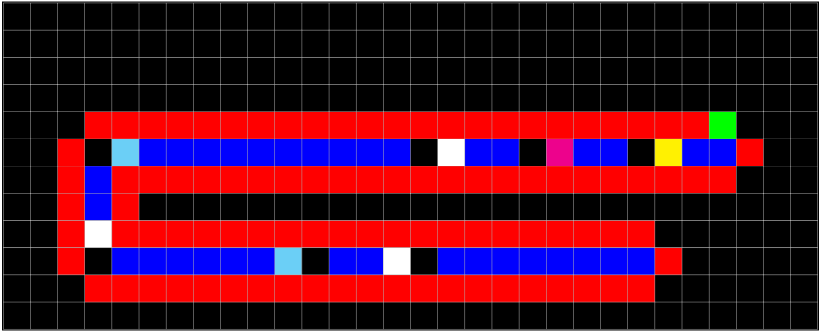


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

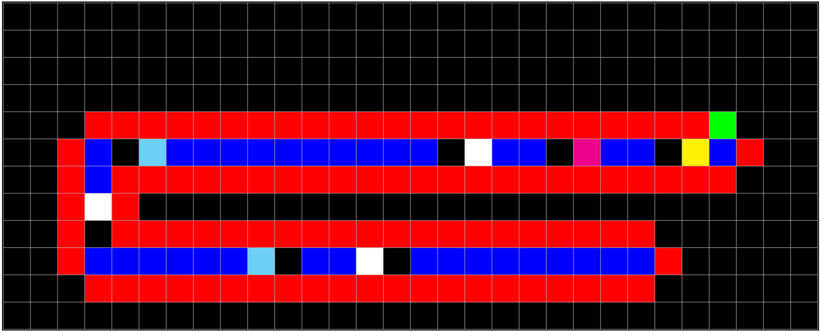


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

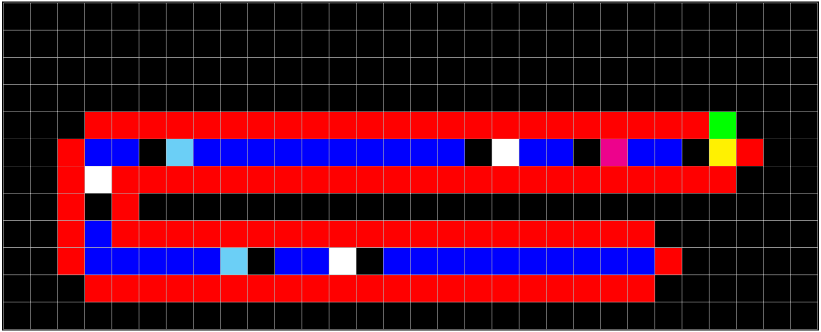


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

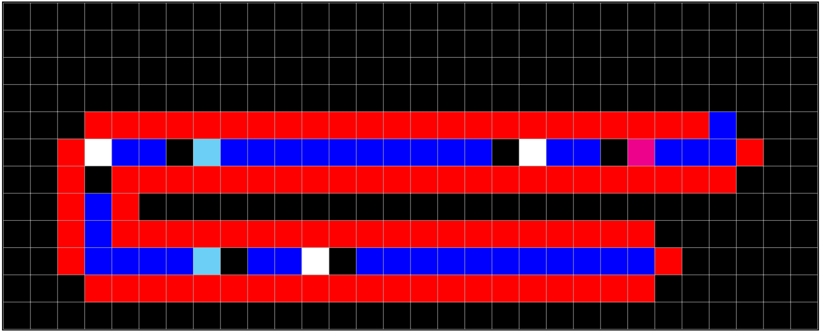


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

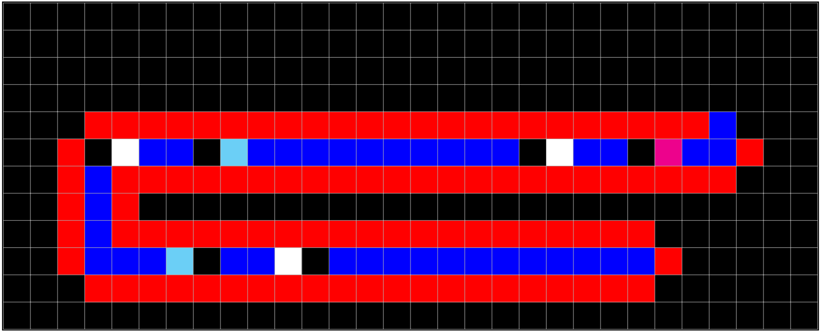


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

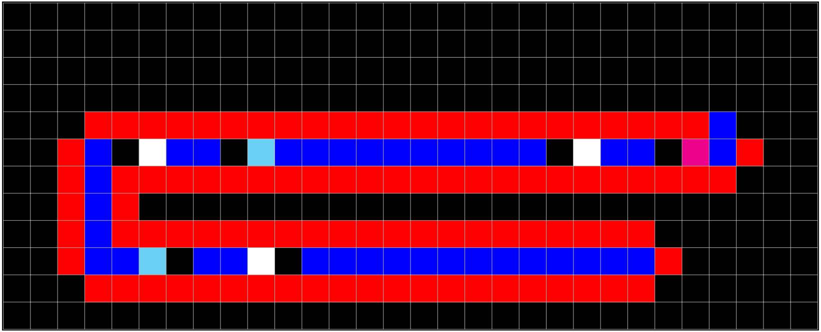


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

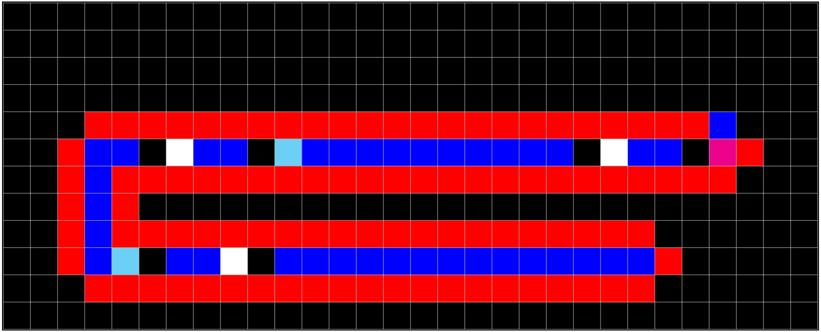


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

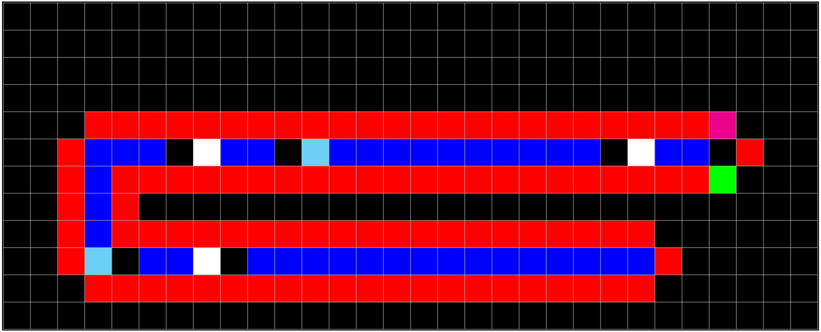


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

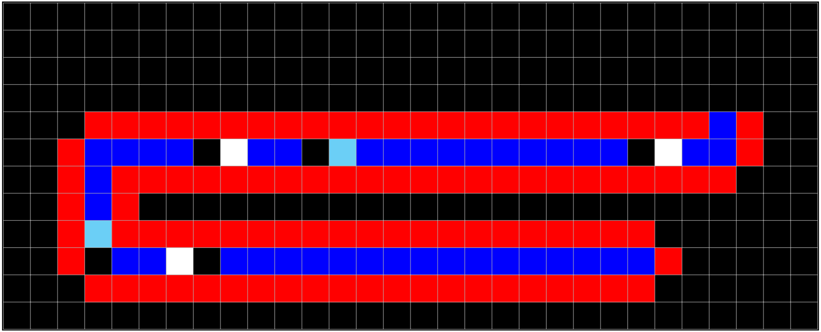


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

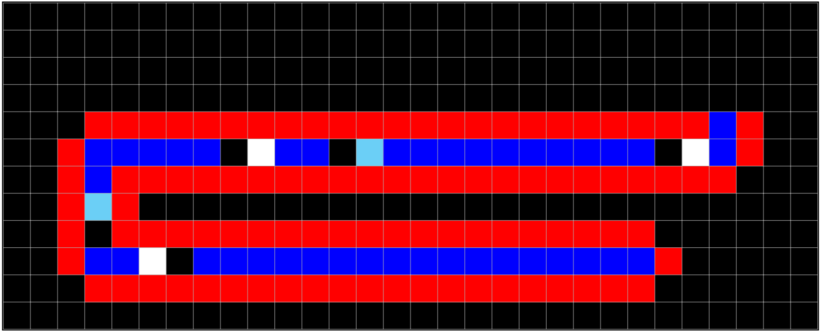


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

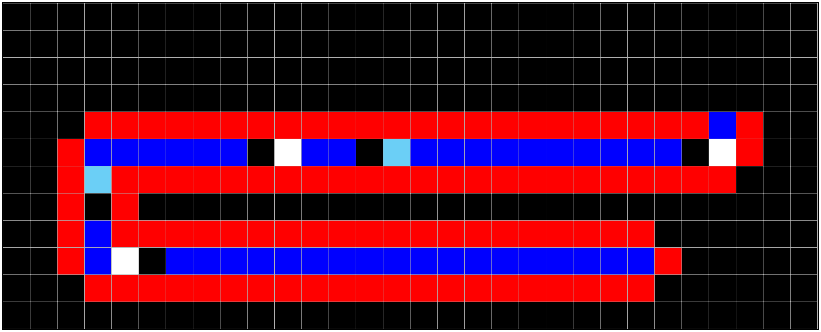


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

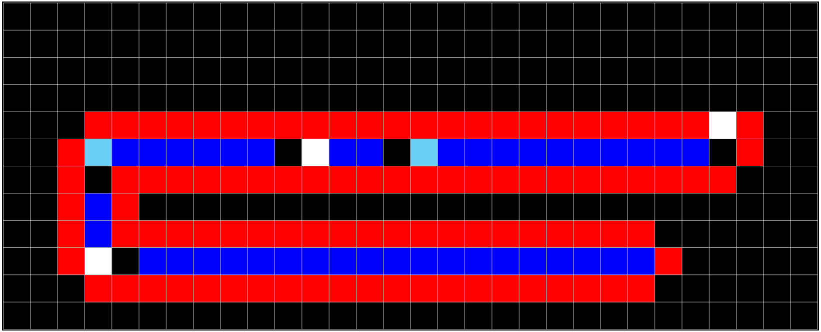


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

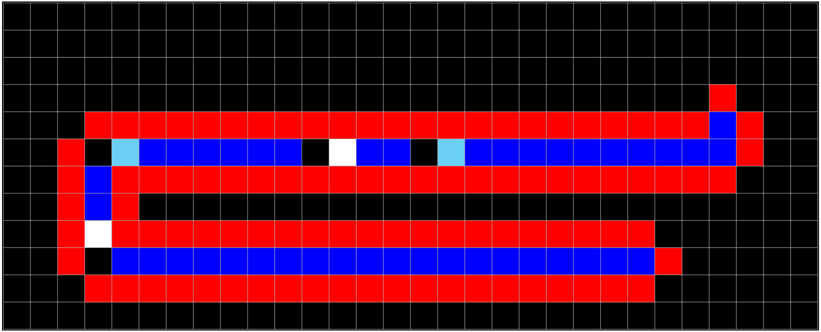


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

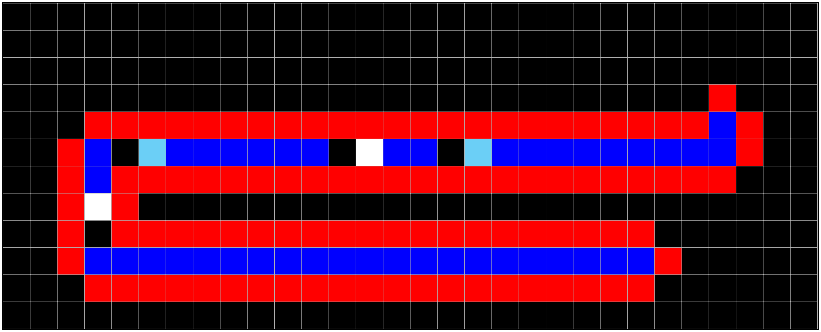


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

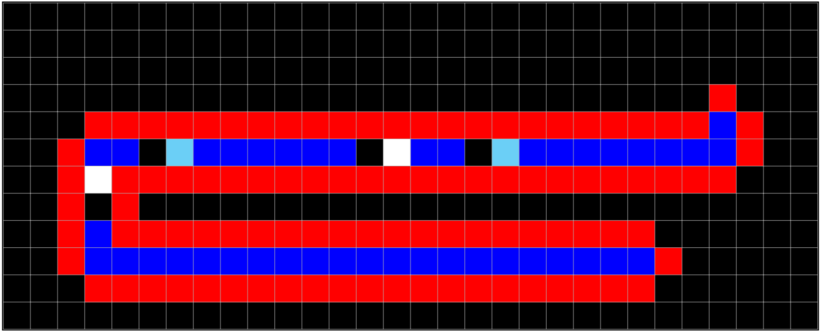


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

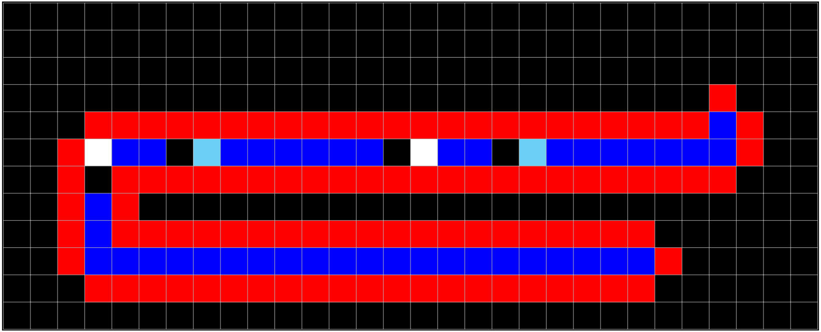


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

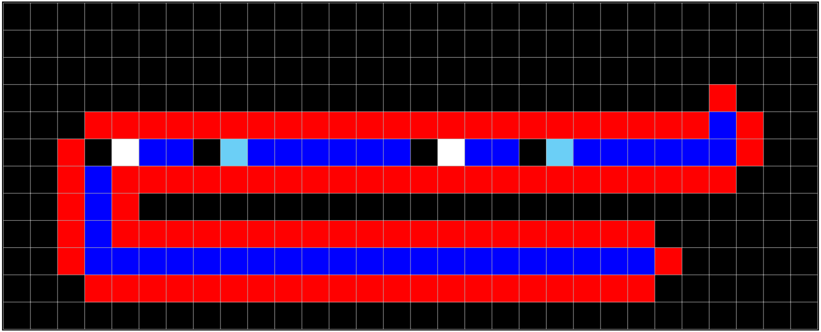


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

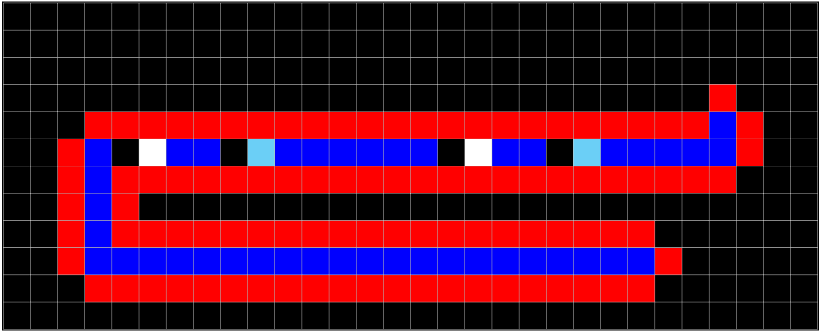


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

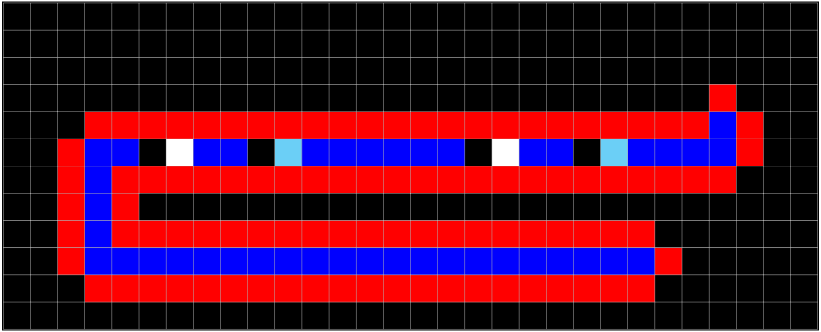


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

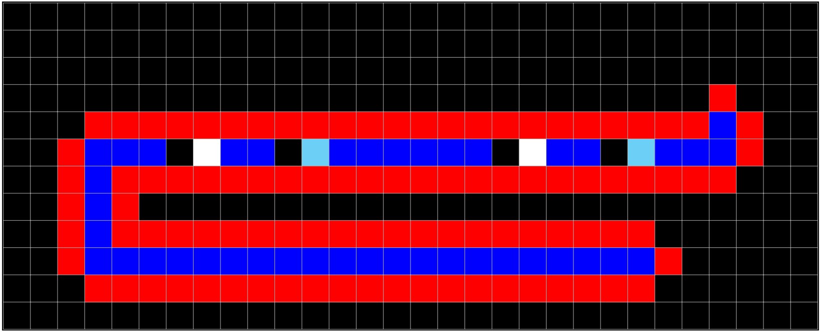


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

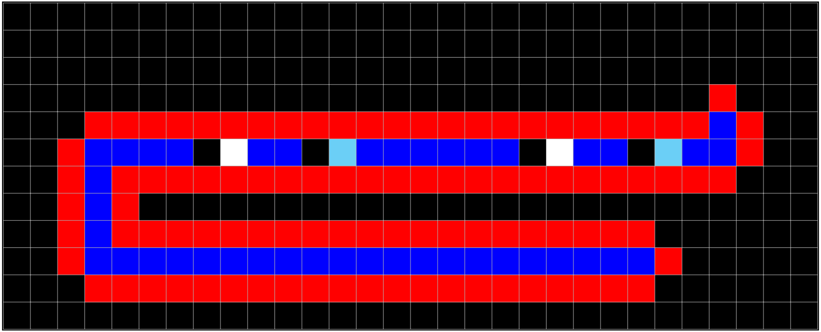


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

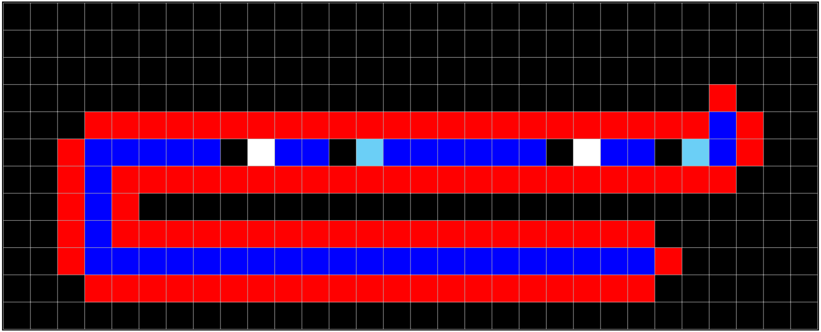


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

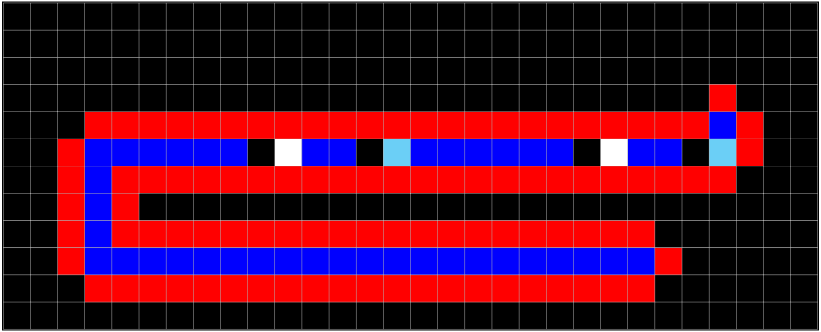


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

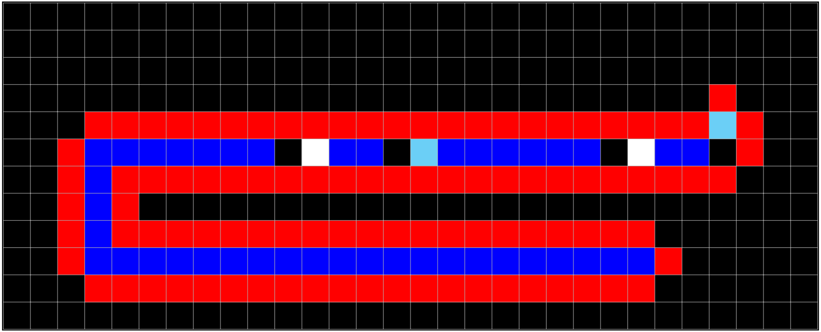


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

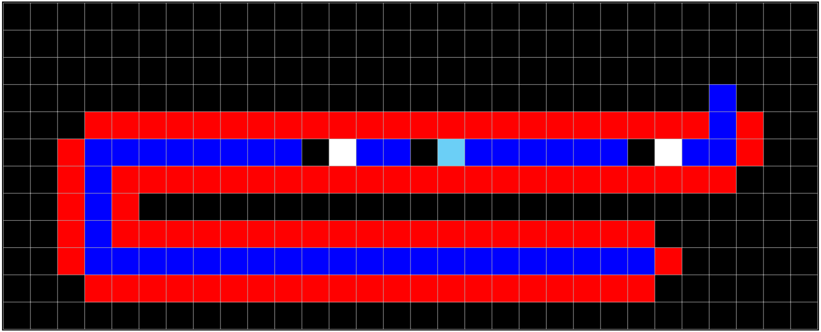


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

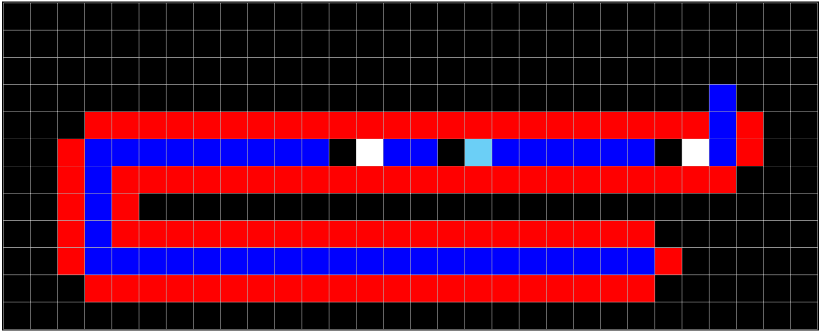


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

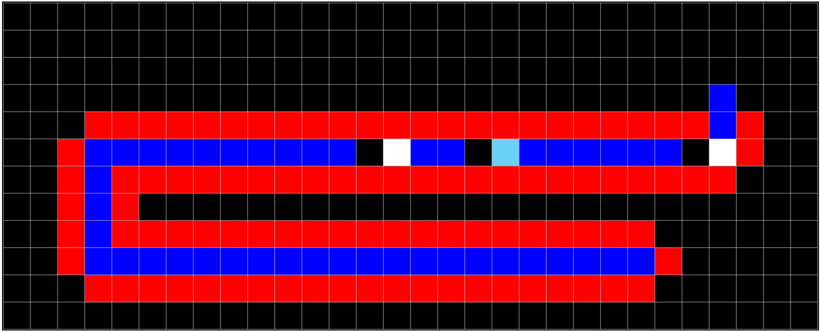


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

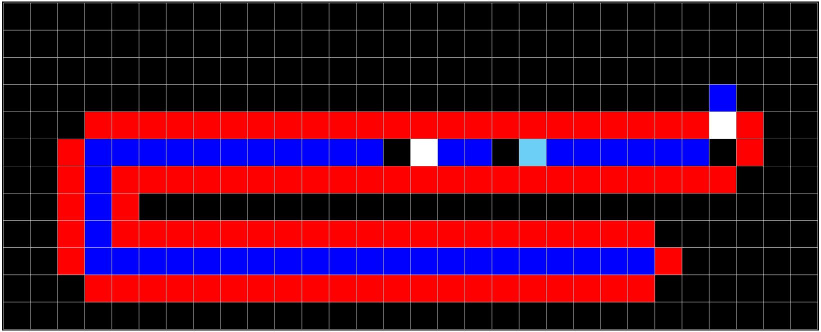


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

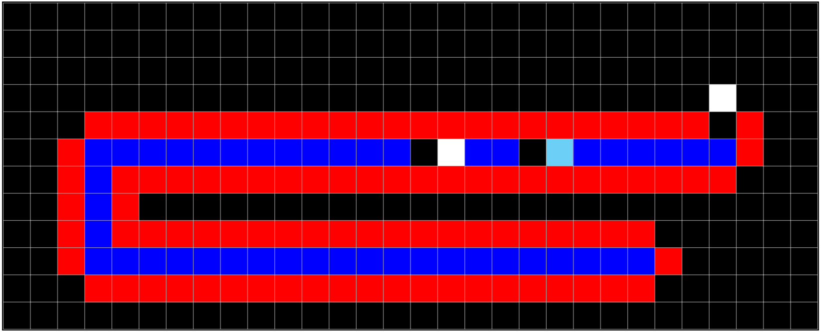


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

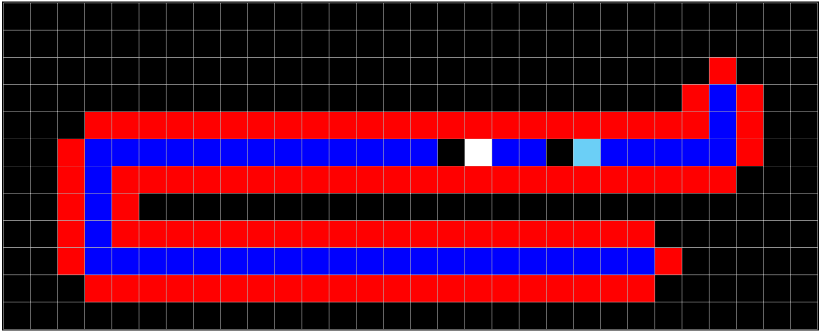


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

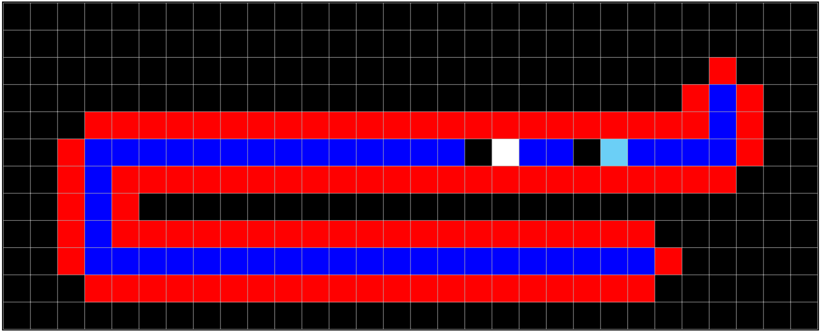


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

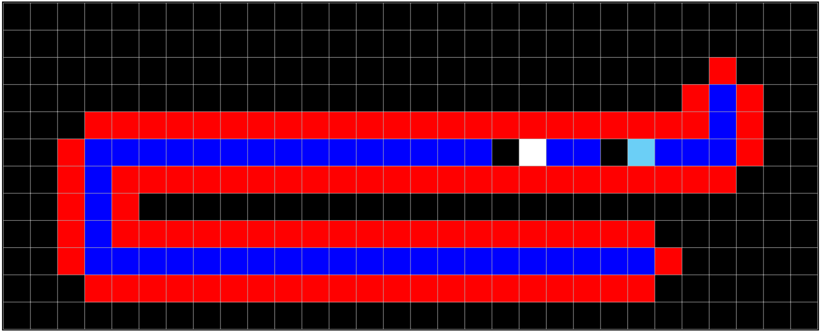


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

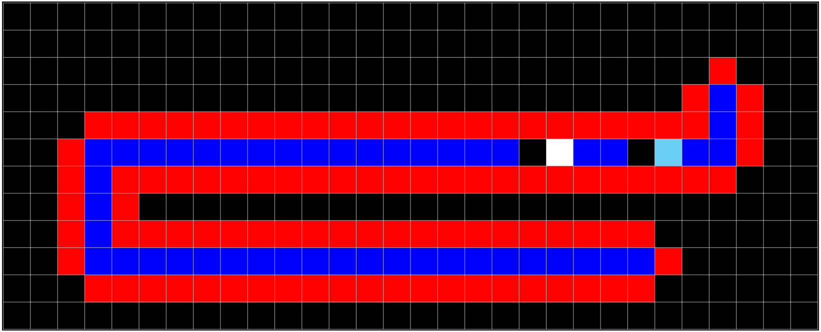


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

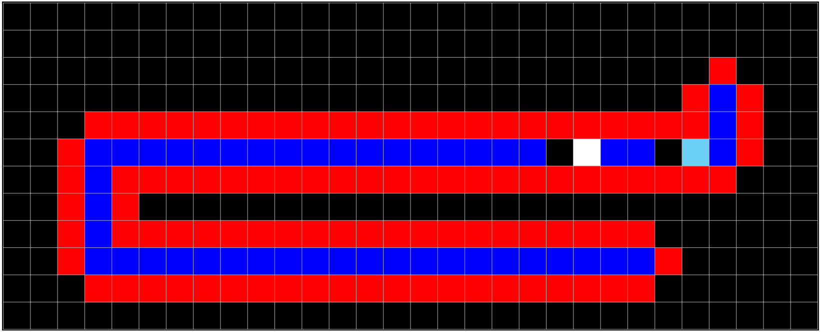


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

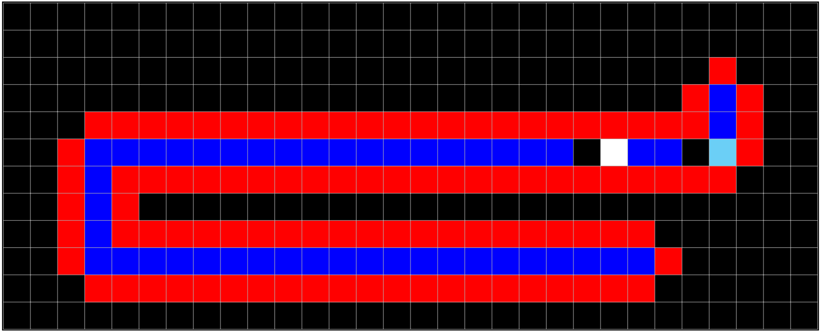


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

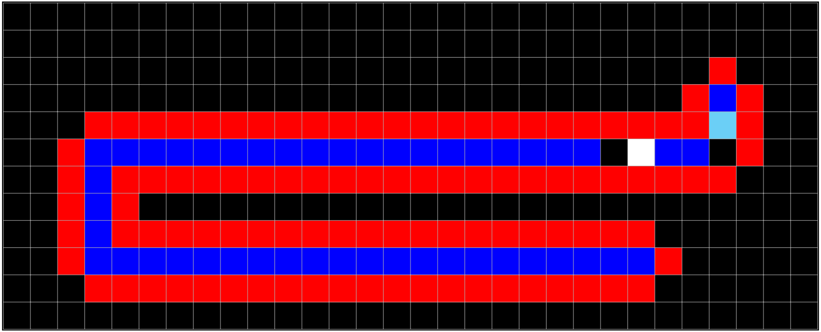


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

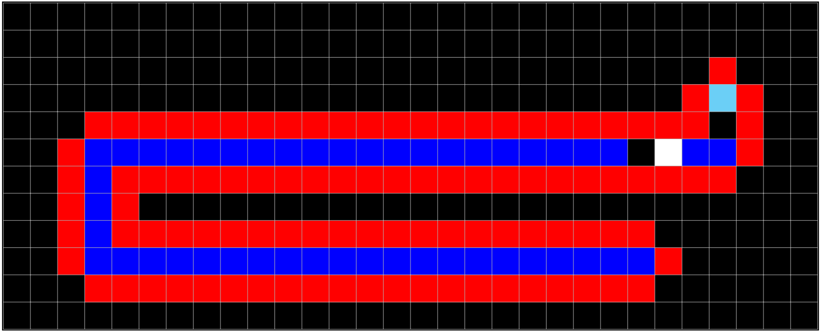


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

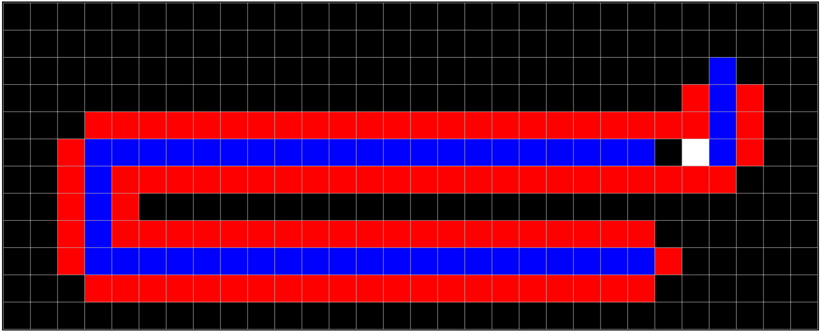


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

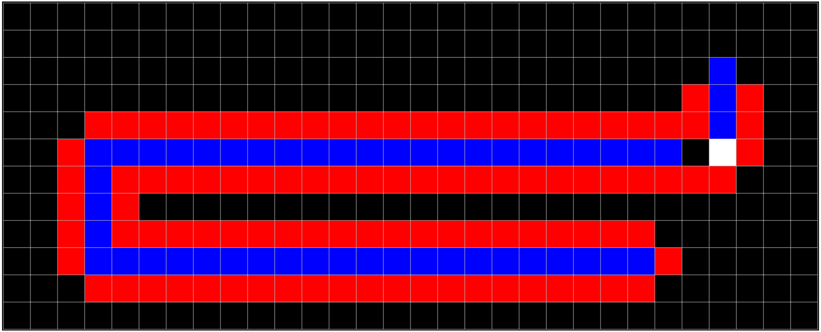


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

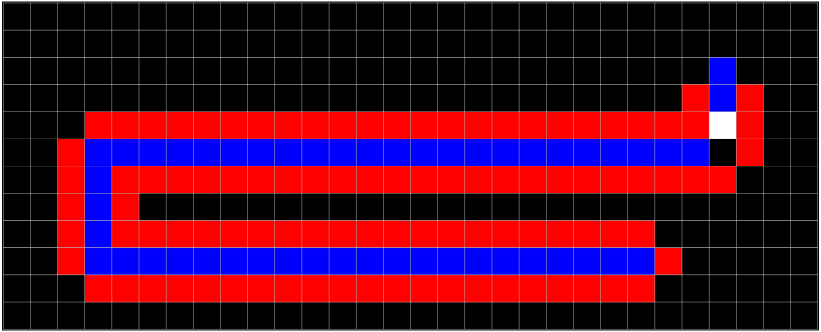


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

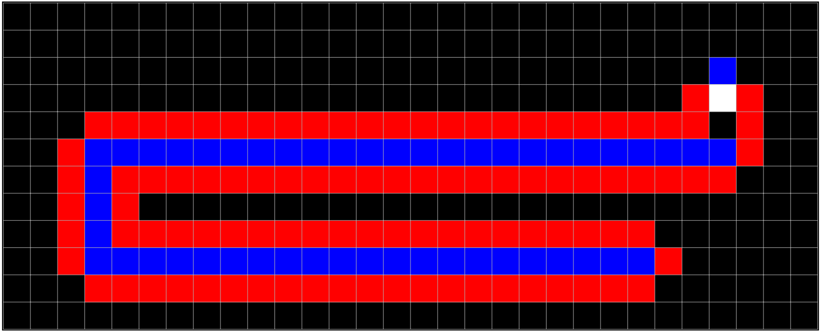


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

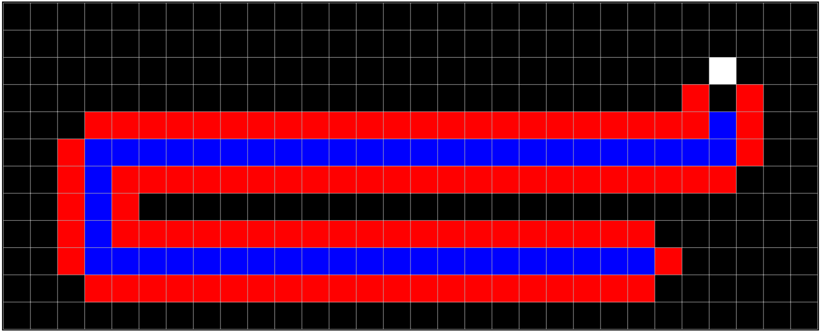


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.

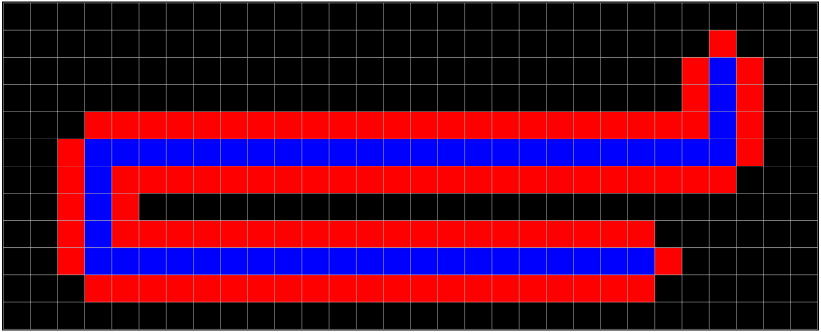


Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

Codd self-reproducing CA

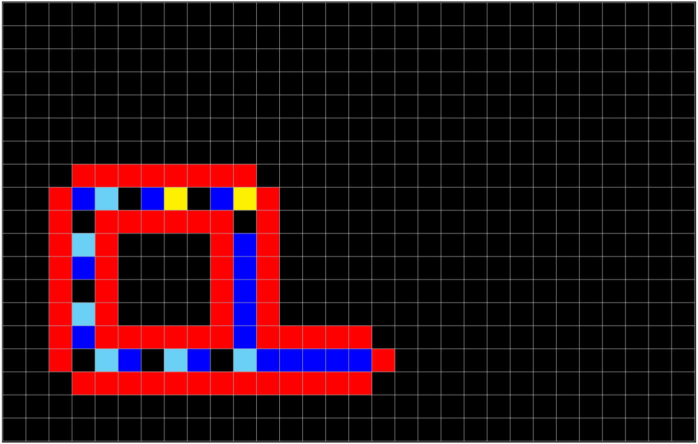
A **8 states self-reproducing** CA with von Neumann neighborhood using sheathed wires.



Implemented by Hutton in 2009, several millions cells, self-reproducing in 1.7×10^{18} steps (estimated).

(Cellular Automata, Codd, 1968)

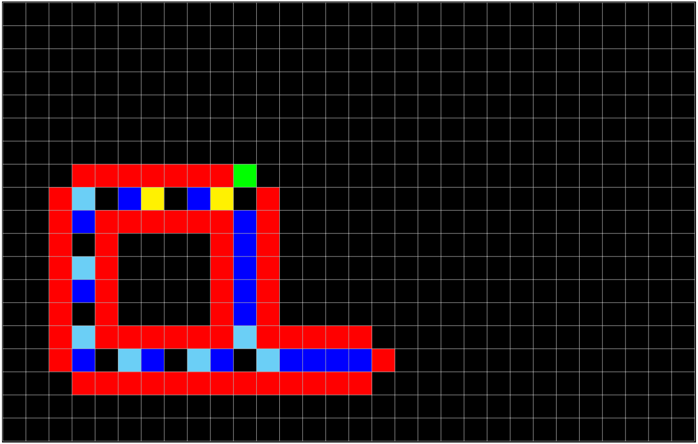
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

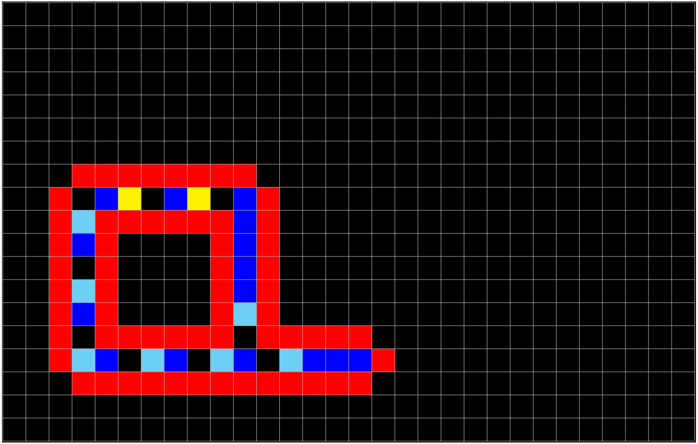
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

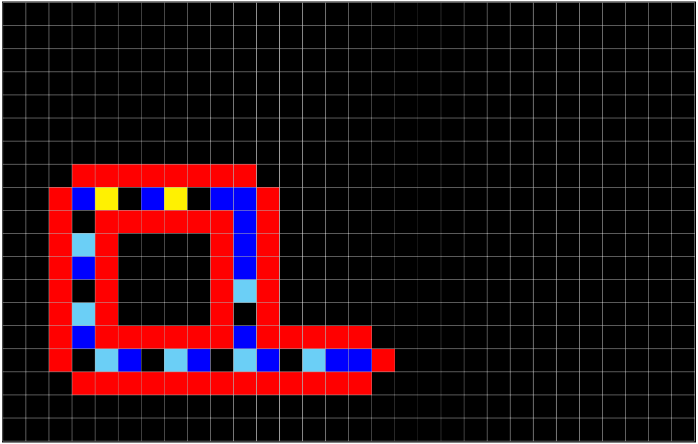
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

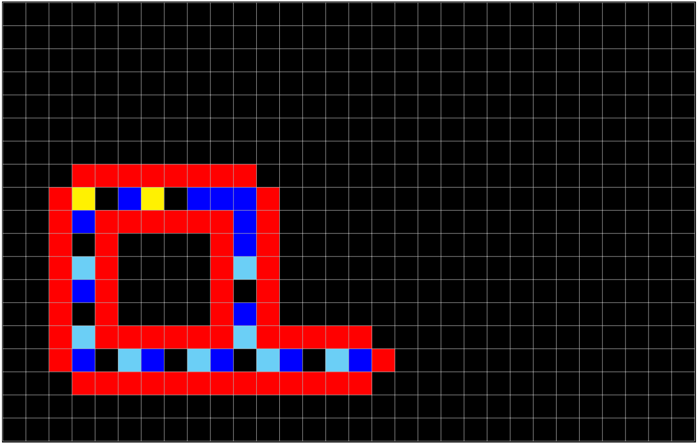
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

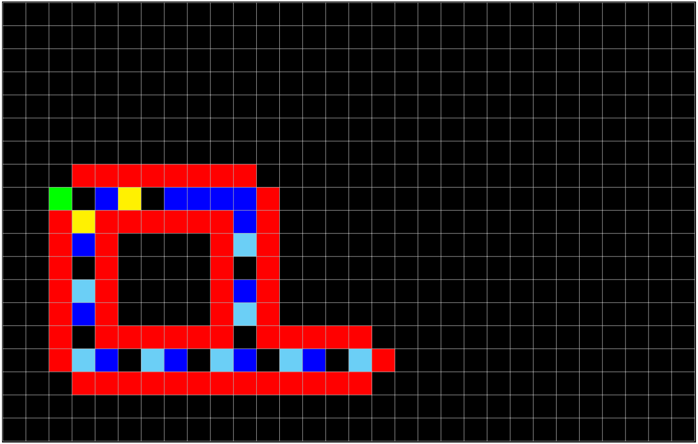
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

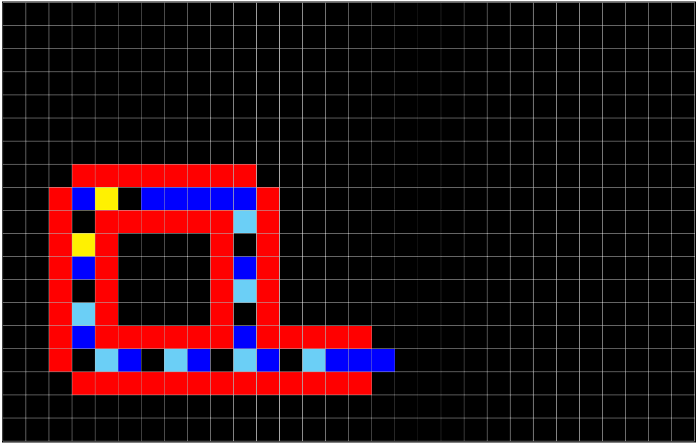
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

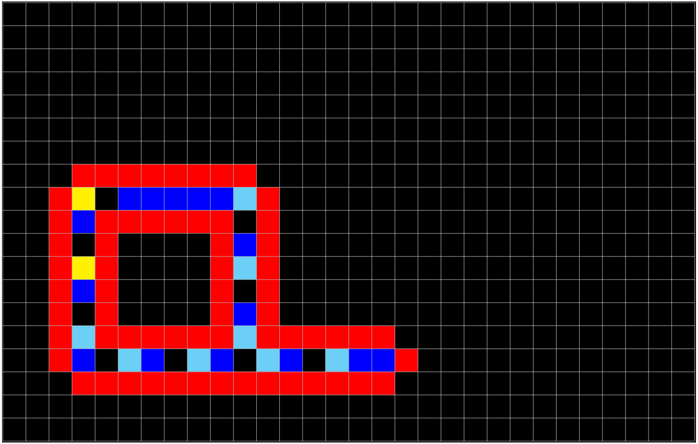
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

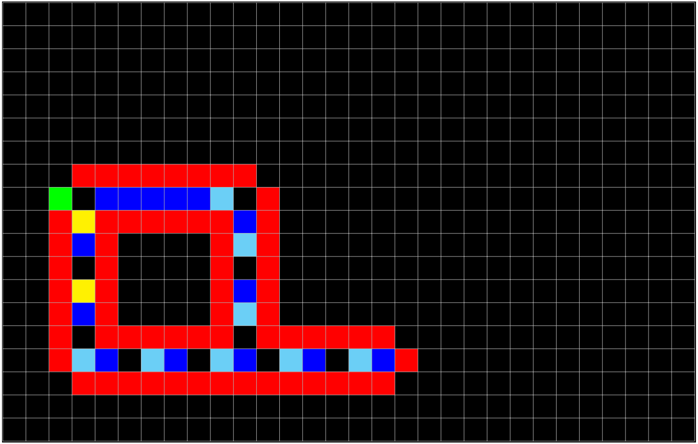
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

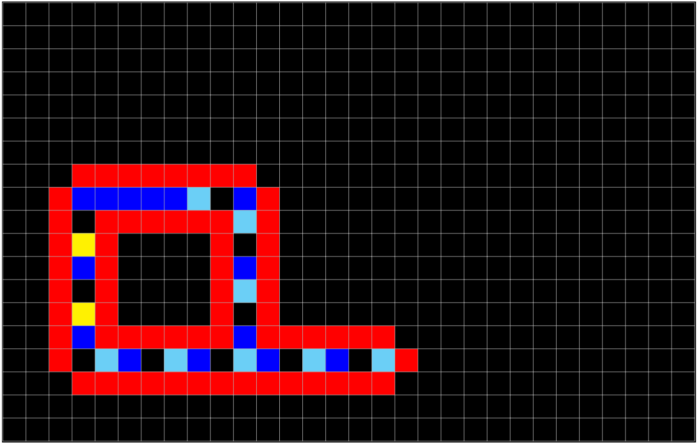
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

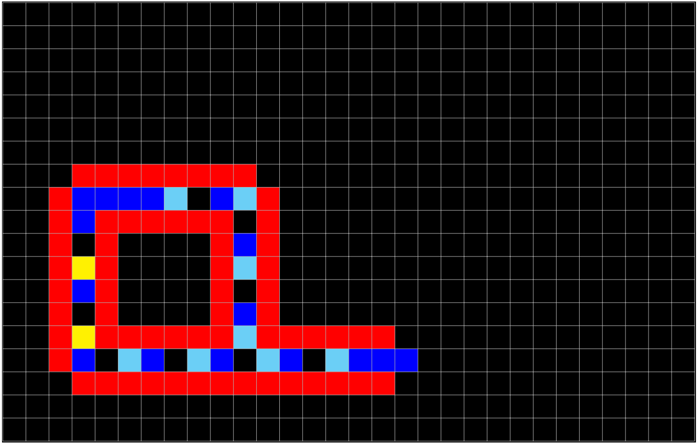
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

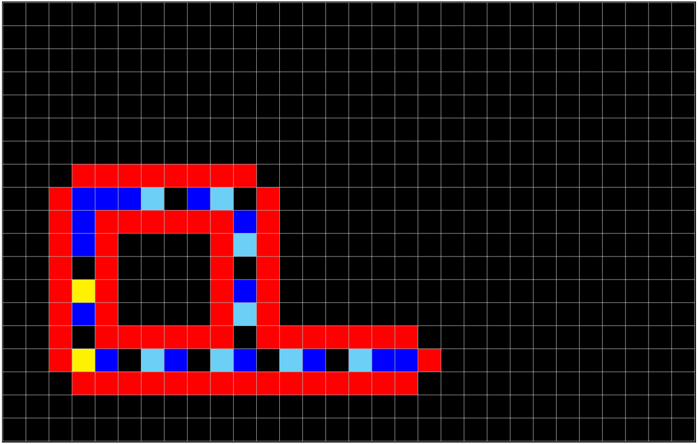
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

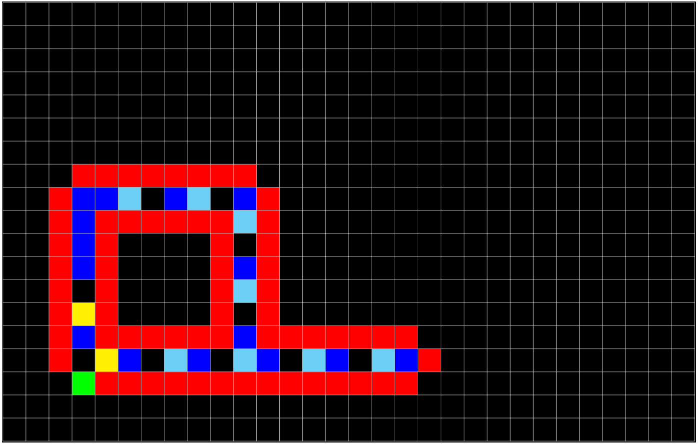
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

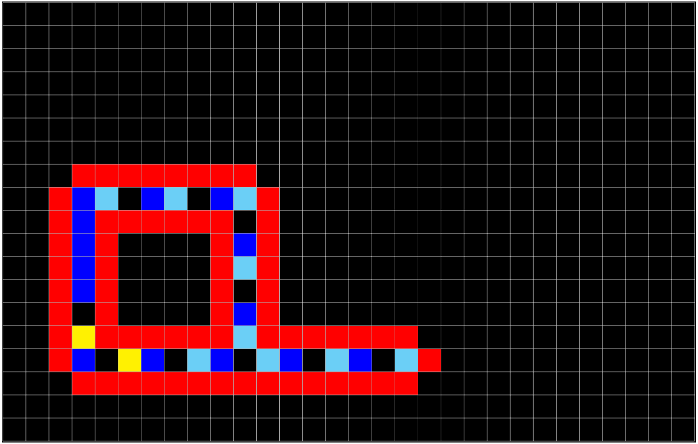
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

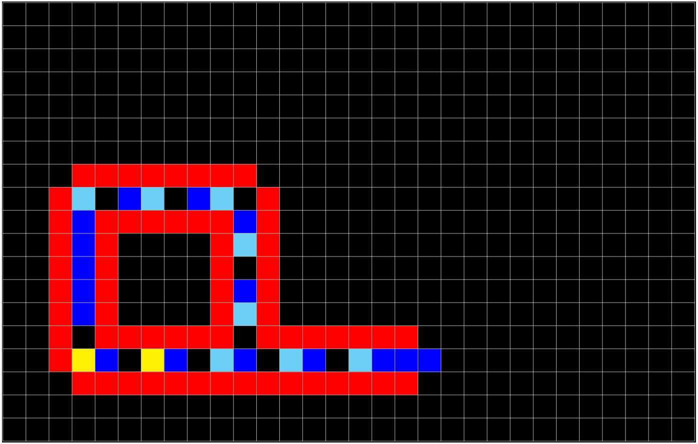
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

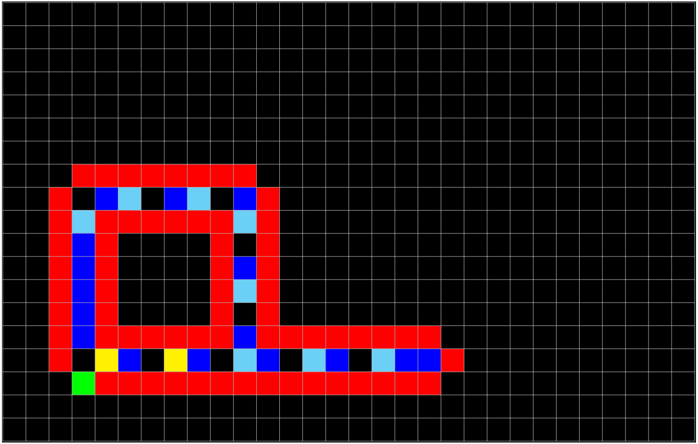
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

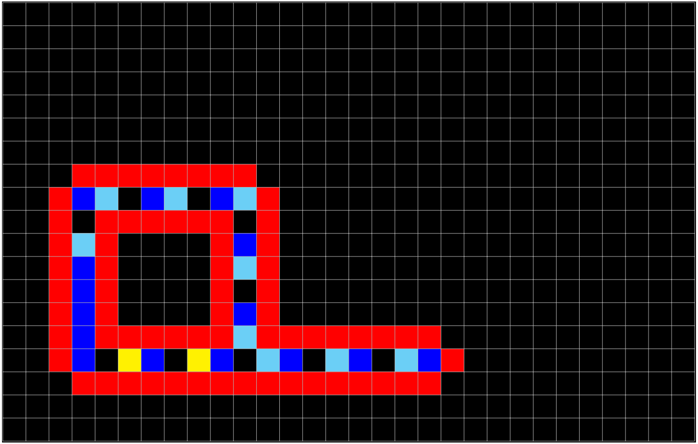
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

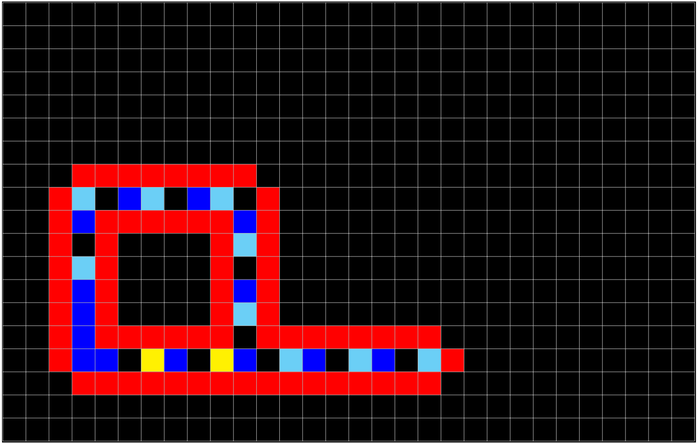
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

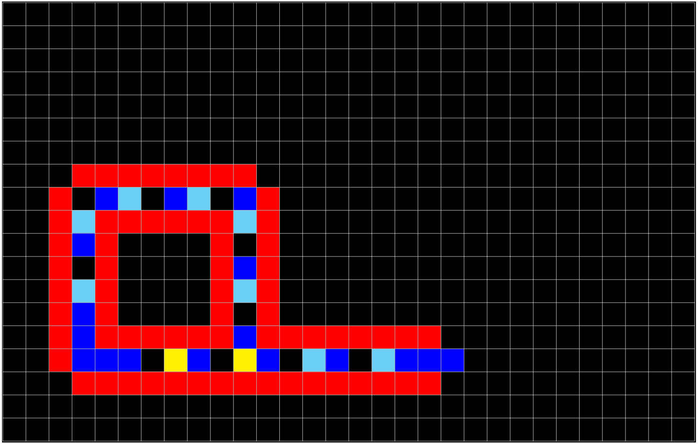
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

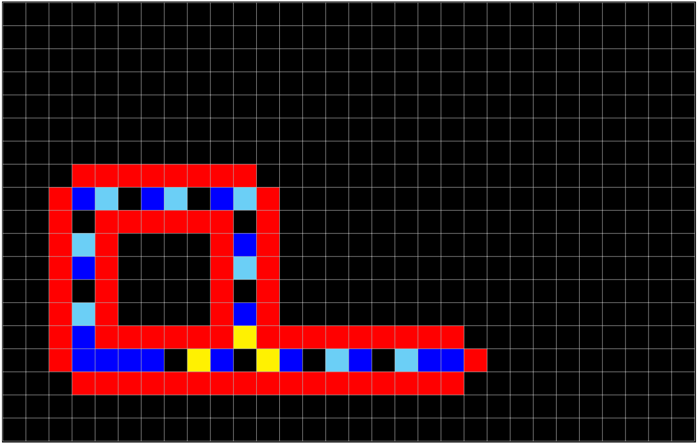
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

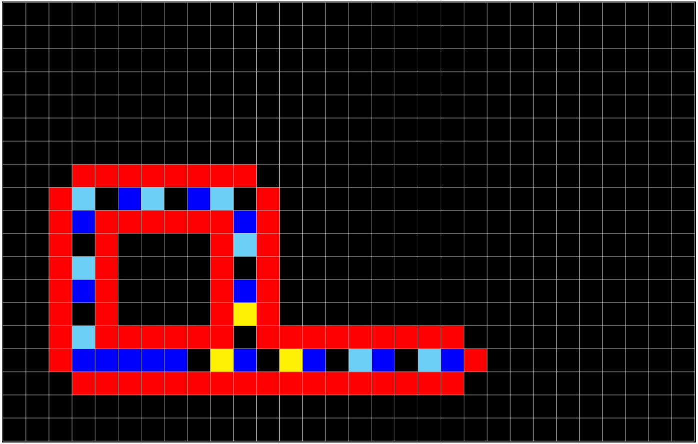
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

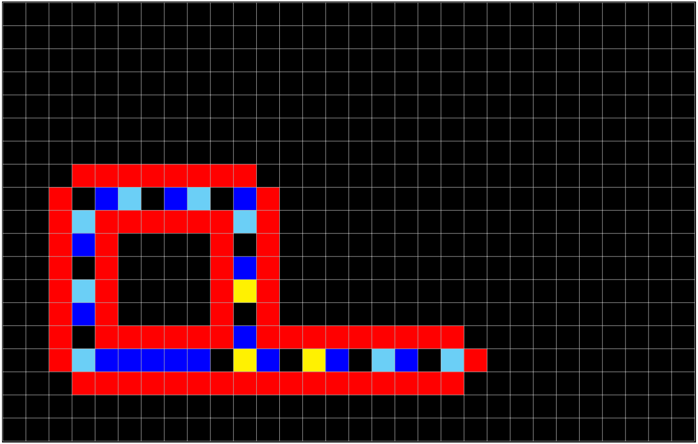
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

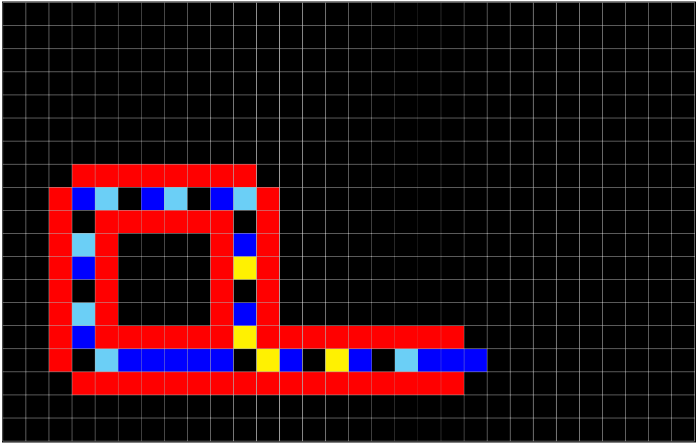
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

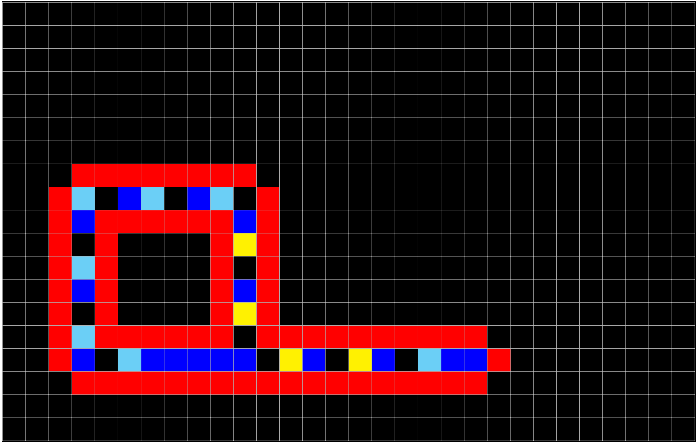
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

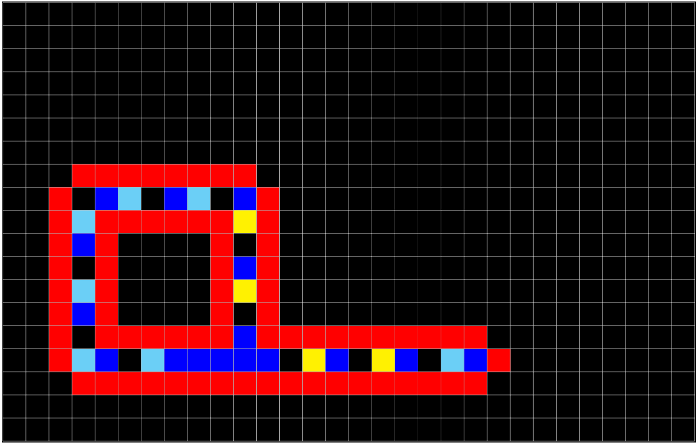
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

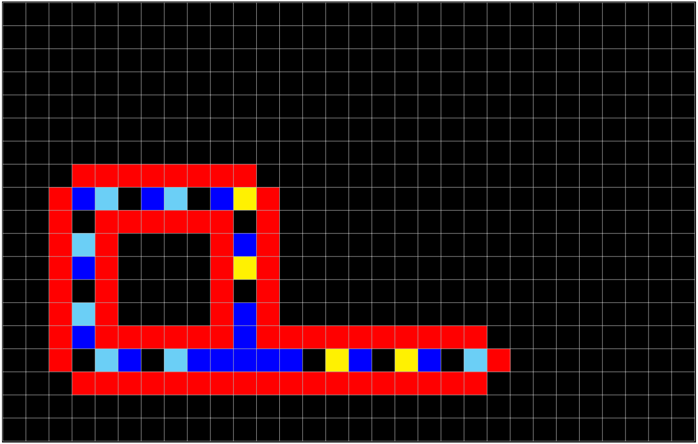
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

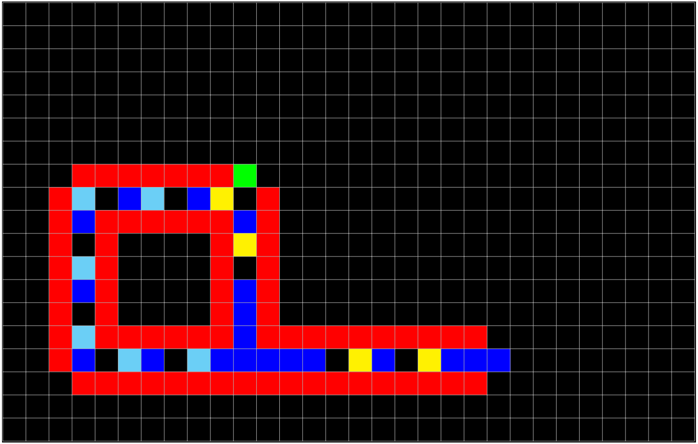
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

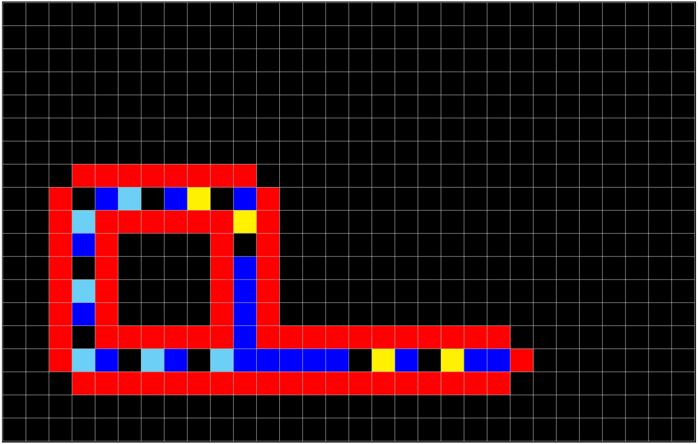
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

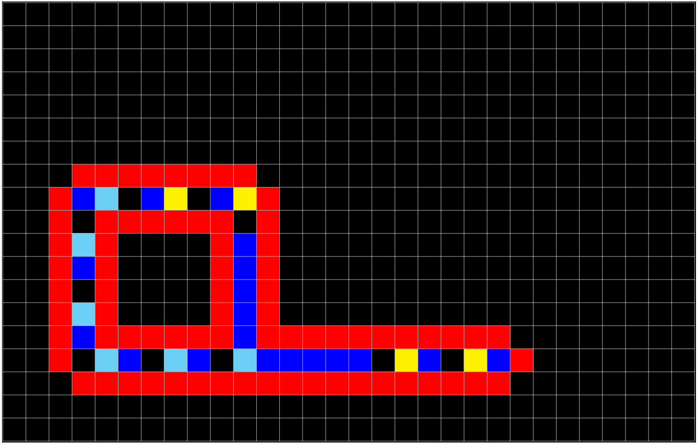
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

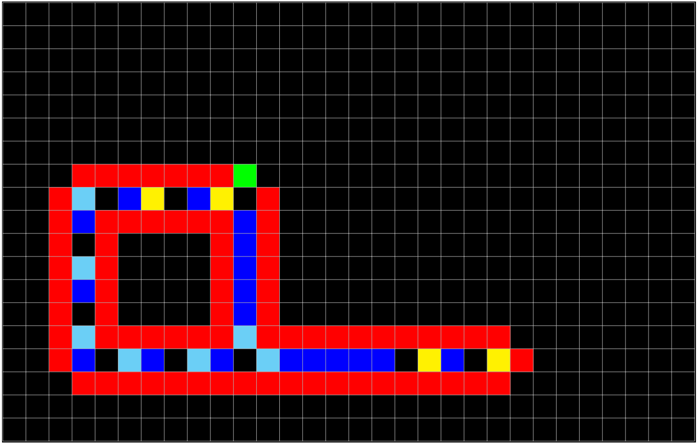
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

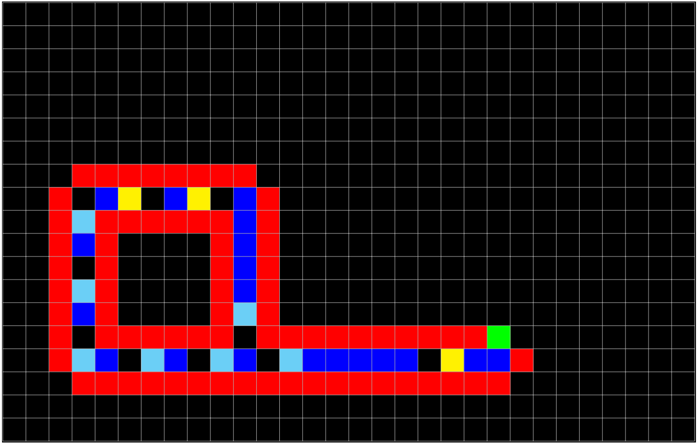
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

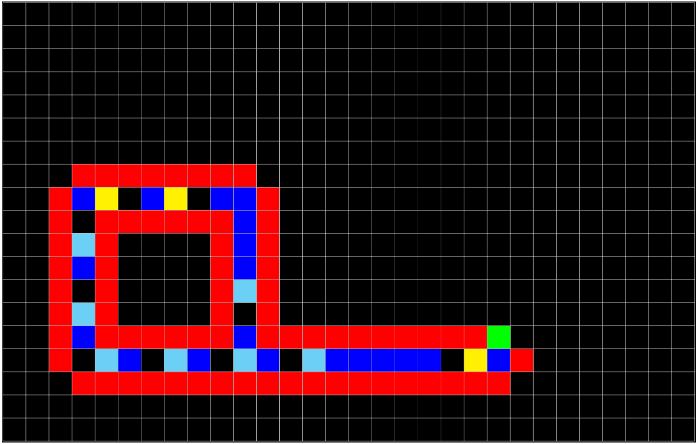
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

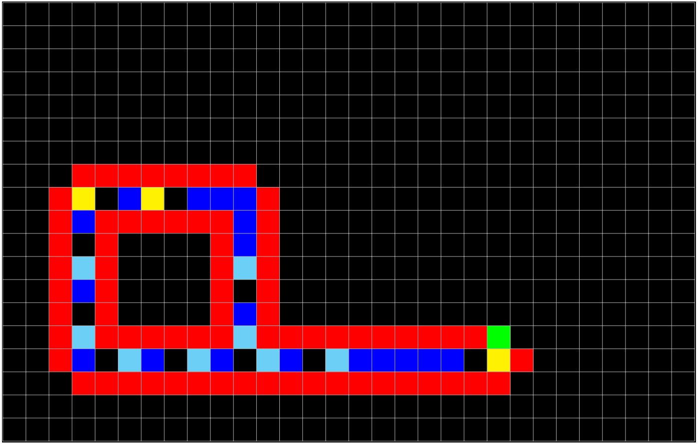
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

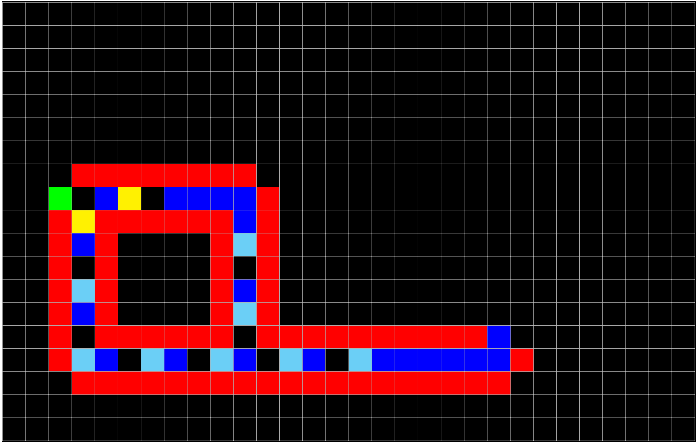
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

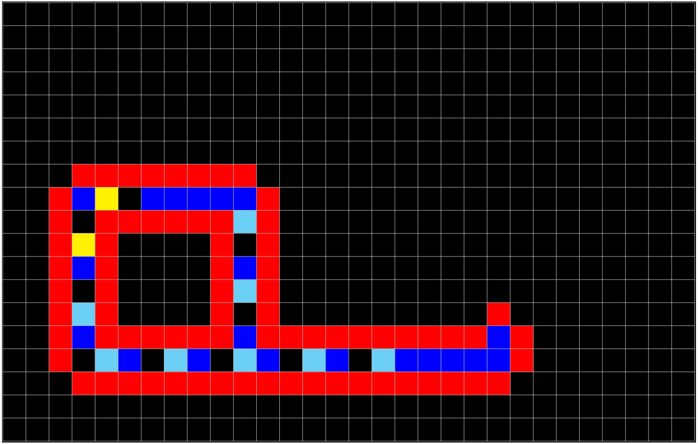
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

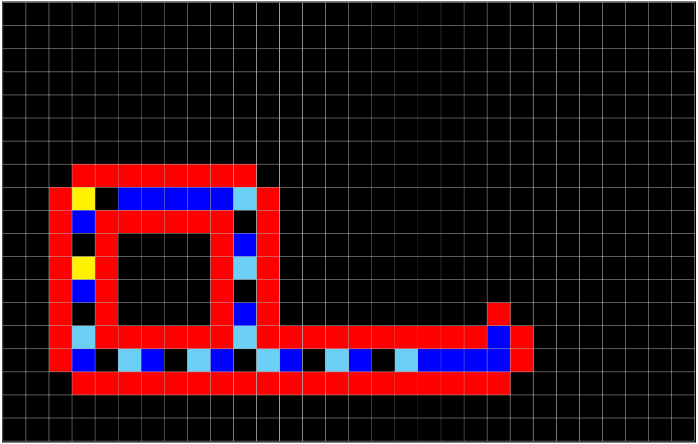
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

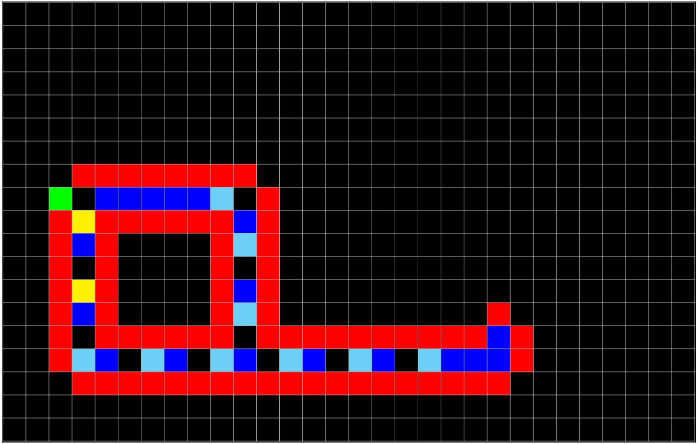
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

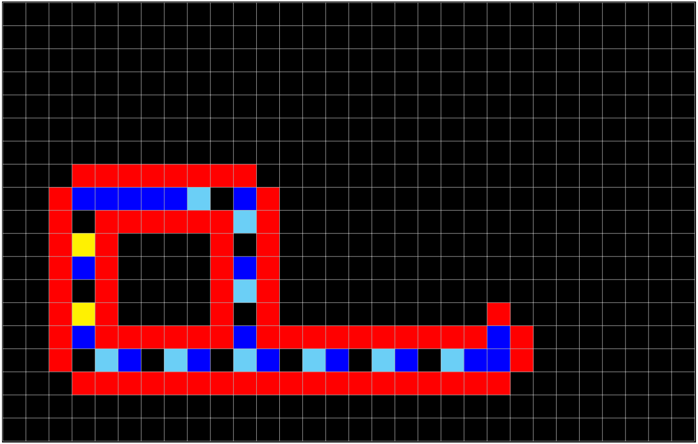
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

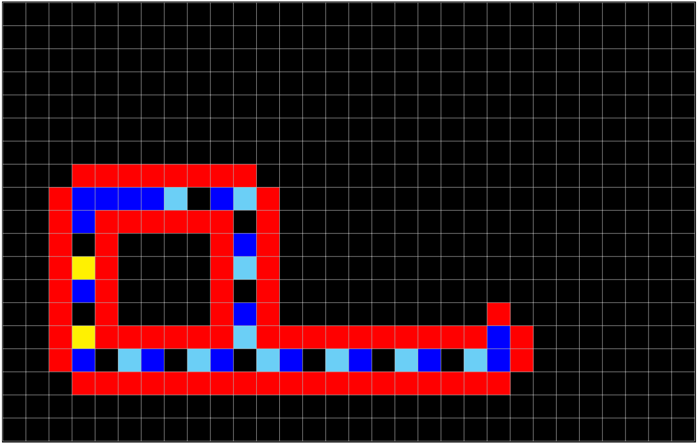
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

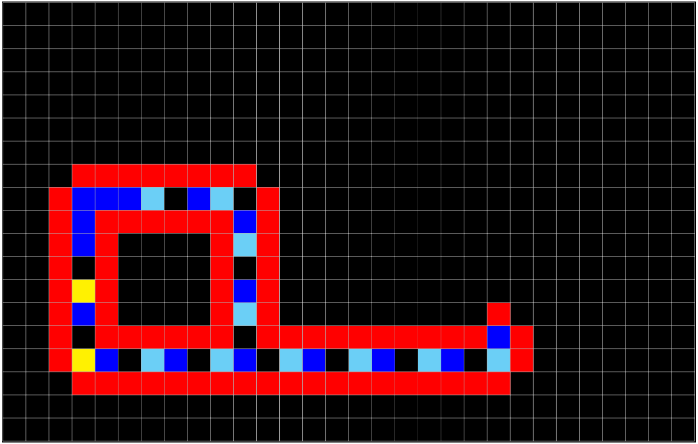
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

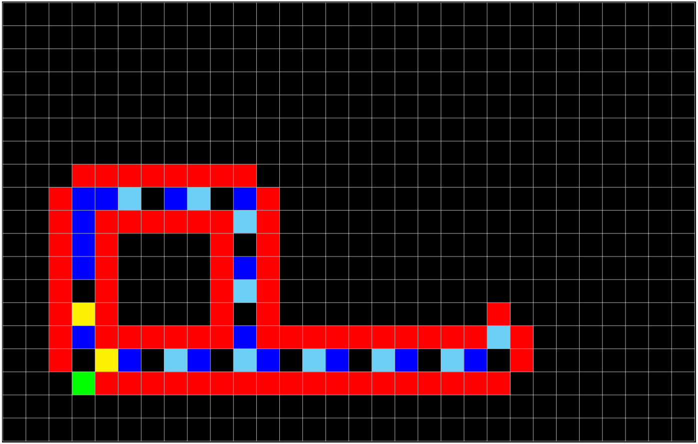
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

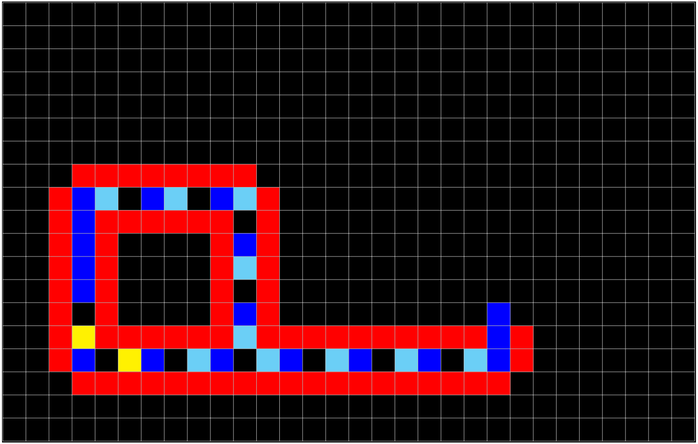
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

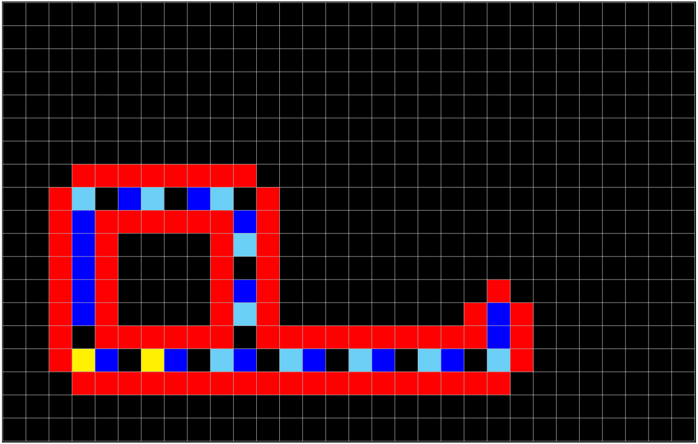
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

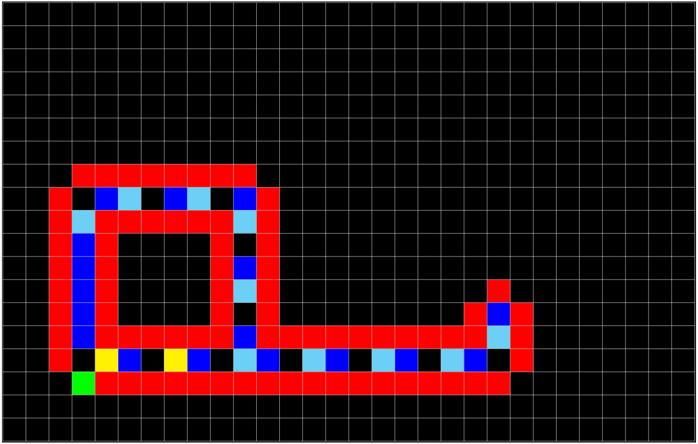
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

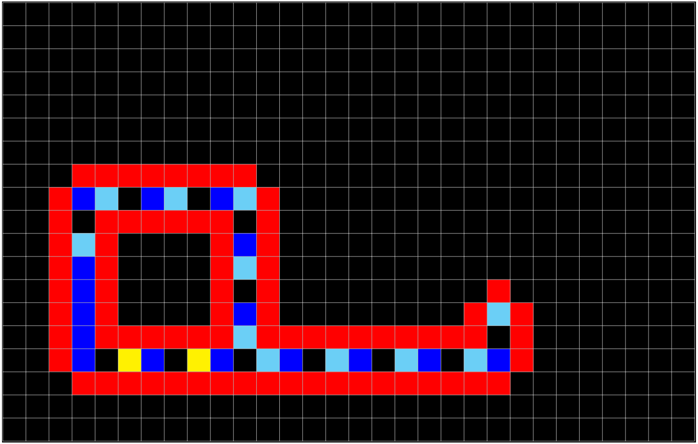
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

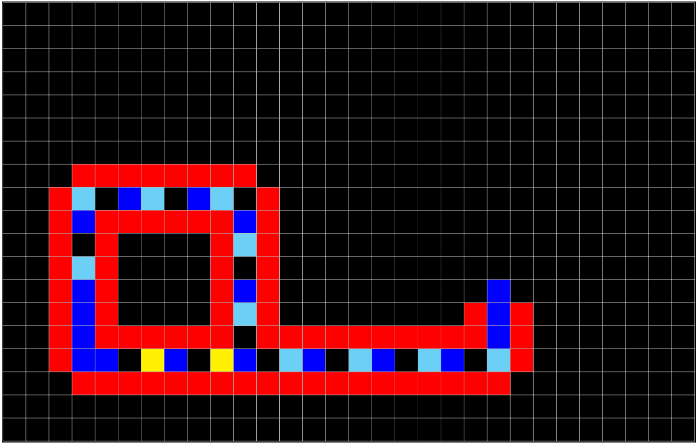
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

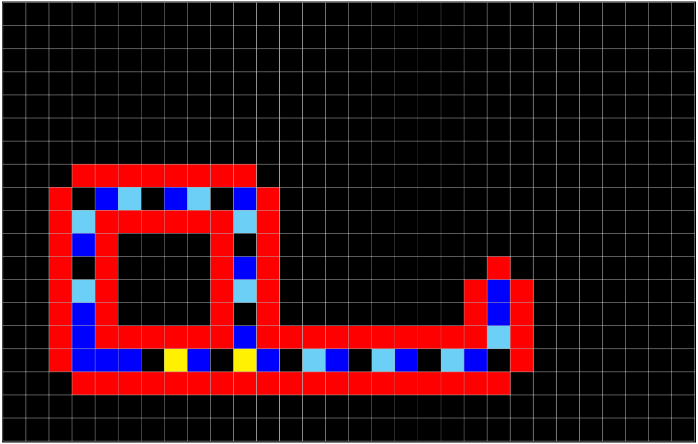
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

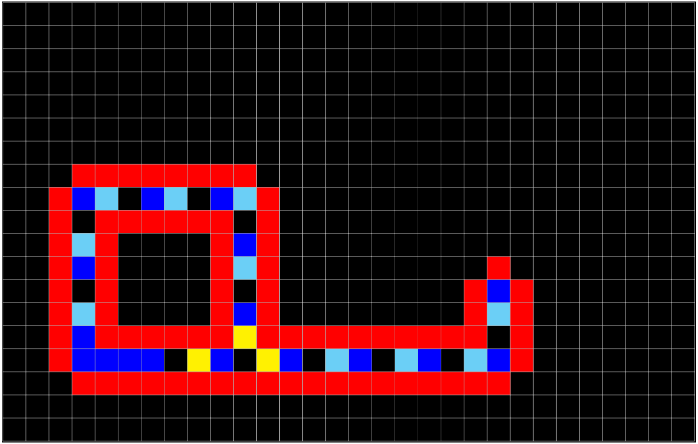
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

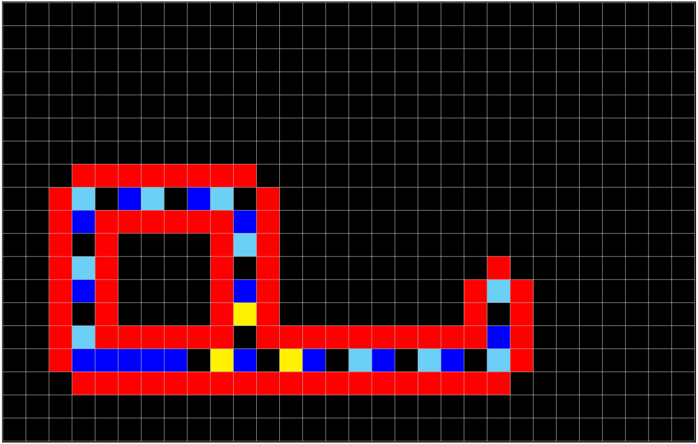
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

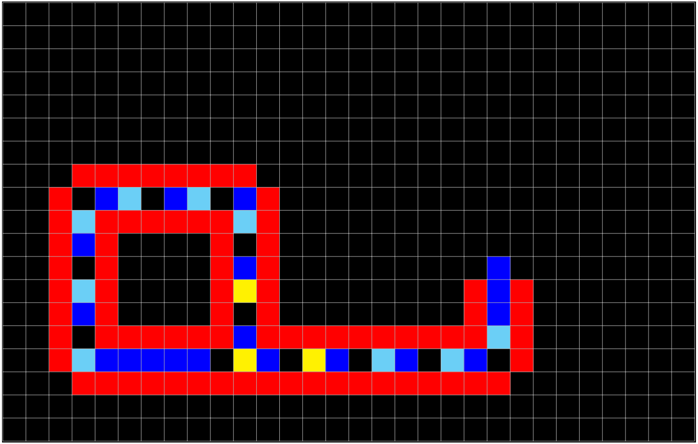
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

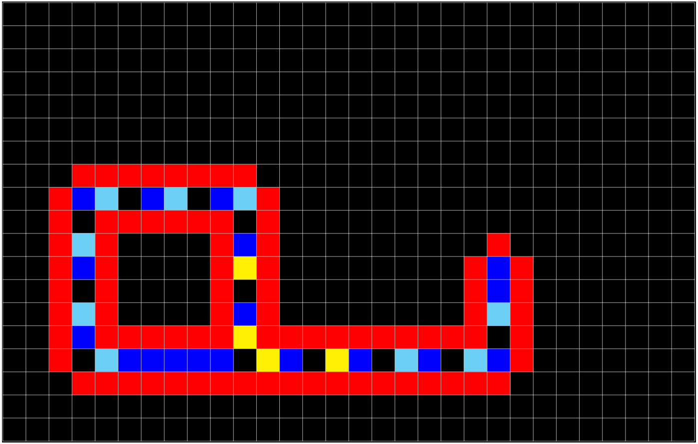
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

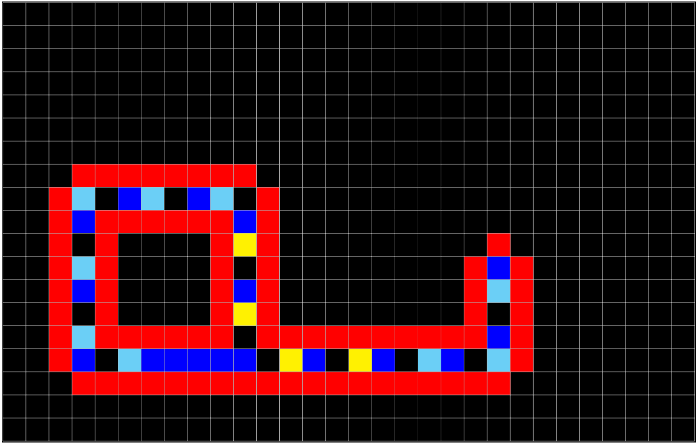
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

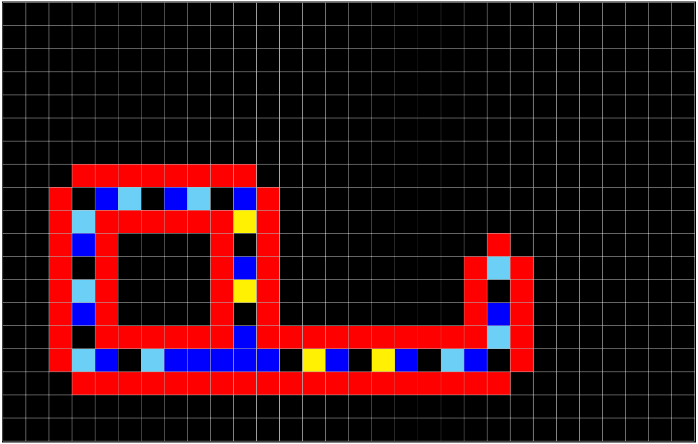
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

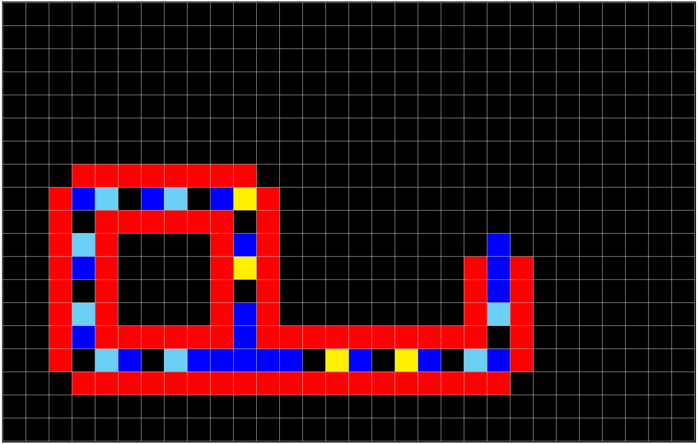
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

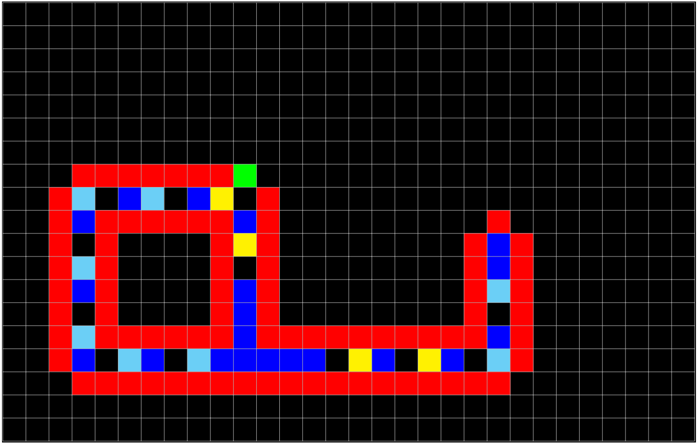
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

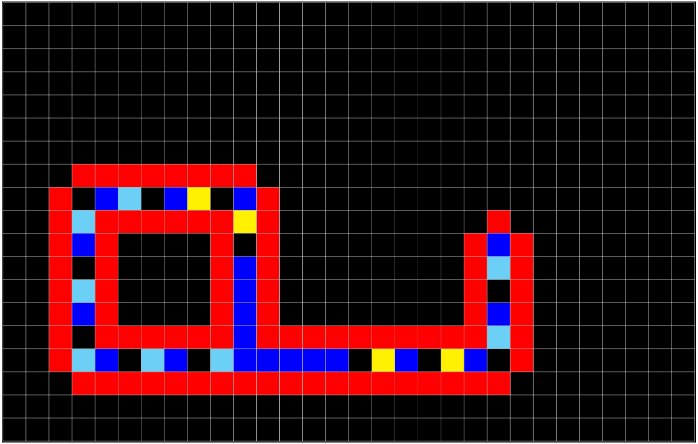
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

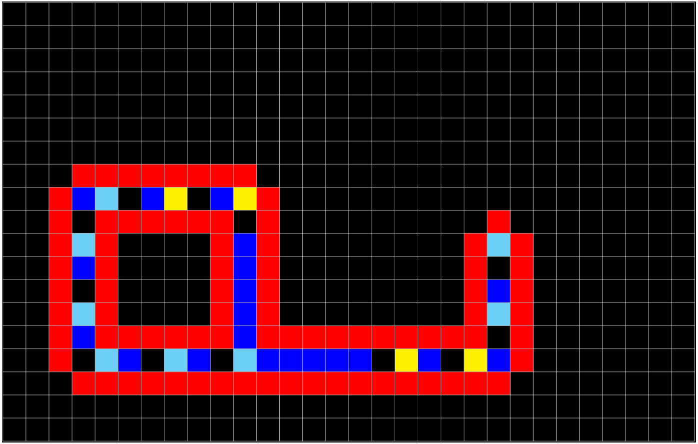
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

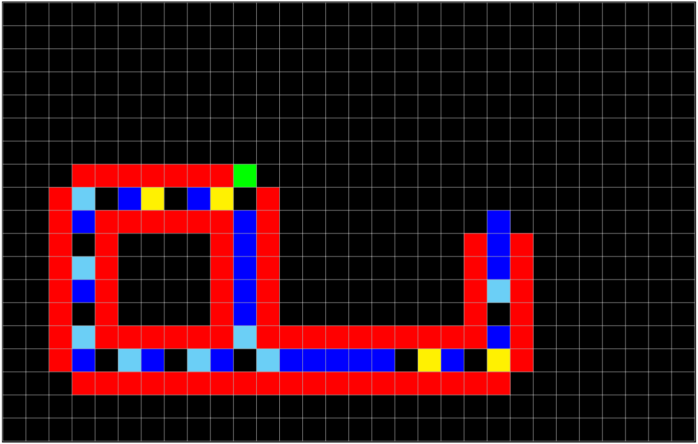
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

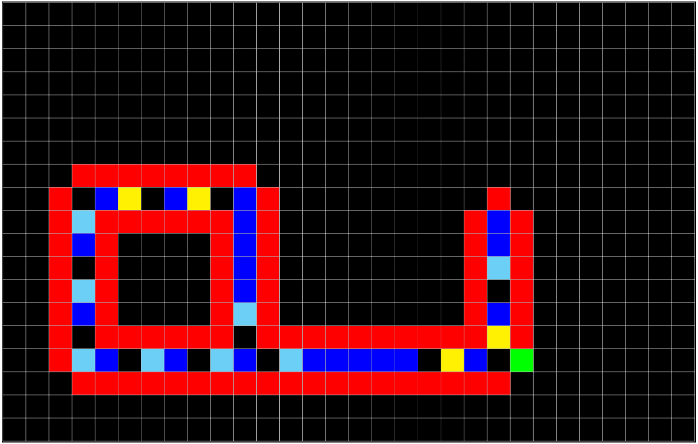
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

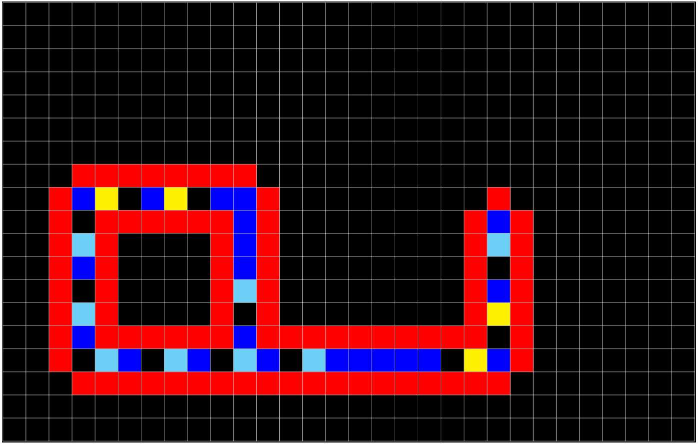
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

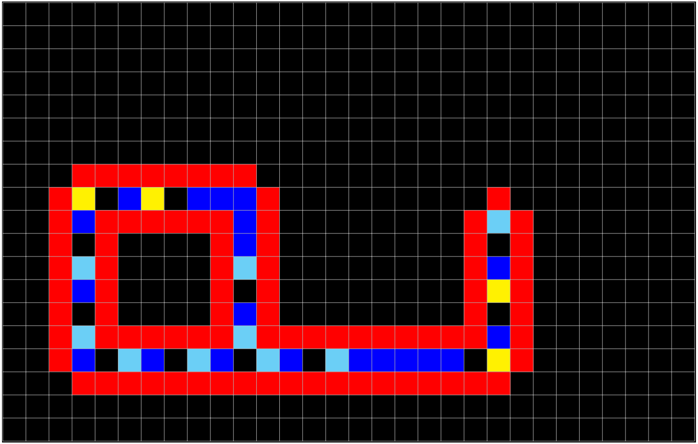
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

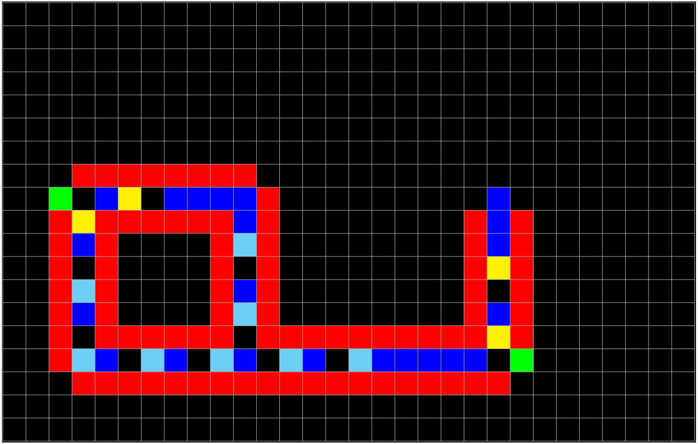
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

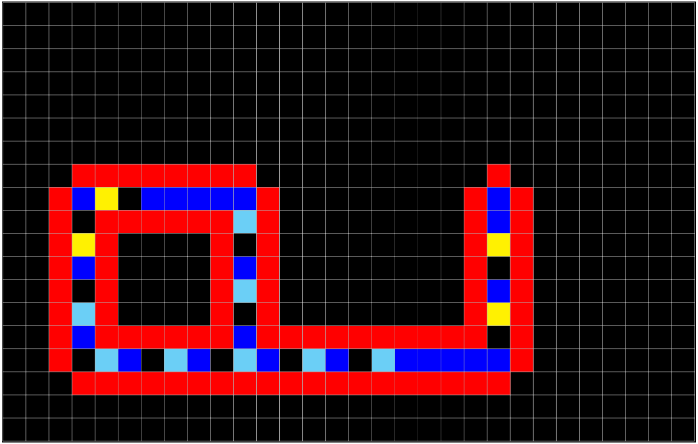
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

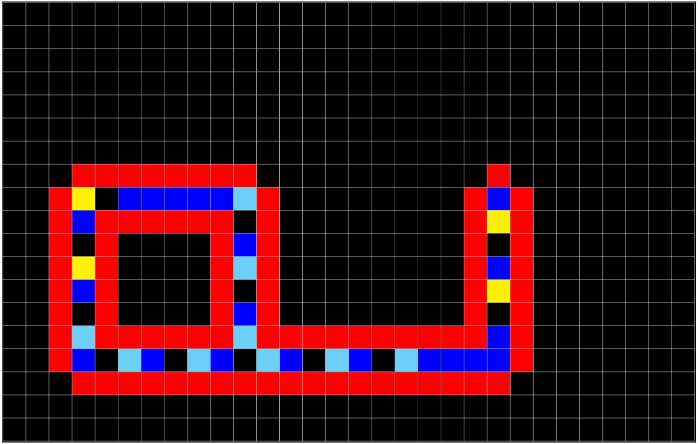
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

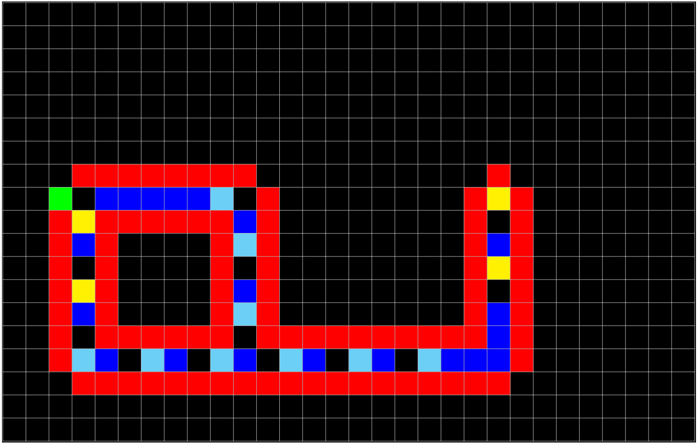
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

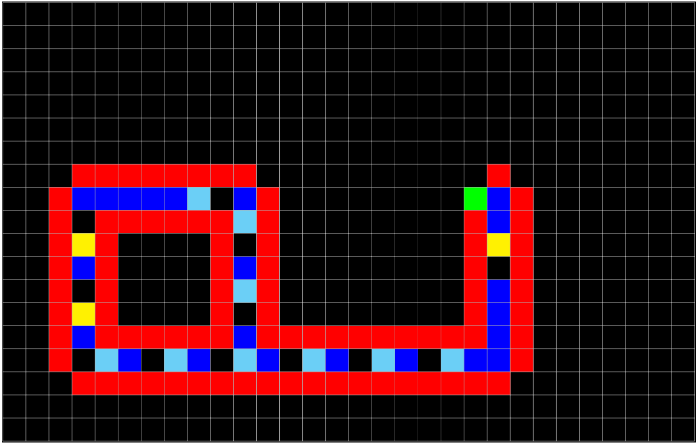
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

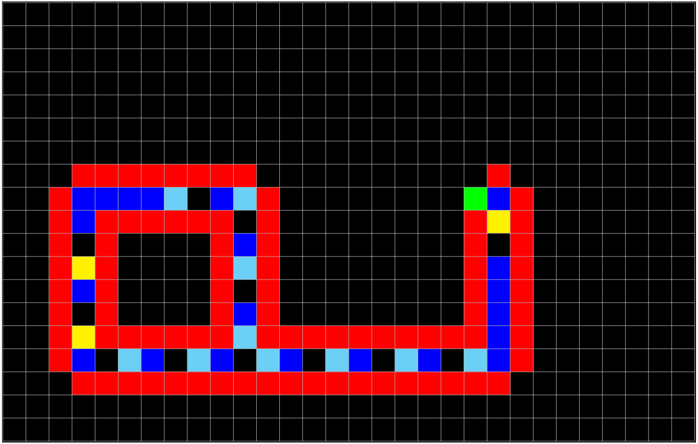
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

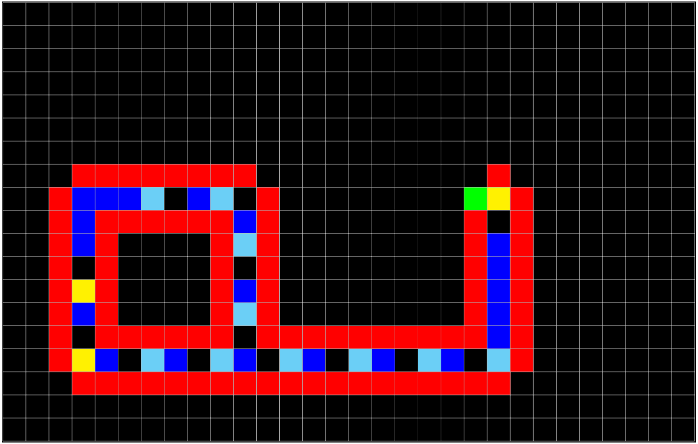
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

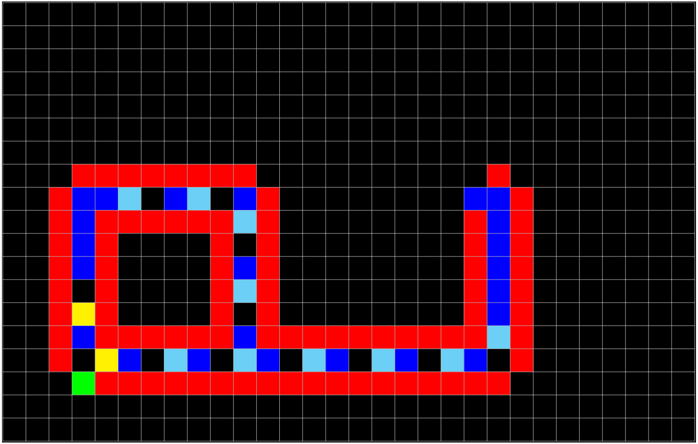
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

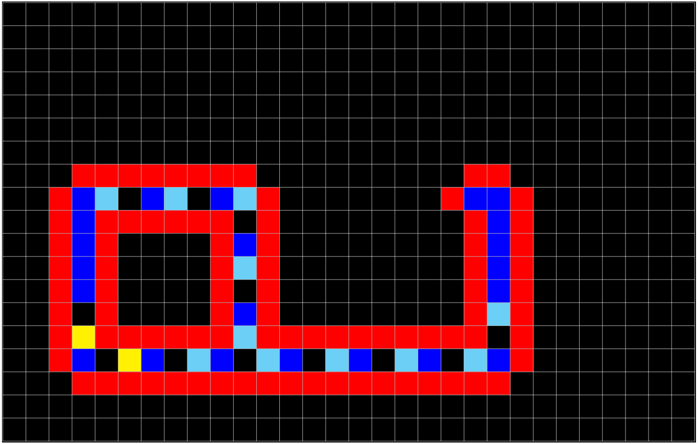
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

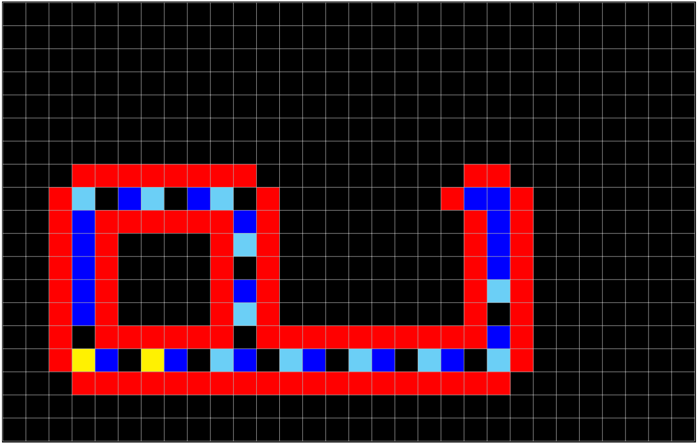
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

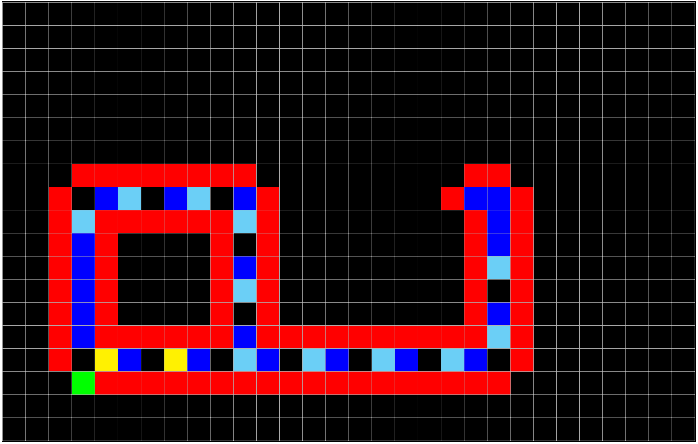
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

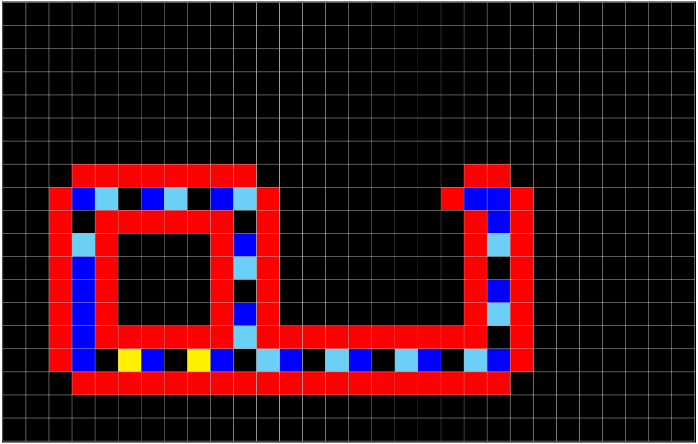
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

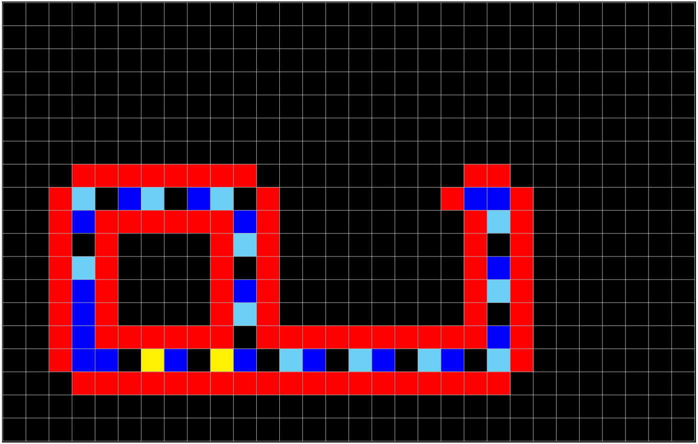
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

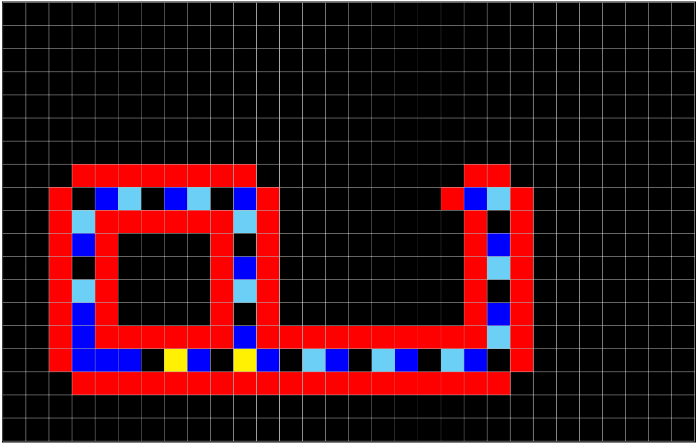
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

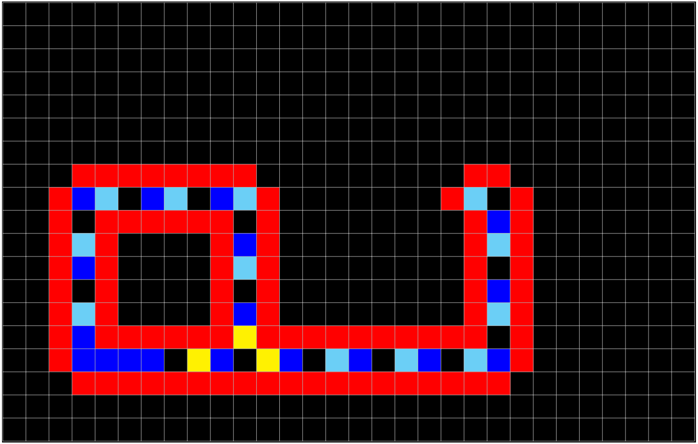
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

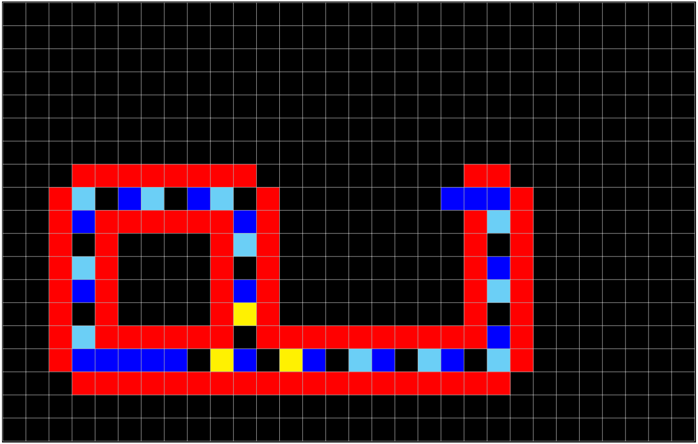
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

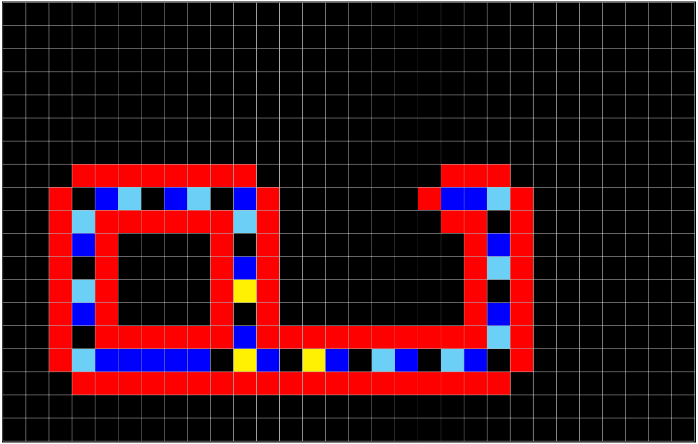
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

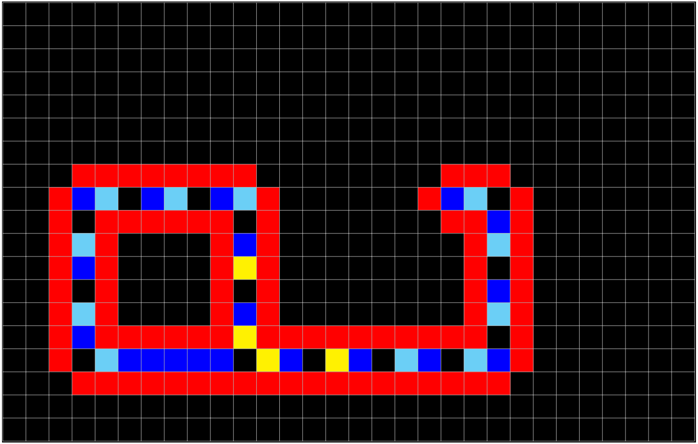
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

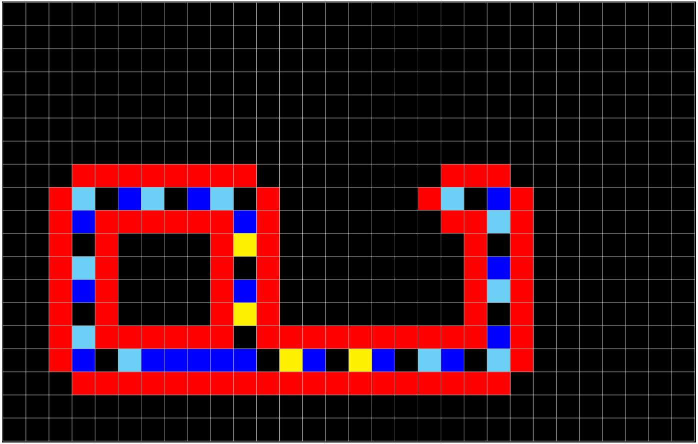
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

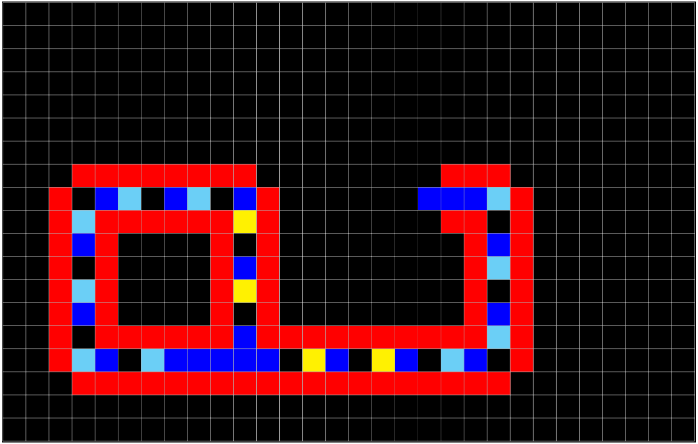
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

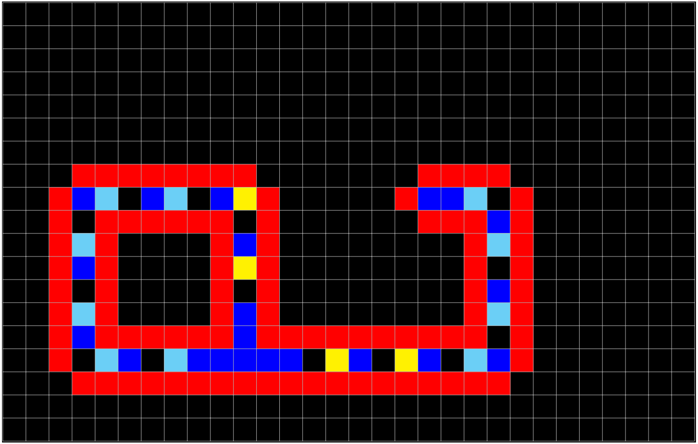
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

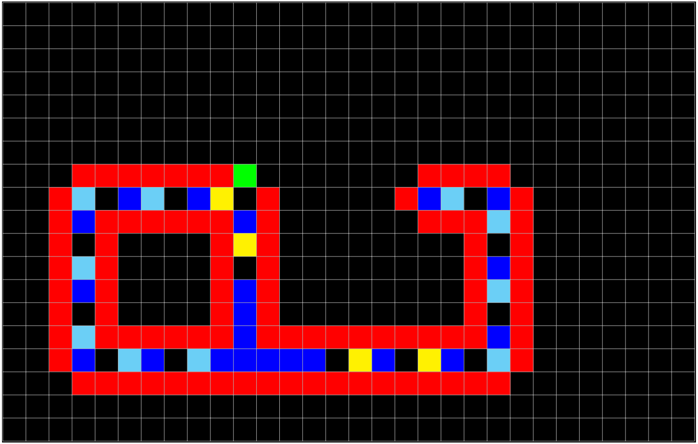
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

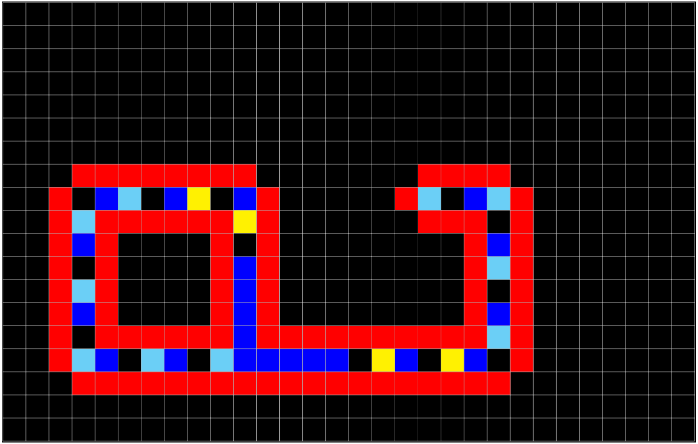
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

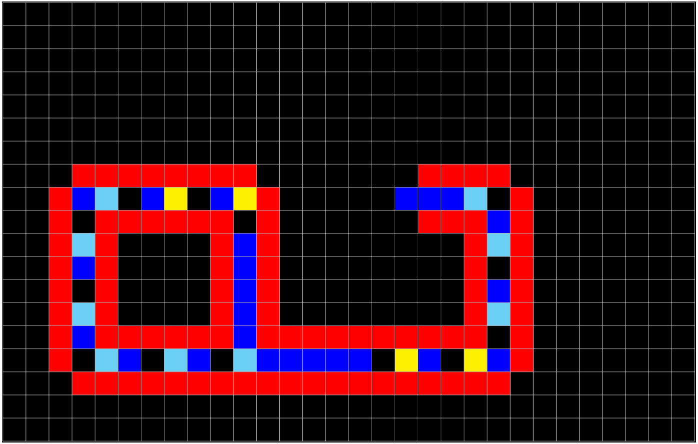
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

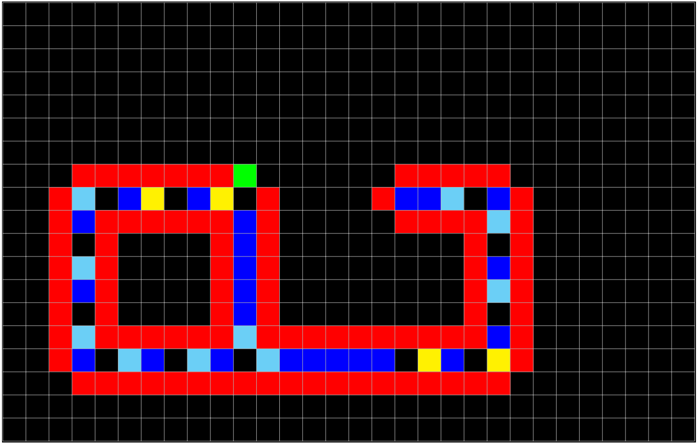
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

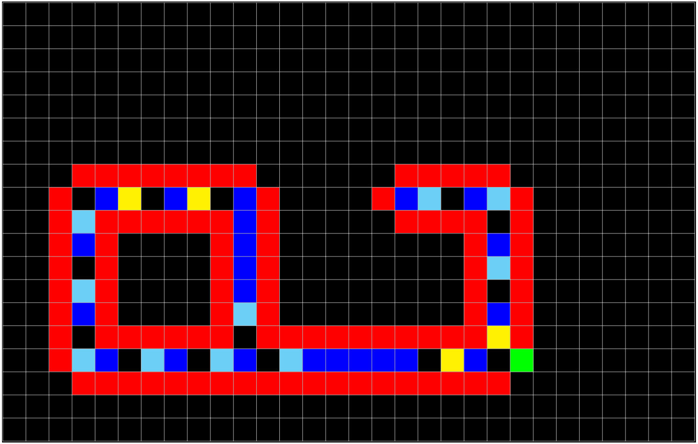
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

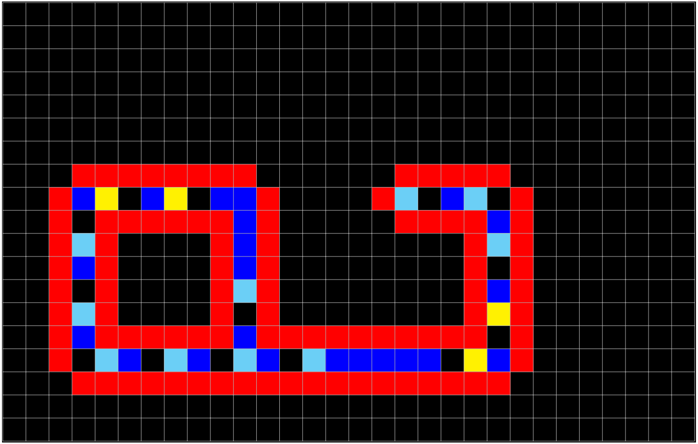
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

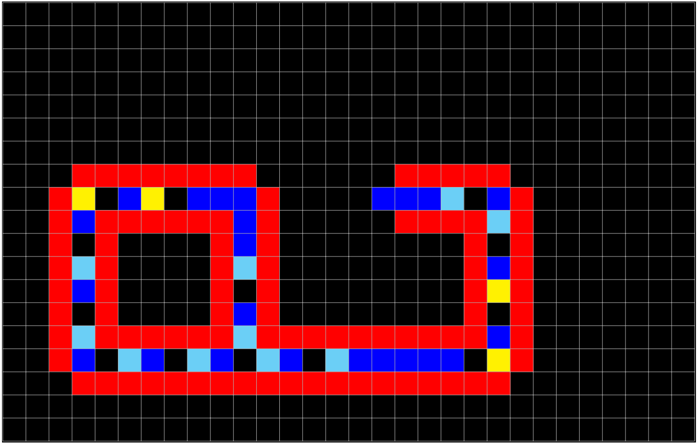
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

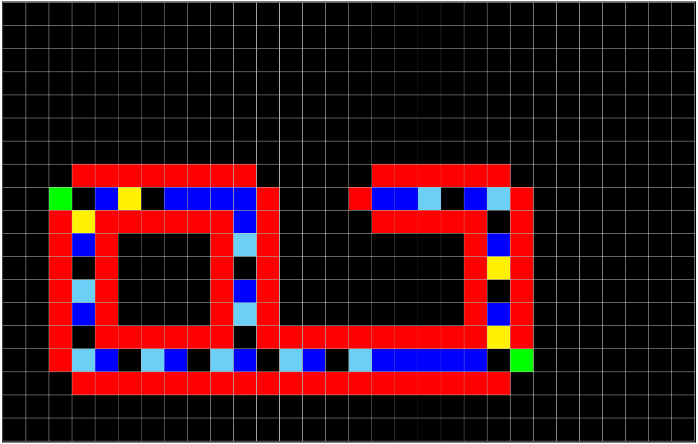
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

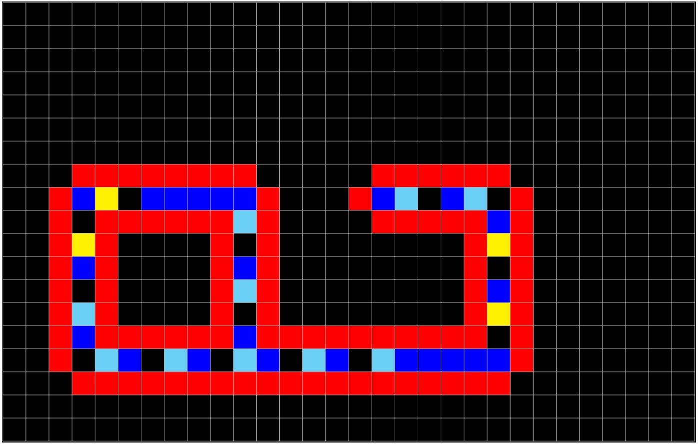
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

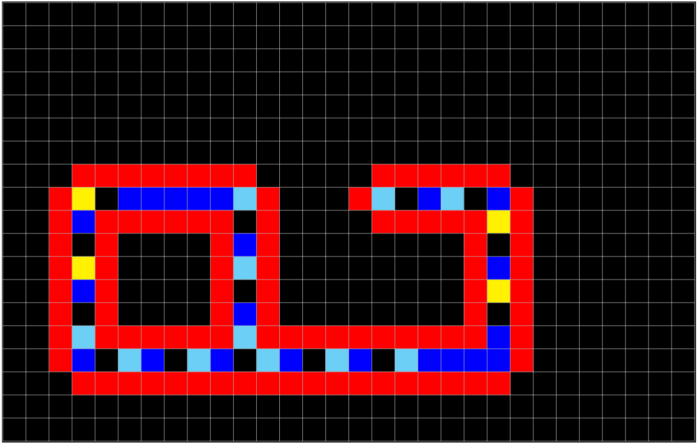
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

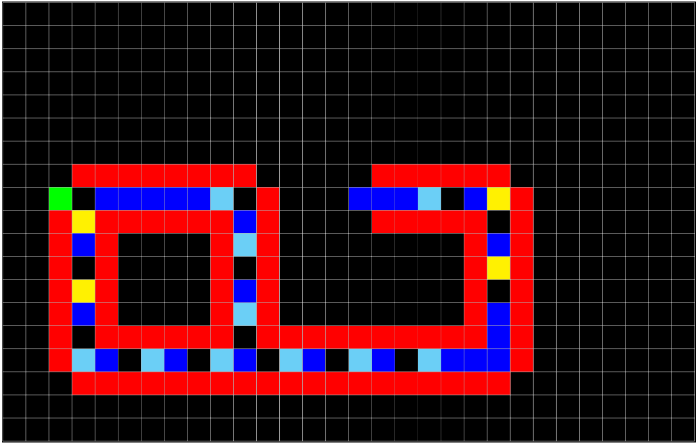
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

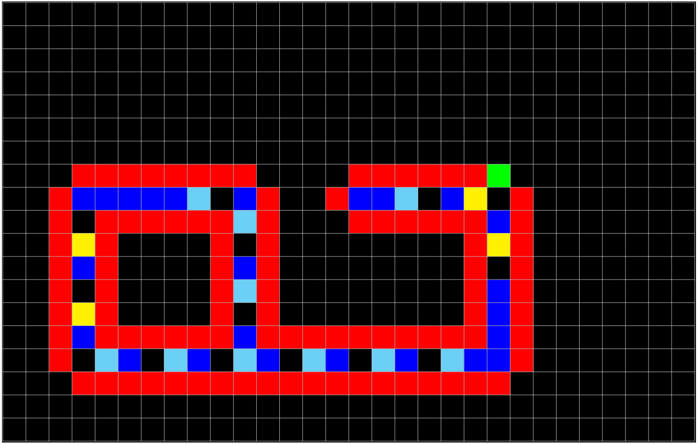
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

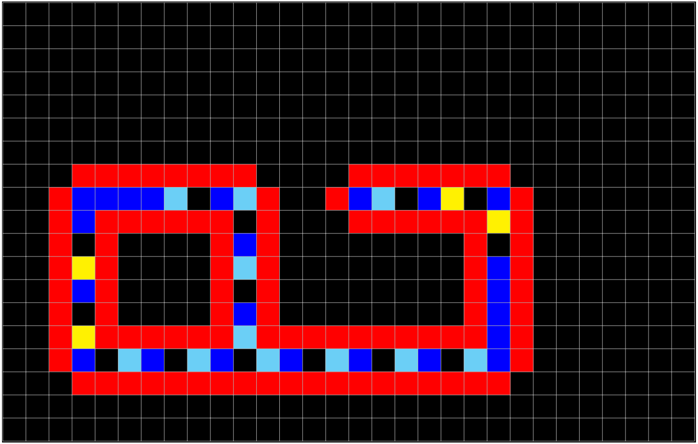
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

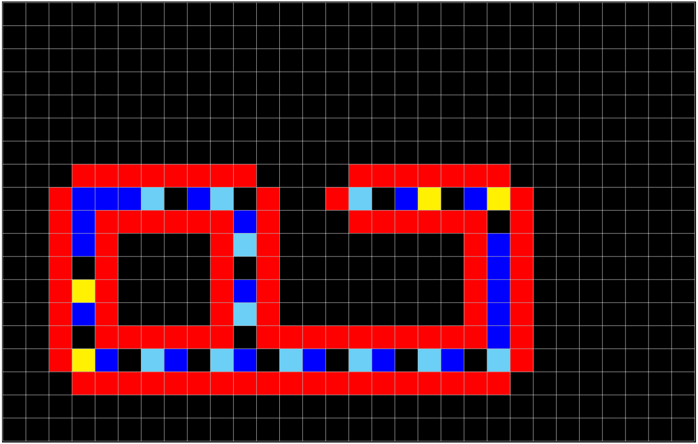
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

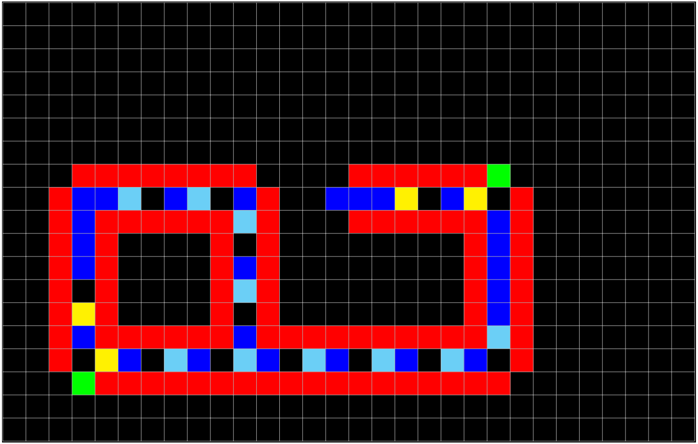
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

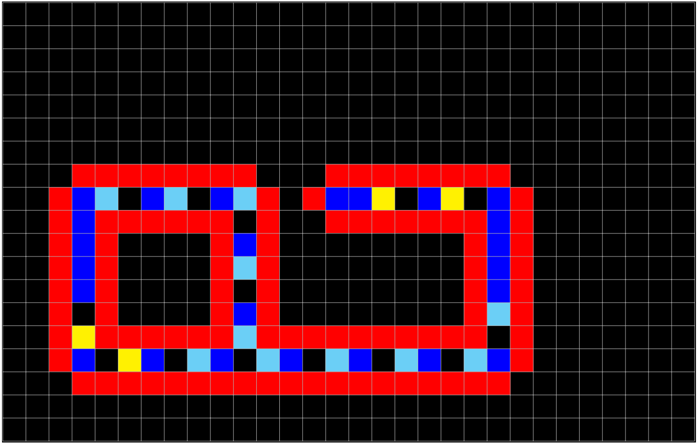
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

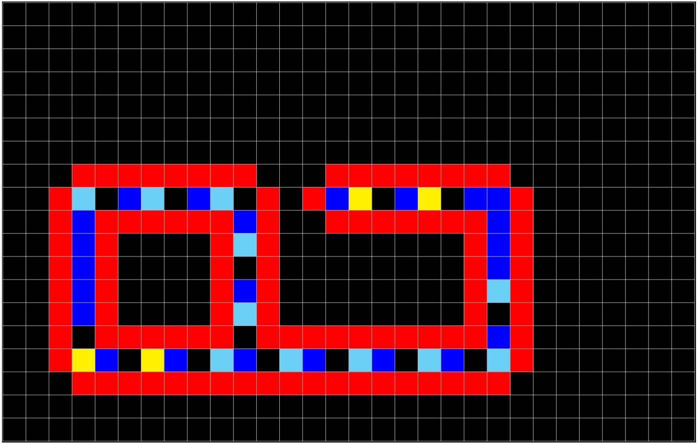
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

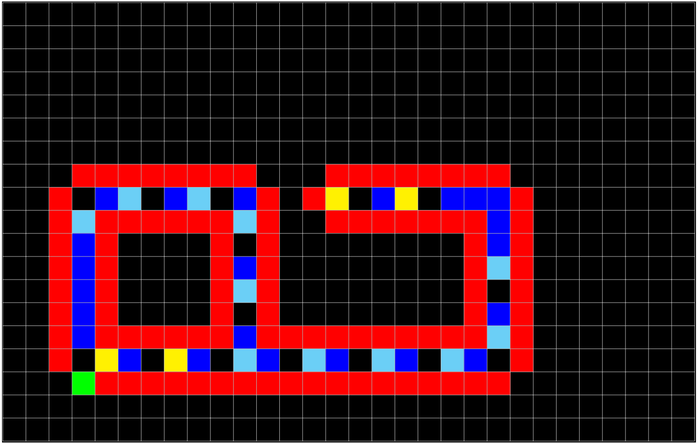
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

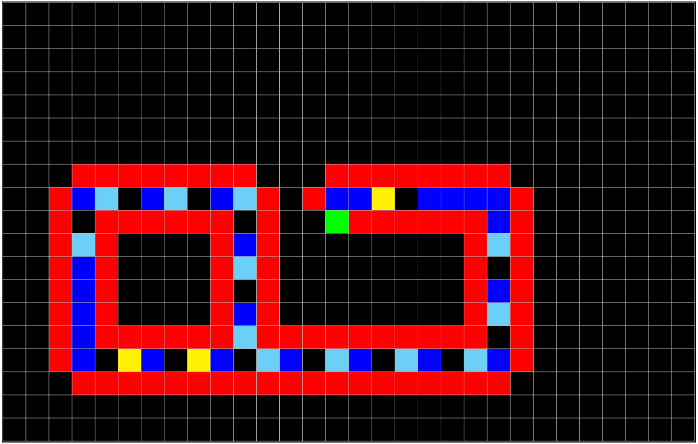
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

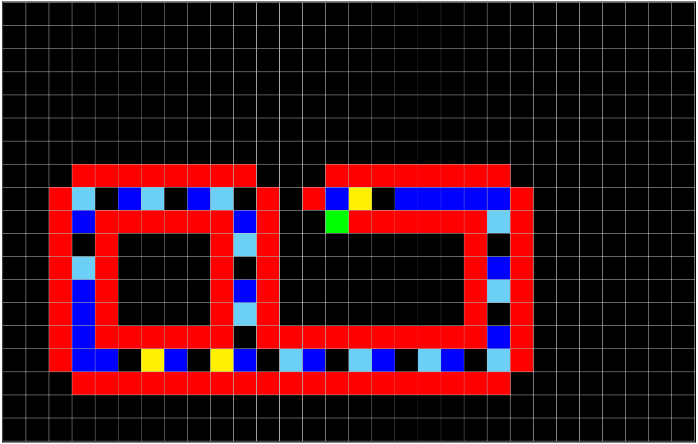
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

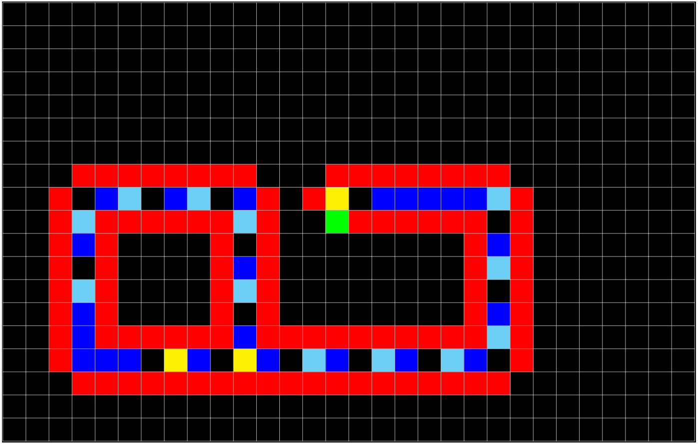
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

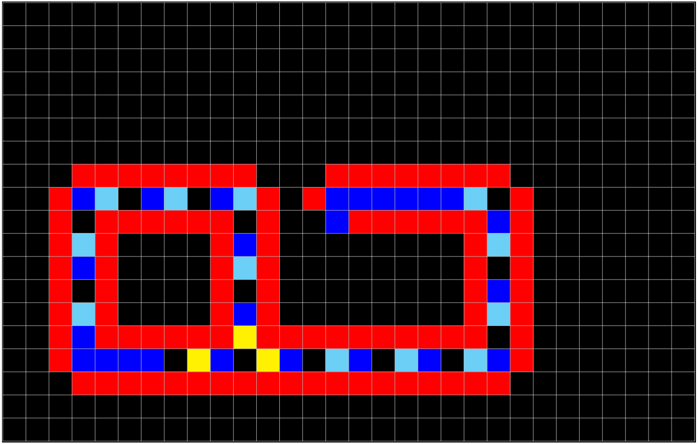
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

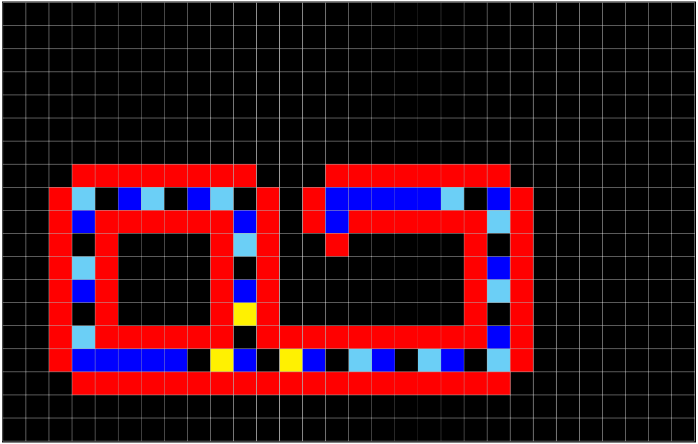
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

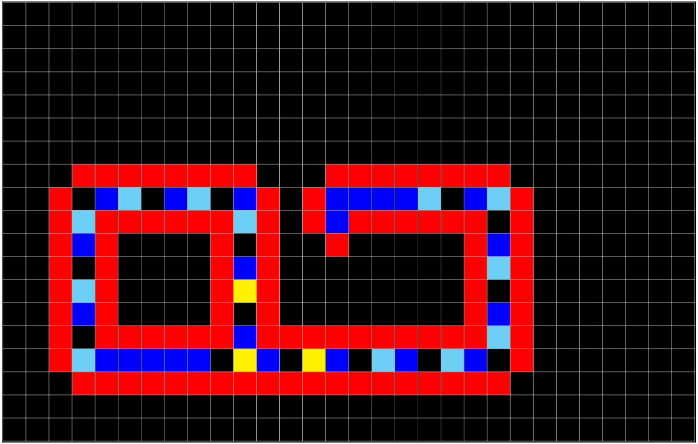
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

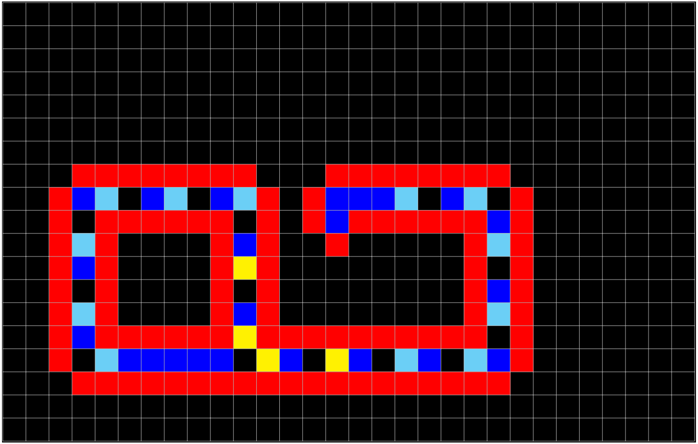
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

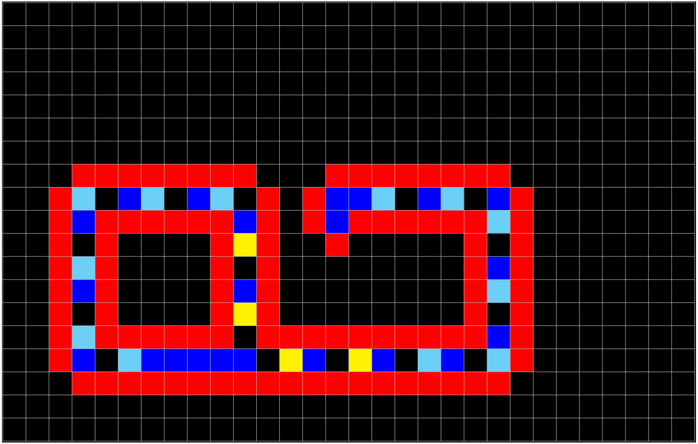
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

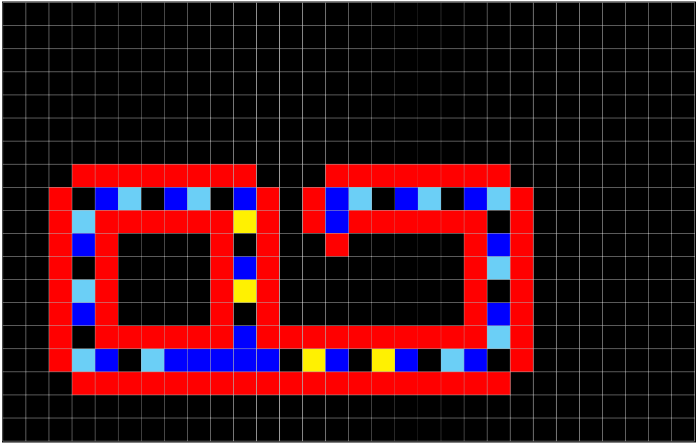
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

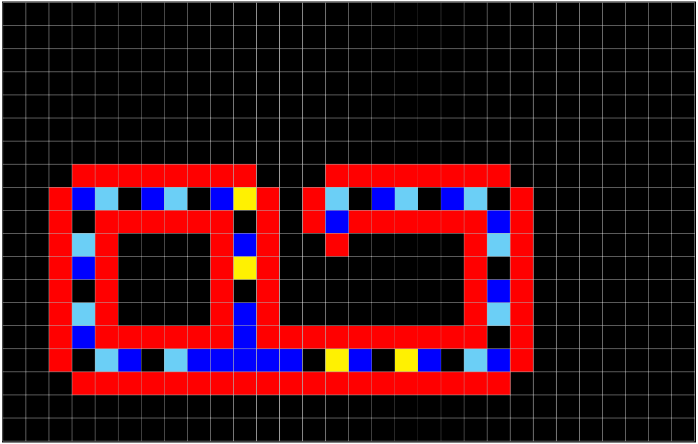
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

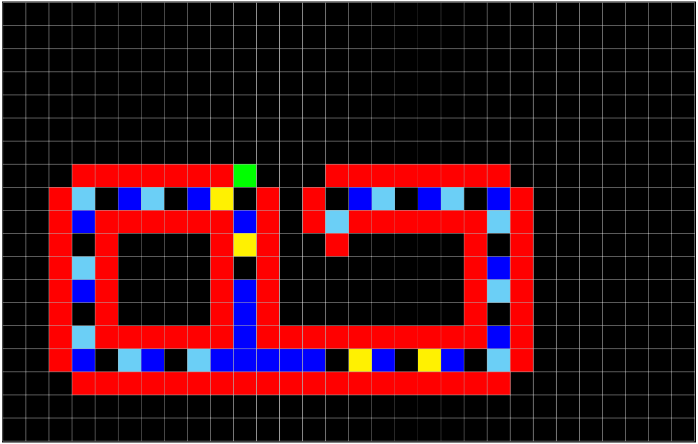
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

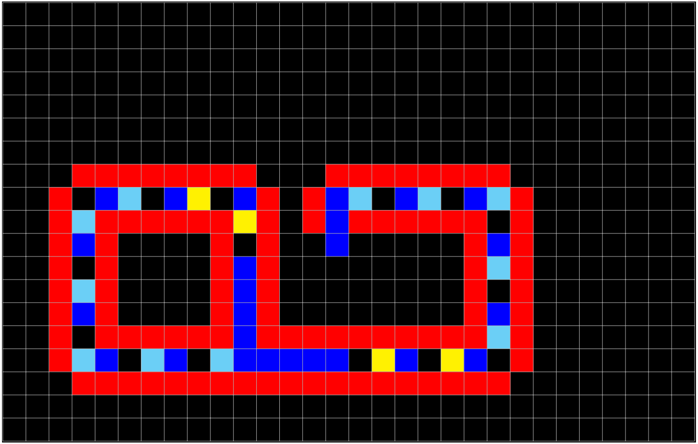
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

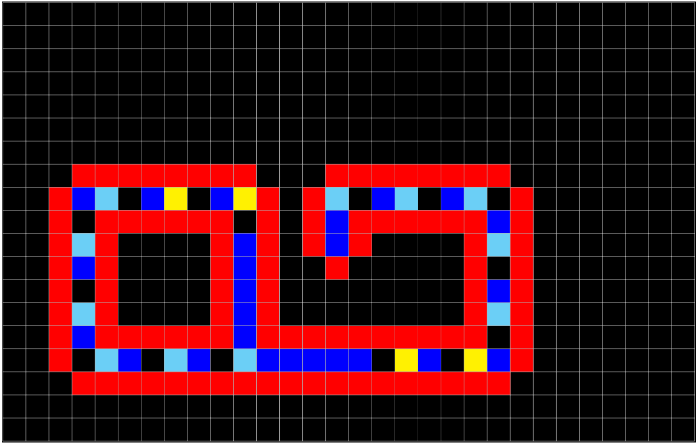
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

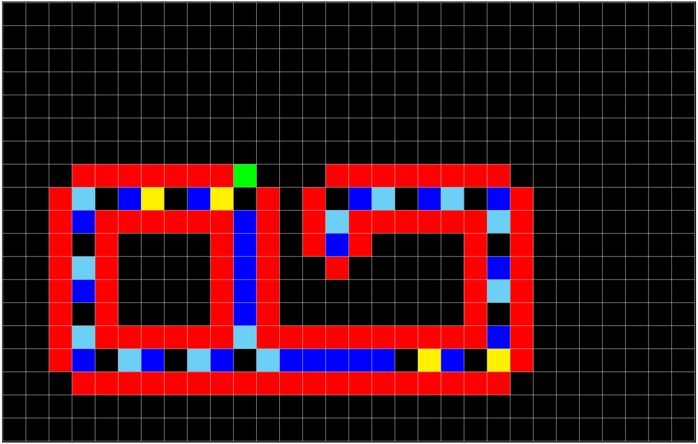
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

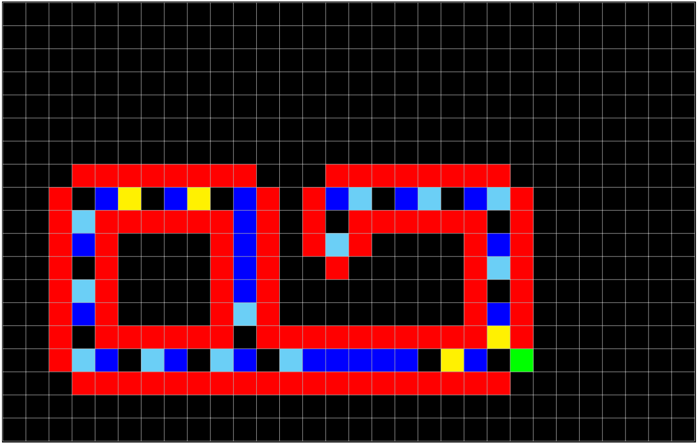
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

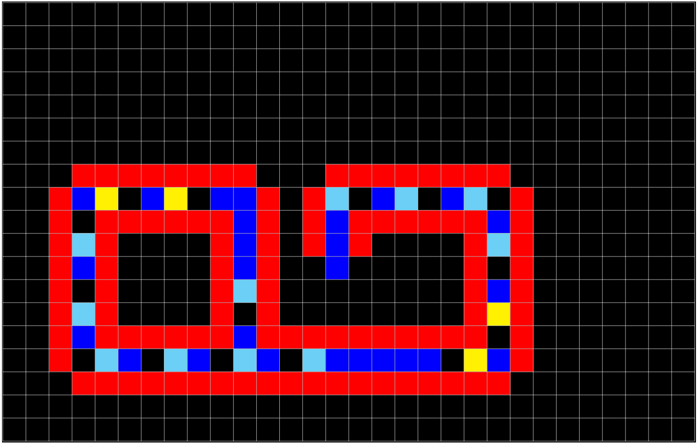
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

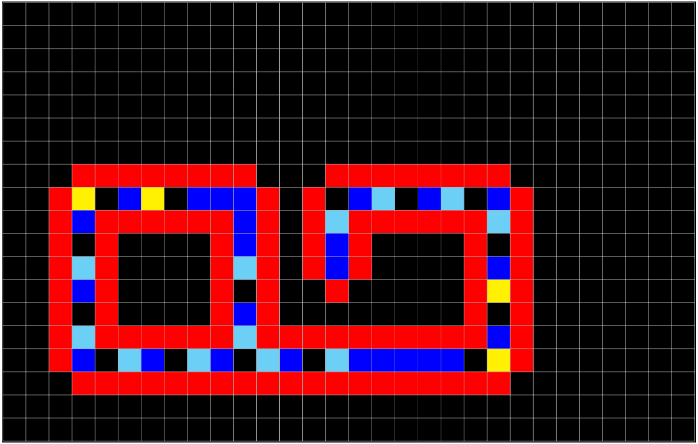
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

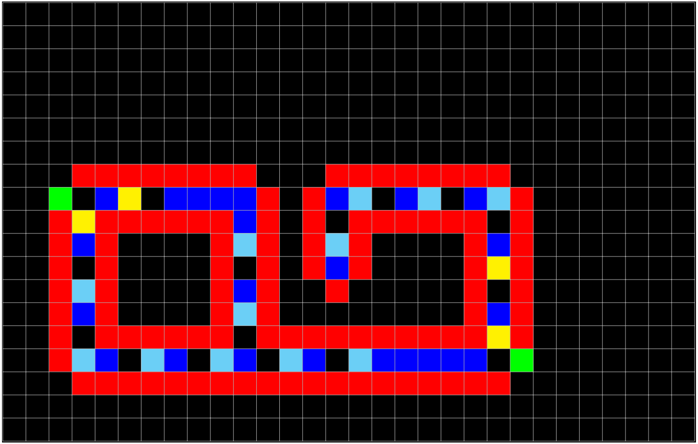
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

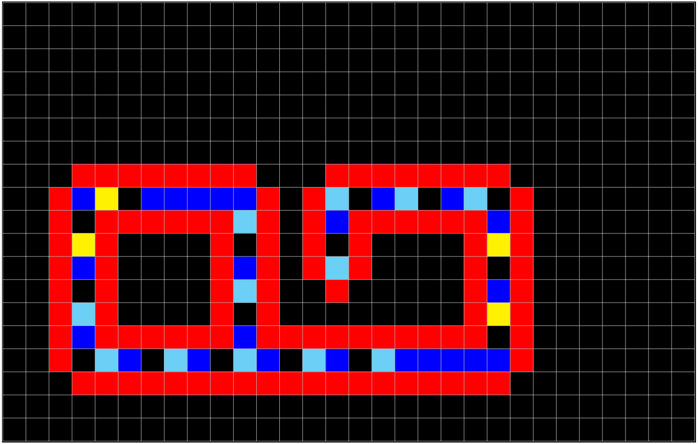
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

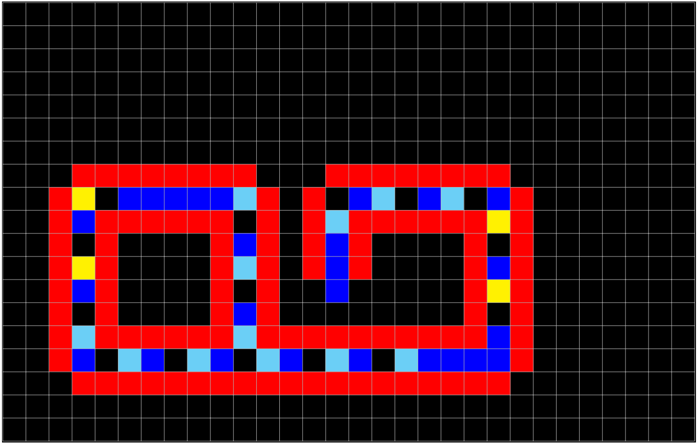
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

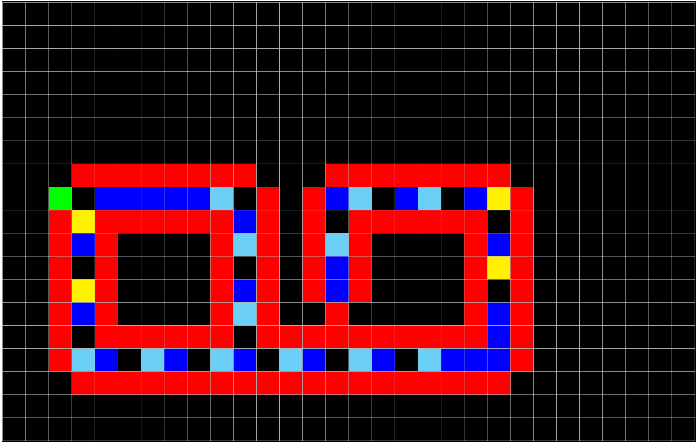
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

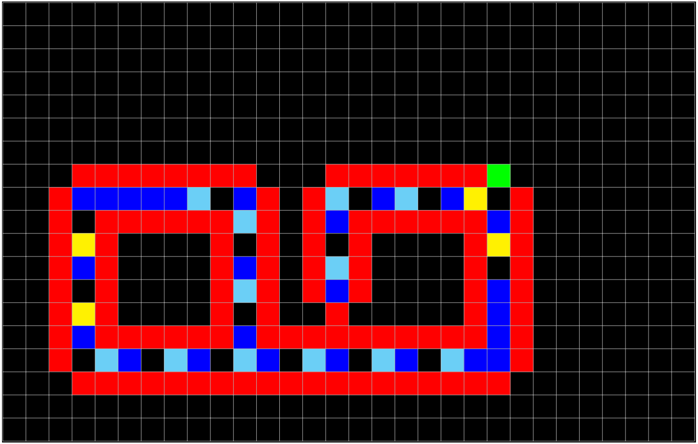
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

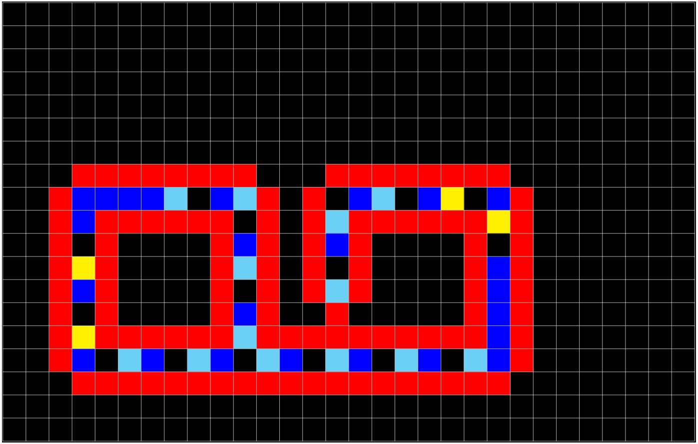
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

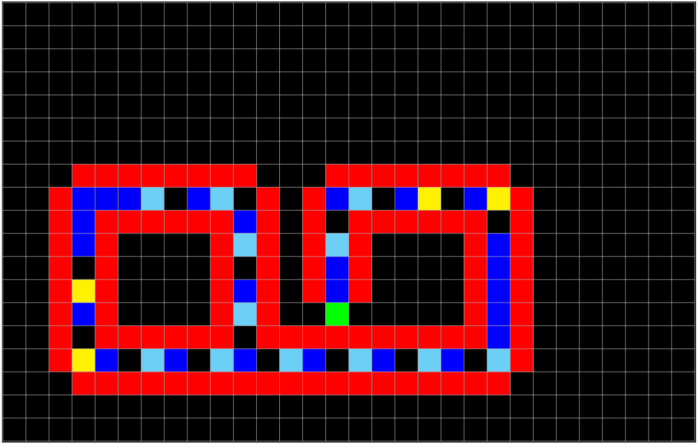
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

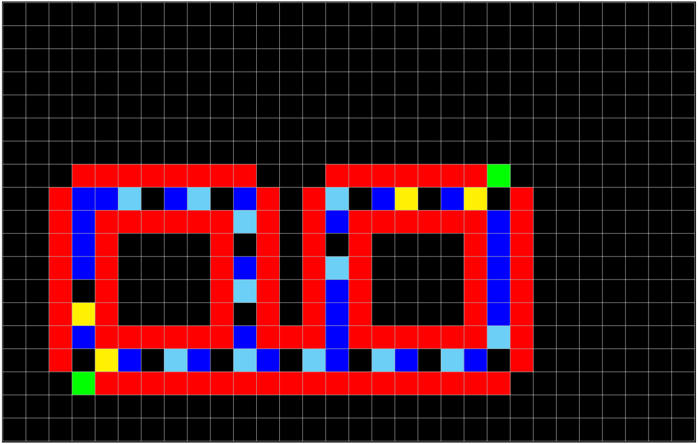
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

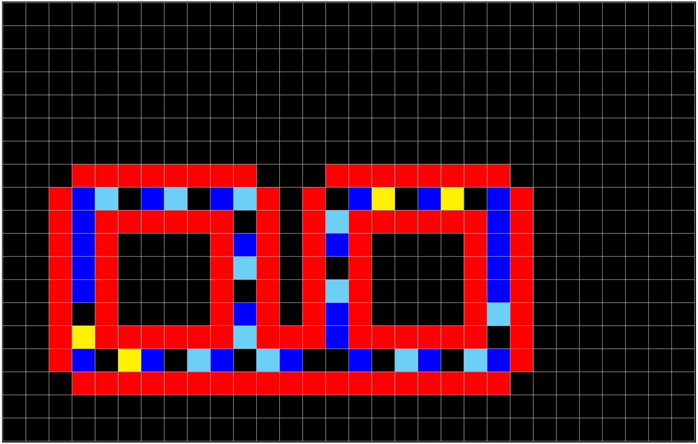
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

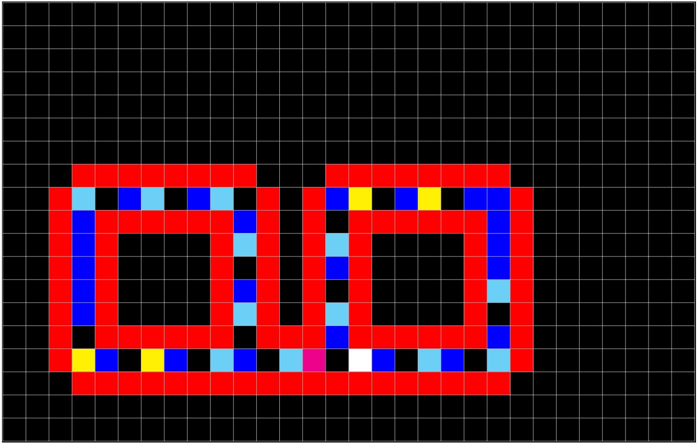
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

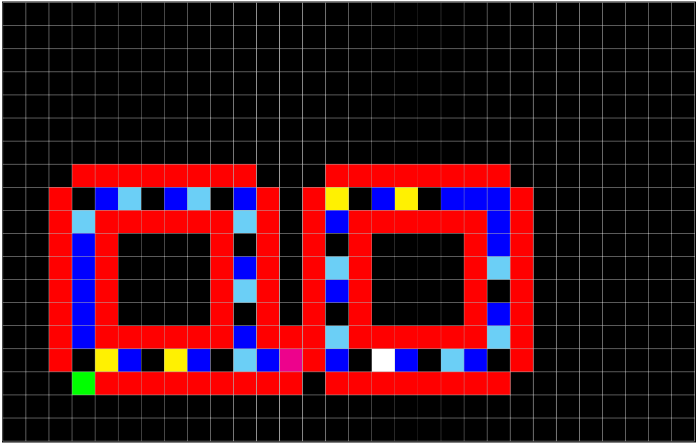
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

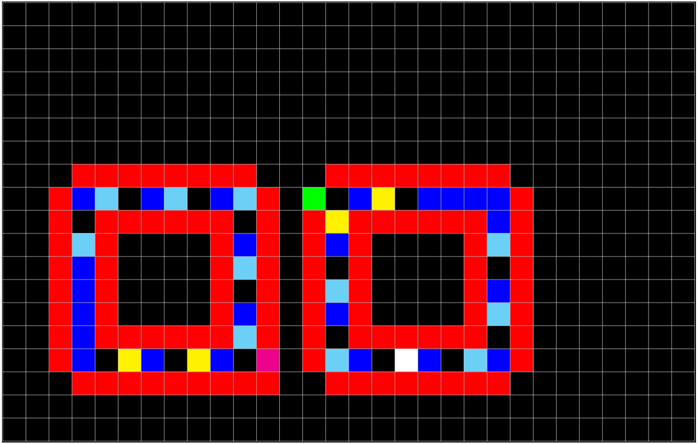
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

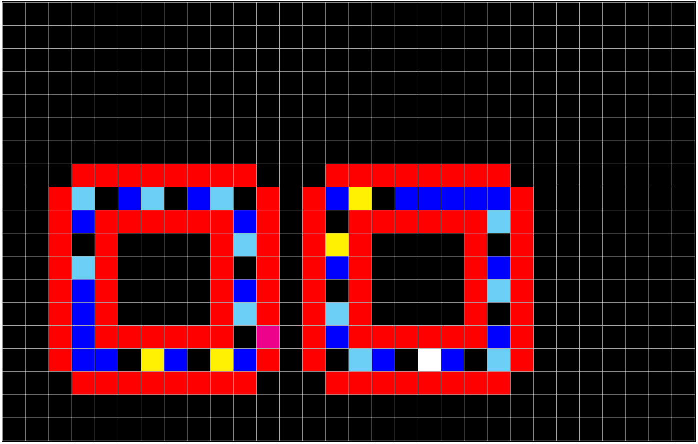
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

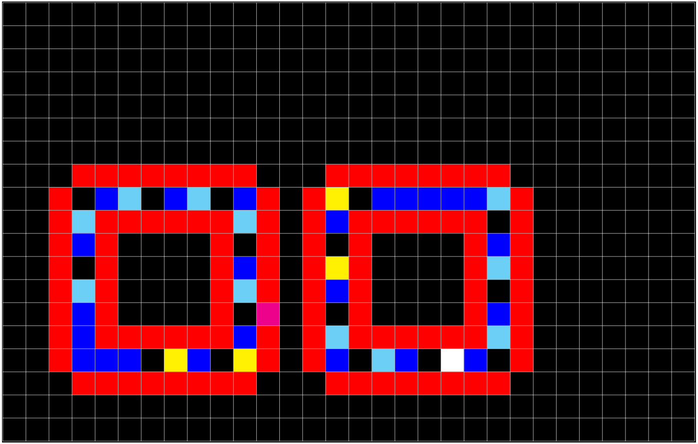
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

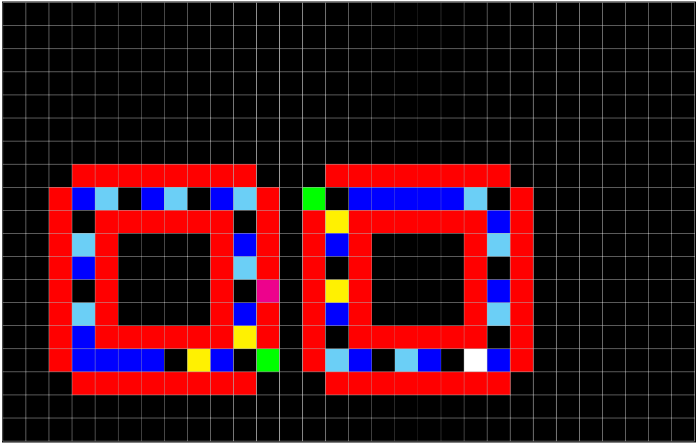
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

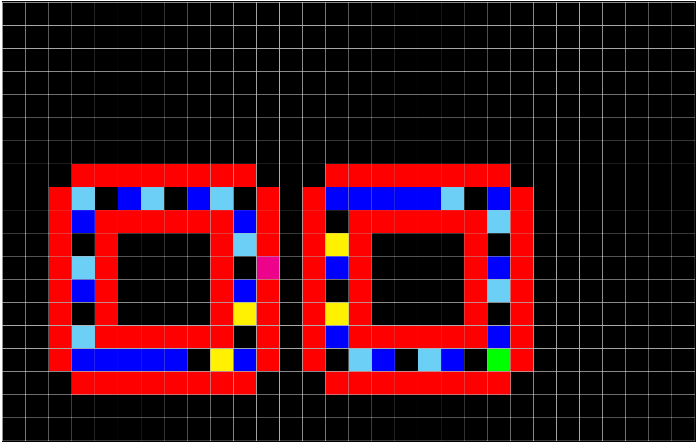
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

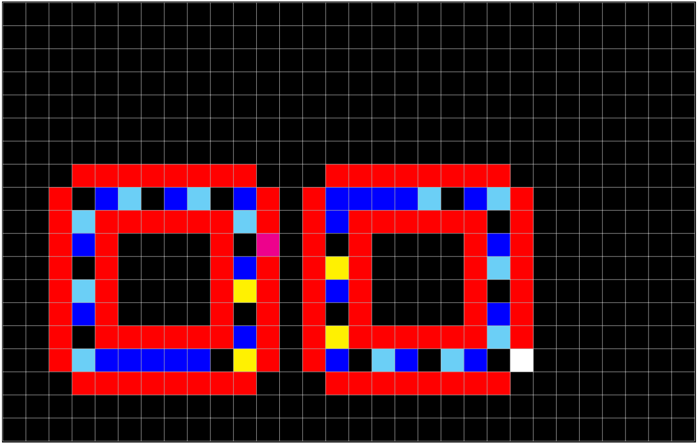
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

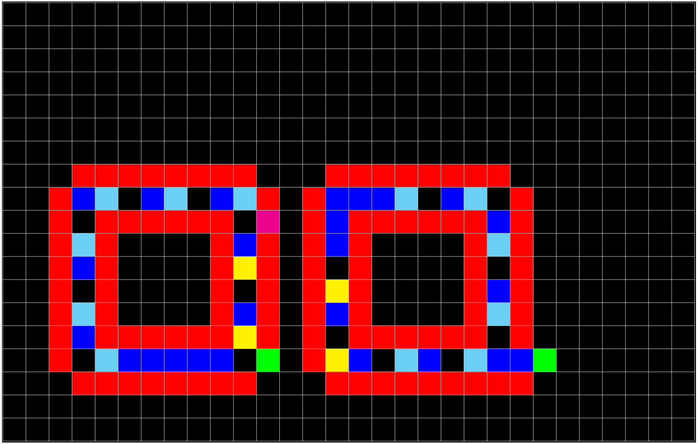
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

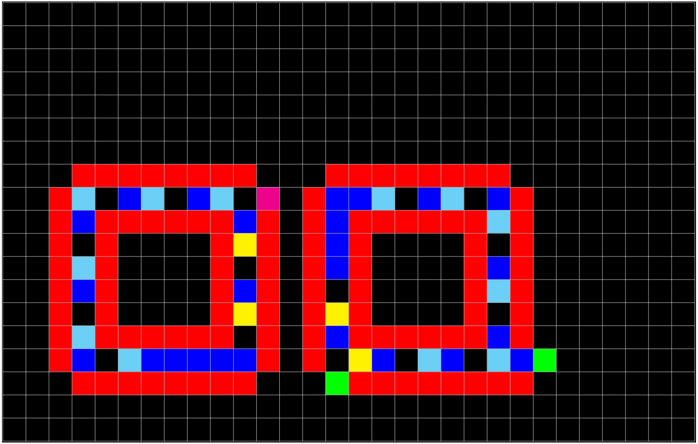
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

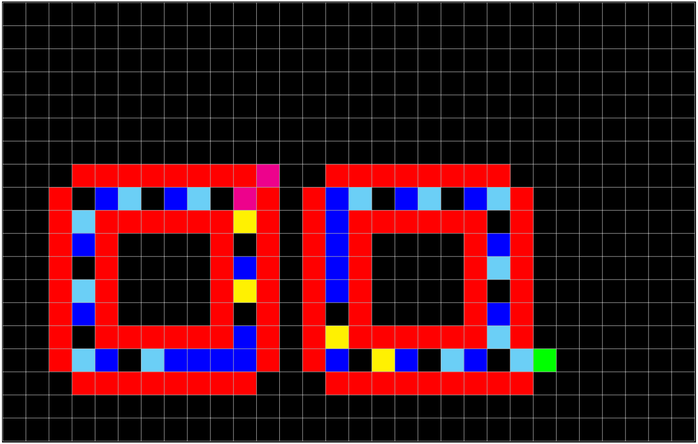
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

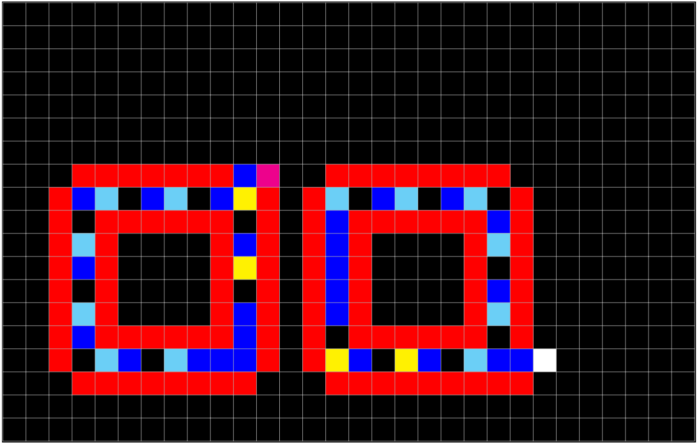
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

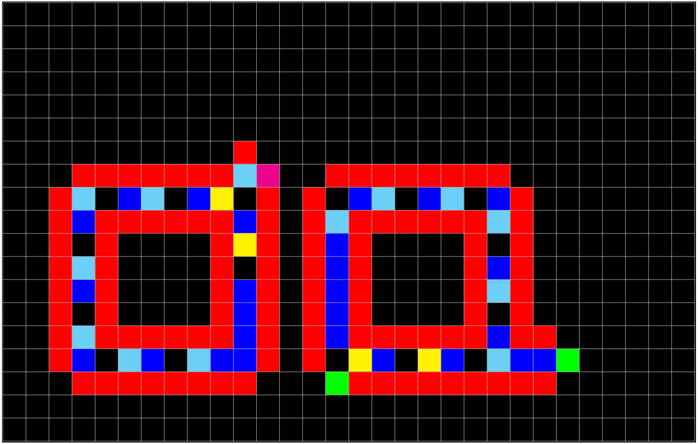
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

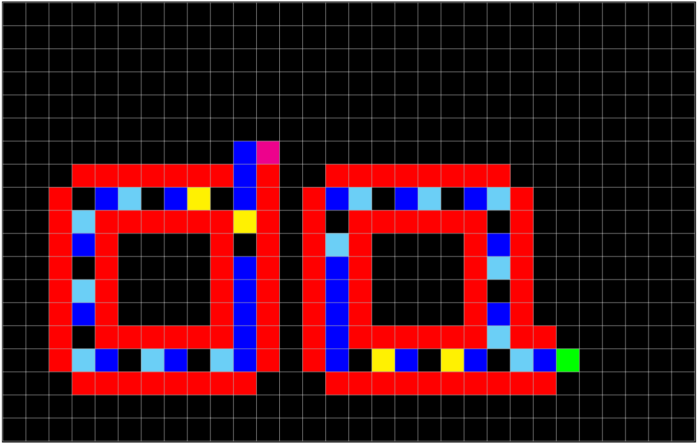
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

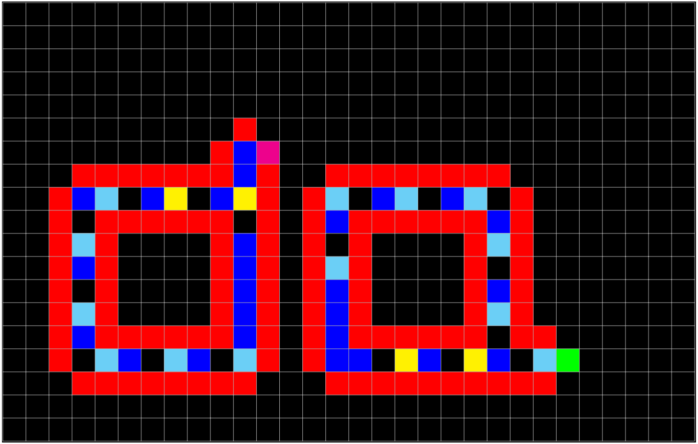
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

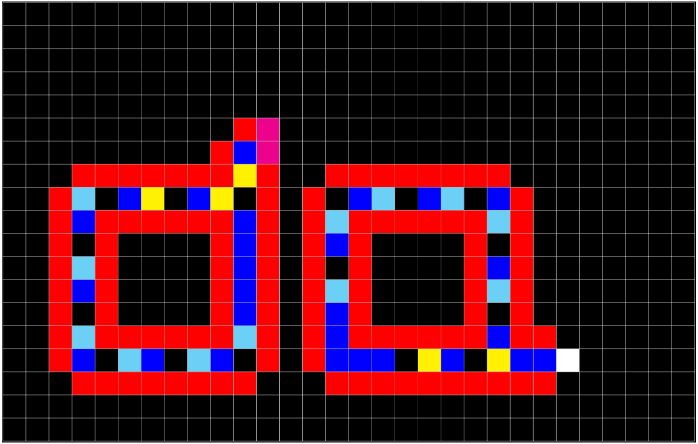
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

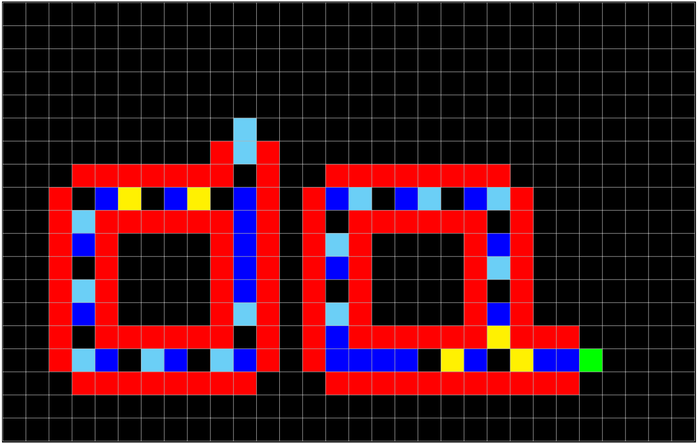
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

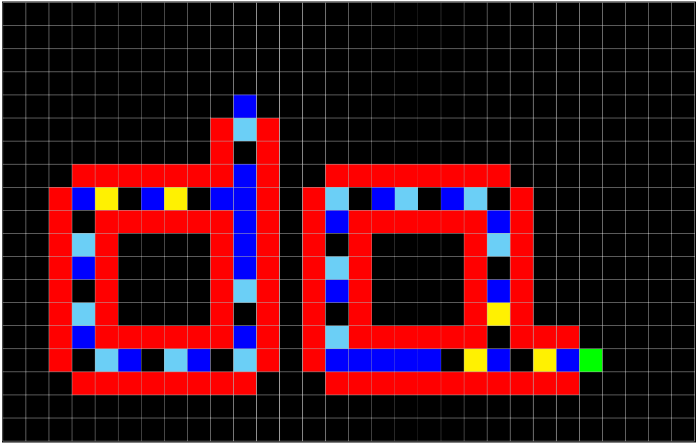
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

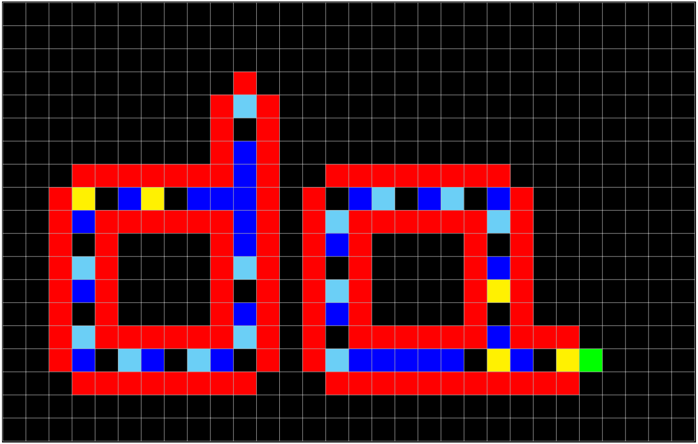
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

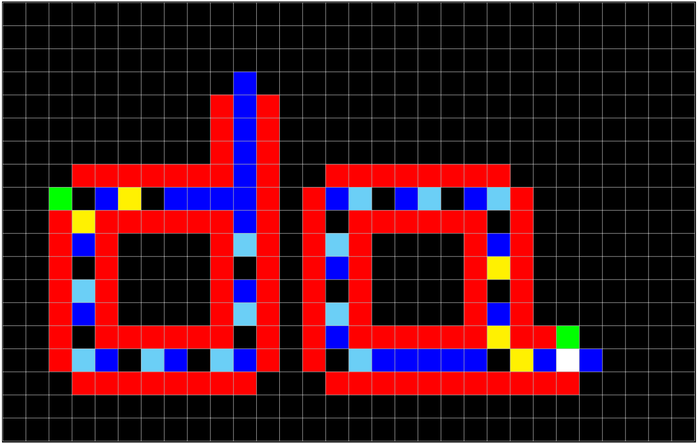
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

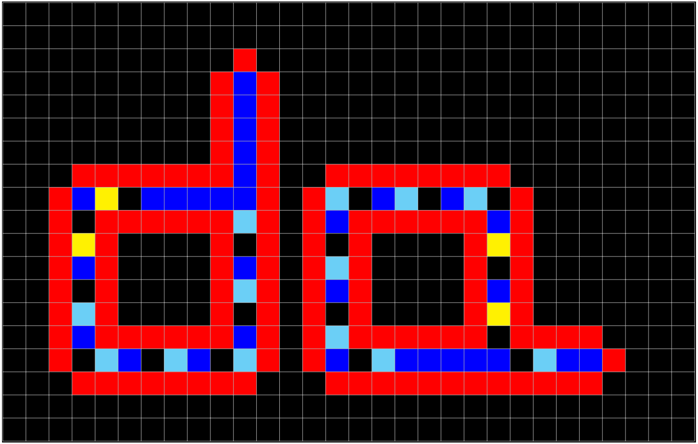
Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

(Langton, 1984)

Langton self-reproducing loops



Langton modifies Codd rule to permit **non universal rotation invariant** self-reproduction by 86 cells in 151 steps.

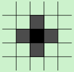
(Langton, 1984)

The XOR rule

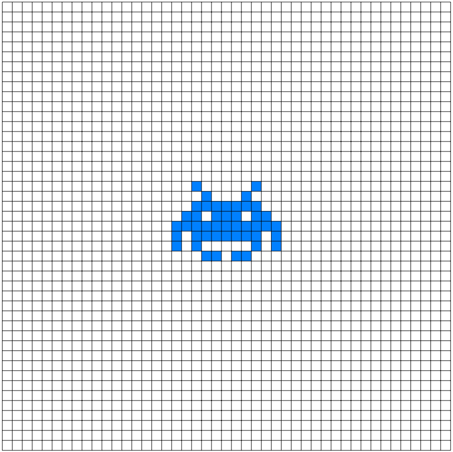
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



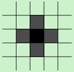
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

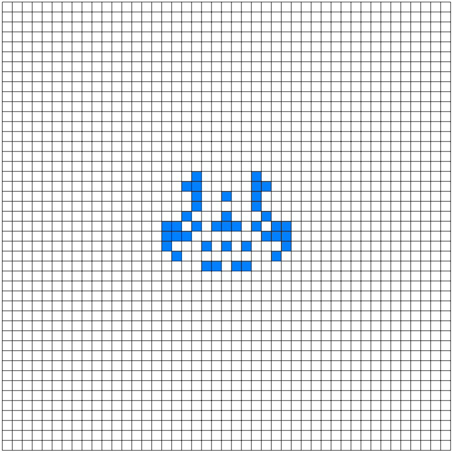
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



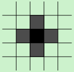
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

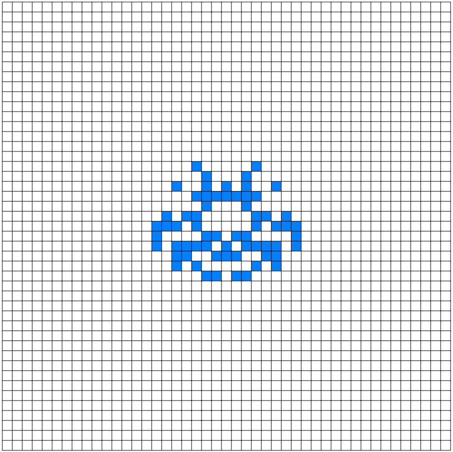
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



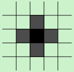
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

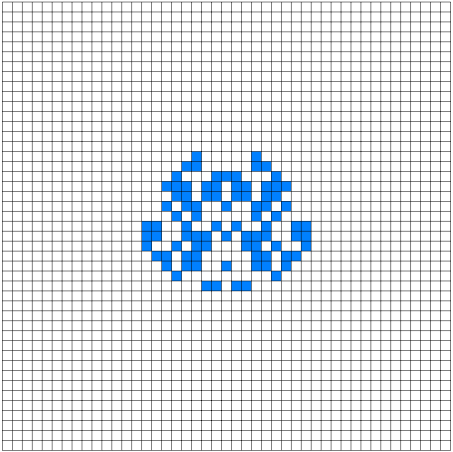
What is a **formal definition** of a self-reproducing CA?

The XOR CA

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



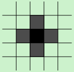
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

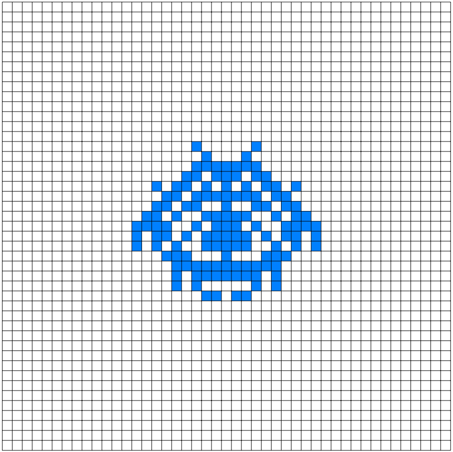
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



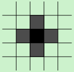
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

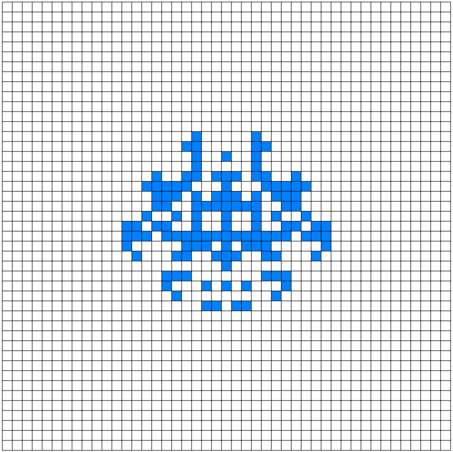
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



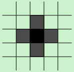
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

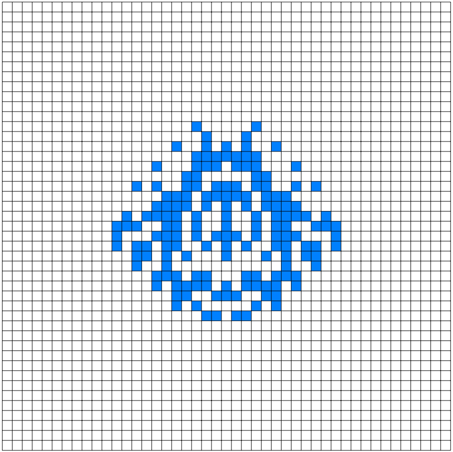
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



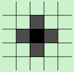
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

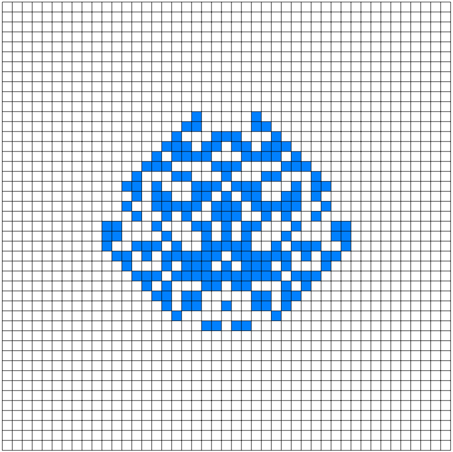
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



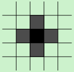
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

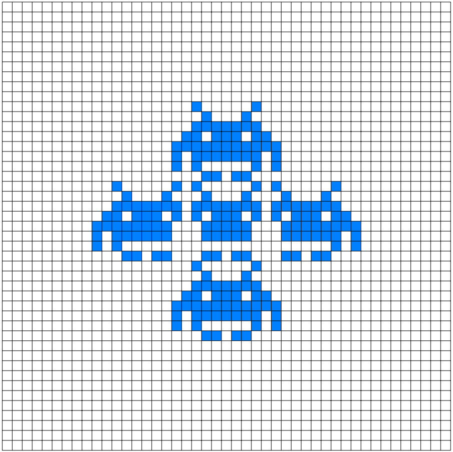
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



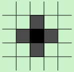
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

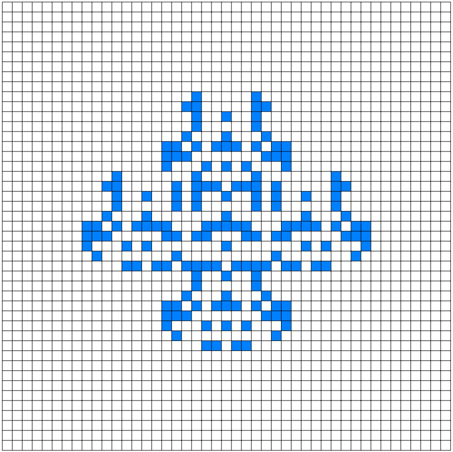
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



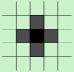
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

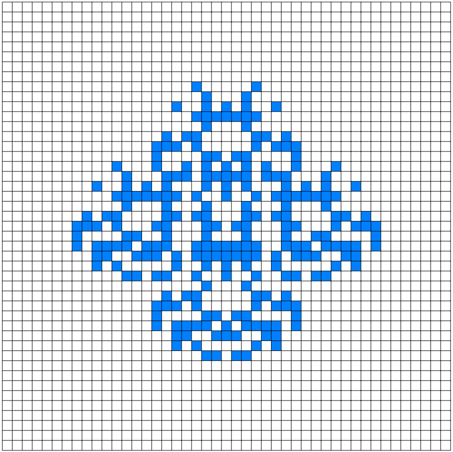
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



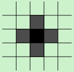
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

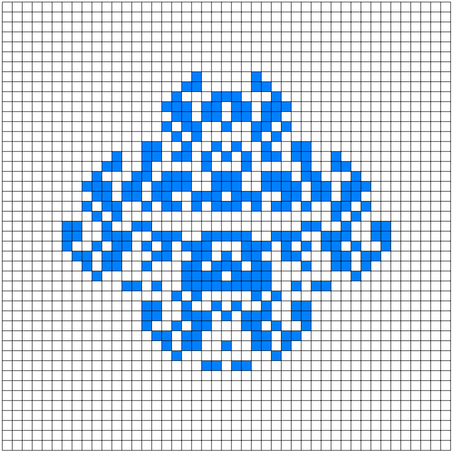
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod{2}$$



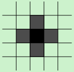
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

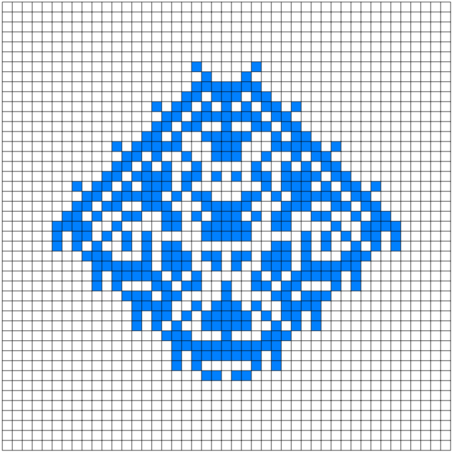
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



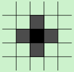
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

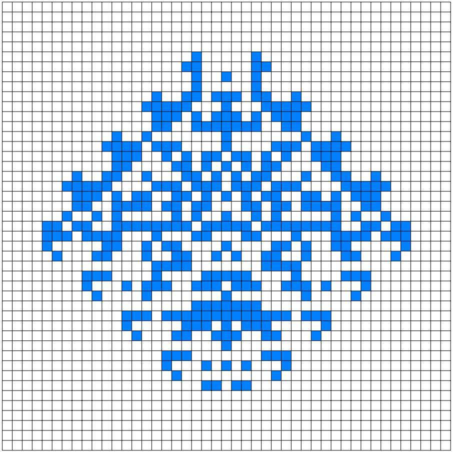
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



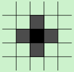
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

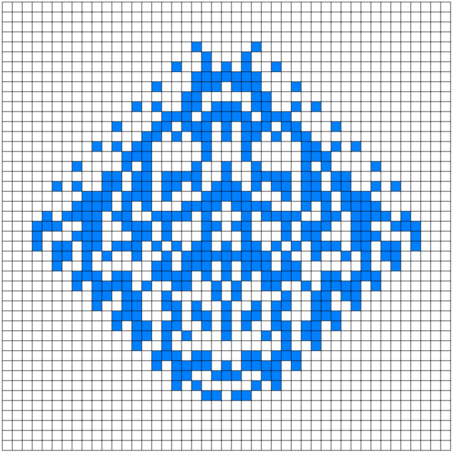
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



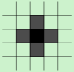
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

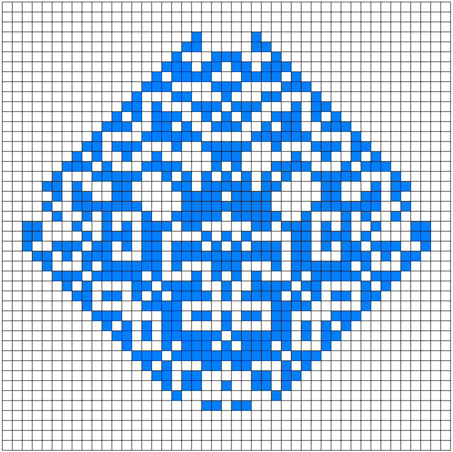
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



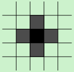
Is the XOR CA a **fair example** of a self-reproducing CA?

The XOR rule

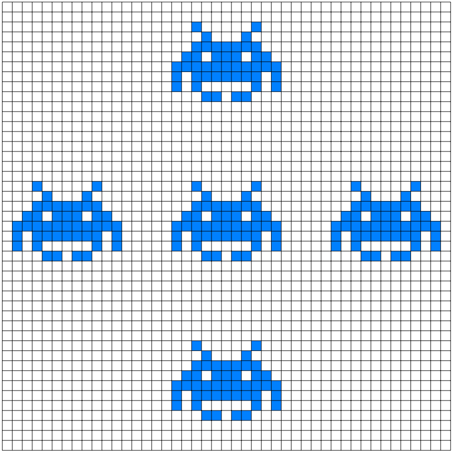
What is a **formal definition** of a self-reproducing CA?

The **XOR CA**

$$S = \mathbb{Z}_2$$

$N =$ 

$$f(x_i) = \sum_i x_i \pmod 2$$



Is the XOR CA a **fair example** of a self-reproducing CA?

An old open problem by Ulam

30 A COLLECTION OF MATHEMATICAL PROBLEMS

2. A problem on matrices arising in the theory of automata

The theory of automata leads to some interesting questions which in the simplest case reduce to matrix theory formulations. Suppose one has an infinite regular system of lattice points in E^n, each capable of existing in various states $S_1, \ldots, S_k$. Each lattice point has a well defined system of m neighbors, and it is assumed that the state of each point at time $t+1$ is uniquely determined by the states of all its neighbors at time t. Assuming that at time t only a finite set of points are active, one wants to know how the activation will spread. In particular, do there exist "universal" systems which are capable of generating arbitrary systems of states. Do there exist subsystems which are able to "reproduce," i.e., to produce other subsystems like the initial ones? In a simple case, one would ask: Does there exist an infinite matrix $A = [a_{ij}]$ of zeros and ones with $\sum_j a_{ij} < B$ for all rows i, such that *every* possible finite matrix of zeros and ones will appear as a main-diagonal submatrix of some power A^p of A? A positive result would provide a simple example of a "universal" and "reproducing" system (in a very limited sense only).

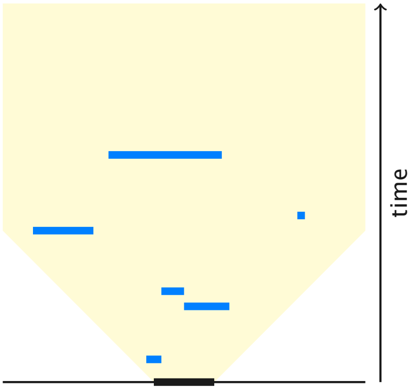
More generally, an analogous question may be asked about matrices whose elements are integers modulo p.

A similar inquiry is pertinent in case of the "recursive functions." Can one obtain all recursive functions by a *prescribed* algorithm operating on a finite set of such functions? More generally, are all expressions in Gödel's system obtainable from a finite system of such expressions and a finite number of rules of composition performed in a prescribed order? That is to say, for example, application of two operations, applied in turn in an order given by one sequence of two symbols.

Perhaps there exists a logical analogue of our universal matrix model.

(Ulam, 1960)

Open Pb Does there exist a CA and a finite configuration that **generates every possible finite pattern**?



Outline of the tutorial

Part I Computing inside cellular space

Engineering *CA and configurations to achieve computational tasks. Universalities. Massively parallel computing.*

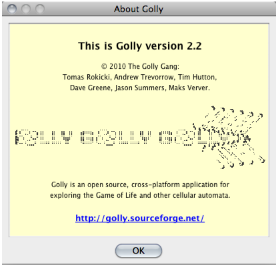
Part II Computing properties of cellular automata

Analyzing *given CA to decide both immediate and dynamical properties. Classical results.*

Part III Computation and reduction: undecidability results

Reducing *instances of undecidable problems to CA and configurations to prove undecidability results. Lots of properties of CA are undecidable.*

Going further



Section I Cellular Automata `http://golly.sf.net/`

`http://www.lif.univ-mrs.fr/~nollinge/acuc/`

Part I

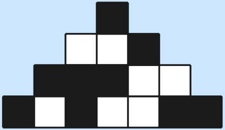
Computing inside the cellular space

Part I
Computing
inside the cellular space

1. Cellular automata

2. A universal model of computation

3. A model of parallel computation



Cellular automata

Definition A **CA** is a tuple (d, S, N, f) where S is a **finite set of states**, $N \subseteq_{\text{finite}} \mathbb{Z}^d$ is the finite **neighborhood** and $f : S^N \to S$ is the **local rule** of the cellular automaton.

A **configuration** $c \in S^{\mathbb{Z}^d}$ is a coloring of $\mathbb{Z}^d$ by S.

The **global map** $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ applies f uniformly and locally:

$$\forall c \in S^{\mathbb{Z}}, \forall z \in \mathbb{Z}^d, \qquad F(c)(z) = f(c_{|z+N}).$$

A **space-time diagram** $\Delta \in S^{\mathbb{Z}^d \times \mathbb{N}}$ satisfies, for all $t \in \mathbb{N}$,

$$\Delta(t+1) = F(\Delta(t)).$$

Space-time diagram

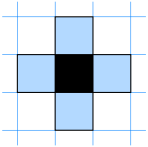
$S = \{0, 1, 2\}$, $r = 1$, $f(x, y, z) = \lfloor 6450288690466/3^{9x+3y+z} \rfloor \pmod 3$

2D cellular automata

Typical **high dimension CA** in this tutorial: $c \in S^{\mathbb{Z}^2}$.

Classical **von Neumann** neighborhood :

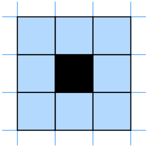
$$N_{\mathsf{vN}} = \{0\} \times \{-1,0,1\} \cup \{-1,0,1\} \times \{0\}$$



von Neumann

Classical **Moore** neighborhood :

$$N_{\mathsf{Moore}} = \{-1,0,1\} \times \{-1,0,1\}$$



Moore

1D cellular automata

More restricted **low dimension CA**, easier to analyze: $c \in S^{\mathbb{Z}}$.

Classical **first neighbors** neighborhood :

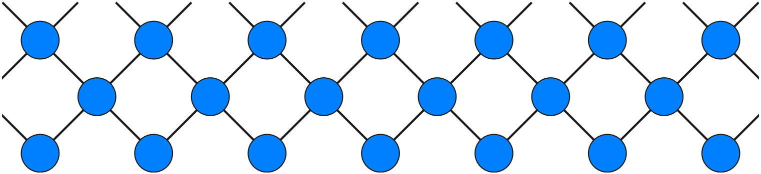
$$N_{\text{first}} = \{-1, 0, 1\}$$

1D

Classical **one-way** neighborhood :

$$N_{\text{OCA}} = \{0, 1\}$$

OCA



Configurations

The set of configurations, $S^{\mathbb{Z}^d}$, is **uncountable**. What reasonnable **countable subset** can we consider?

Recursive configurations are useless, **undecidability** is everywhere (Rice theorem).

Finite configurations with a quiescent state.

Periodic configurations are ultimately periodic.

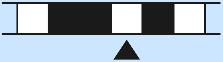
Ultimately periodic configurations compromise.

Thanks to **locality**, one can also consider **partial space-time diagrams** to study all configurations.

Part I
Computing
inside the cellular space

1. Cellular automata

2. A universal model of computation



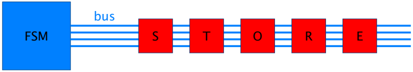
3. A model of parallel computation

Universality in higher dimensions

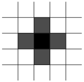
Construction of universal CA appeared with CA as a tool to embed computation into the CA world. First, for **2D CA**

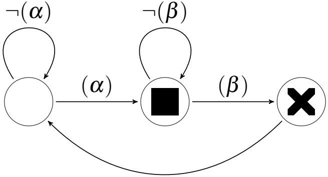
1966	von Neumann	5	29
1968	Codd	5	8
1970	Conway	8	2
1970	Banks	5	2

A natural idea in 2D is to emulate **universal boolean circuits** by embedding ingredients into the CA space: **signals**, **wires**, **turns**, **fan-outs**, **gates**, **delays**, **clocks**, *etc*.



Copper

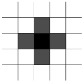
$$\left(\mathbb{Z}^2, \{\square, \blacksquare, \boldsymbol{\times}\},\right.$$  $$\left., \delta\right)$$

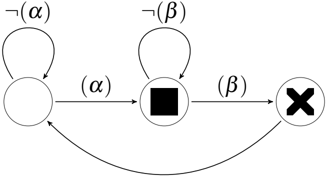


(α) both north and south, or east and west, neighbors in state ✖ or ■ ;

(β) at least two neighbors in state ✖ or ■ and either exactly one neighbor in state ✖ or exactly one neighbor in state ■.

Copper

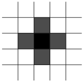
$\left(\mathbb{Z}^2, \{\square, \blacksquare, \times\},\right.$  $\left., \delta\right)$

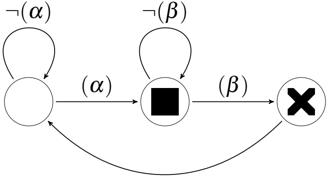


(α) both north and south, or east and west, neighbors in state $\times$ or $\blacksquare$;

(β) at least two neighbors in state $\times$ or $\blacksquare$ and either exactly one neighbor in state $\times$ or exactly one neighbor in state $\blacksquare$.

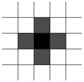
Copper

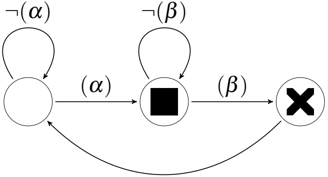
$$\left(\mathbb{Z}^2, \{\square, \blacksquare, \boldsymbol{\times}\},\right.$$  $$\left., \delta\right)$$



- (α) both north and south, or east and west, neighbors in state $\boldsymbol{\times}$ or $\blacksquare$;
- (β) at least two neighbors in state $\boldsymbol{\times}$ or $\blacksquare$ and either exactly one neighbor in state $\boldsymbol{\times}$ or exactly one neighbor in state $\blacksquare$.

Copper

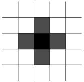
$\Bigg(\mathbb{Z}^2, \{\square, \blacksquare, \times\},$  $, \delta\Bigg)$

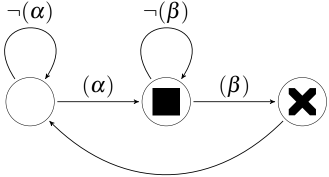


(α) both north and south, or east and west, neighbors in state ✖ or ■ ;

(β) at least two neighbors in state ✖ or ■ and either exactly one neighbor in state ✖ or exactly one neighbor in state ■.

Copper

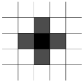
$\Big(\mathbb{Z}^2, \{\square, \blacksquare, \boldsymbol{\times}\},$  $, \delta\Big)$

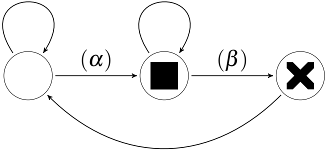


(α) both north and south, or east and west, neighbors in state $\boldsymbol{\times}$ or $\blacksquare$;

(β) at least two neighbors in state $\boldsymbol{\times}$ or $\blacksquare$ and either exactly one neighbor in state $\boldsymbol{\times}$ or exactly one neighbor in state $\blacksquare$.

Copper

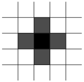
$\Big(\mathbb{Z}^2, \{\square, \blacksquare, \boldsymbol{\times}\},$  $, \delta\Big)$

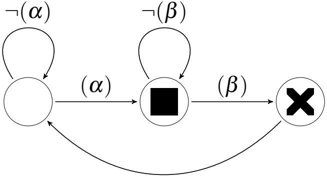


(α) both north and south, or east and west, neighbors in state $\boldsymbol{\times}$ or $\blacksquare$;

(β) at least two neighbors in state $\boldsymbol{\times}$ or $\blacksquare$ and either exactly one neighbor in state $\boldsymbol{\times}$ or exactly one neighbor in state $\blacksquare$.

Copper

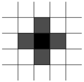
$$\left(\mathbb{Z}^2, \{\square, \blacksquare, \boldsymbol{\times}\}, \right.$$  $$\left., \delta\right)$$

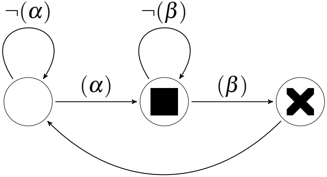


(α) both north and south, or east and west, neighbors in state $\boldsymbol{\times}$ or $\blacksquare$;

(β) at least two neighbors in state $\boldsymbol{\times}$ or $\blacksquare$ and either exactly one neighbor in state $\boldsymbol{\times}$ or exactly one neighbor in state $\blacksquare$.

Copper

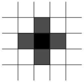
$\left(\mathbb{Z}^2, \{\square, \blacksquare, \text{✖}\},\right.$  $\left., \delta\right)$

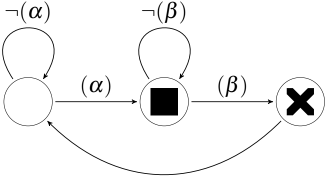


(α) both north and south, or east and west, neighbors in state ✖ or ■ ;

(β) at least two neighbors in state ✖ or ■ and either exactly one neighbor in state ✖ or exactly one neighbor in state ■.

Copper

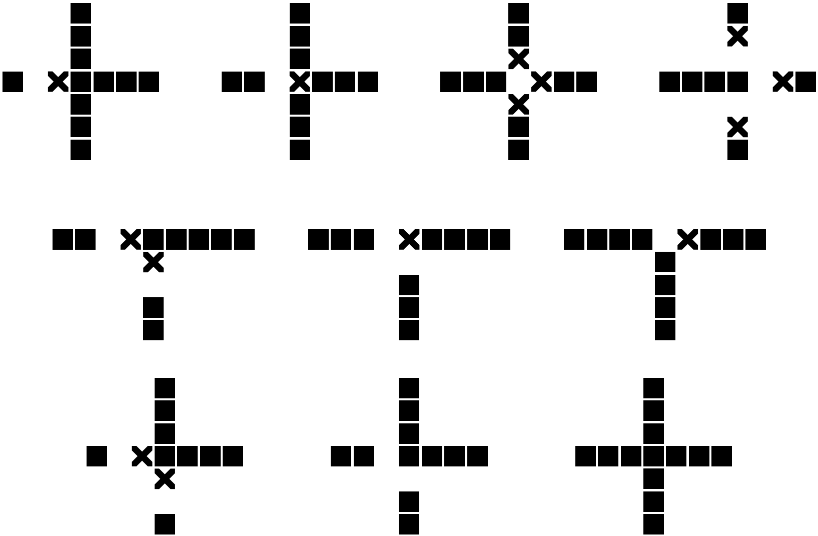
$$\left(\mathbb{Z}^2, \{\square, \blacksquare, \boldsymbol{\times}\},\right.$$

$$\left., \delta\right)$$



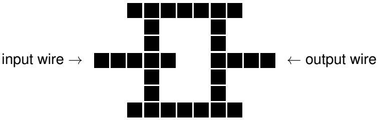
(α) both north and south, or east and west, neighbors in state $\boldsymbol{\times}$ or $\blacksquare$;

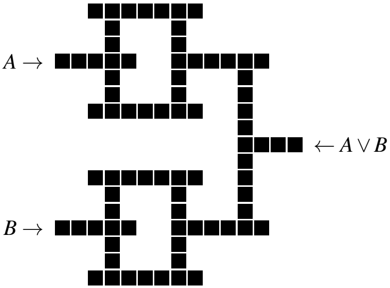
(β) at least two neighbors in state $\boldsymbol{\times}$ or $\blacksquare$ and either exactly one neighbor in state $\boldsymbol{\times}$ or exactly one neighbor in state $\blacksquare$.

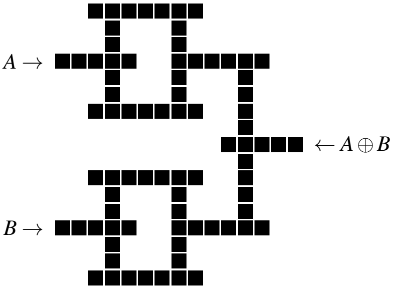
Copper: Intersections



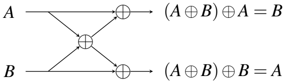
Copper: Gates

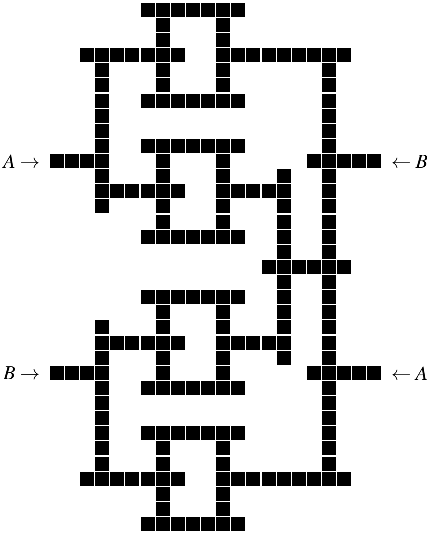




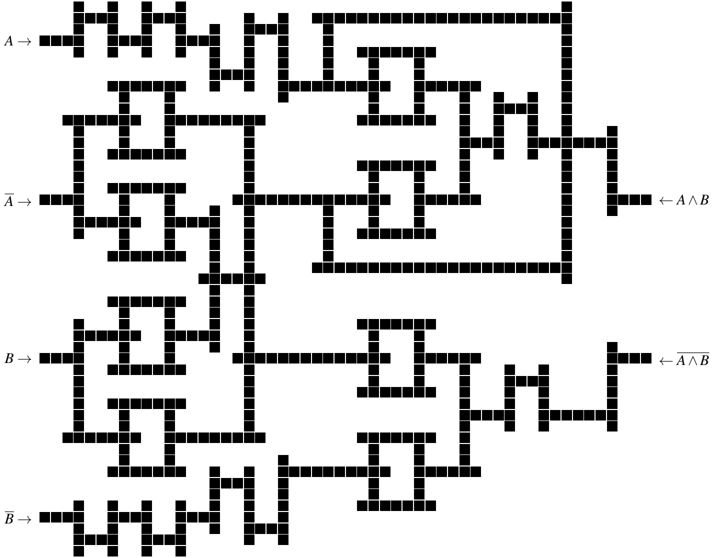


Copper: Xing


A
B
$(A \oplus B) \oplus A = B$
$(A \oplus B) \oplus B = A$

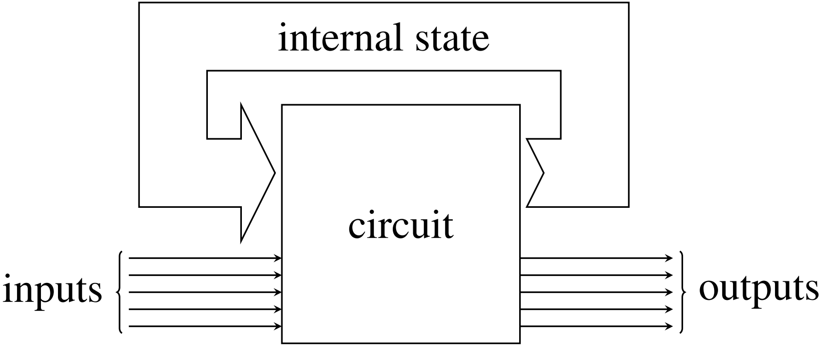

$A \rightarrow$
$\leftarrow B$
$B \rightarrow$
$\leftarrow A$

Copper: AND



$A \rightarrow$
$\overline{A} \rightarrow$
$B \rightarrow$
$\overline{B} \rightarrow$
$\leftarrow A \wedge B$
$\leftarrow \overline{A \wedge B}$

Copper: FSM



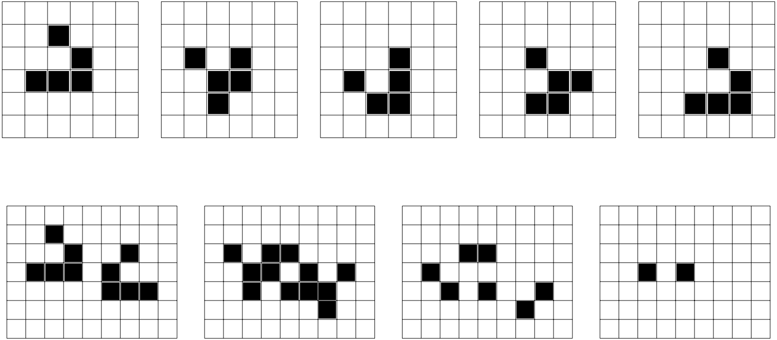
Theorem Copper is **universal for boolean circuits**.

Simulating a universal device requires an **ultimately periodic configuration** of **infinitely many** non quiescent cells.

The Game of Life

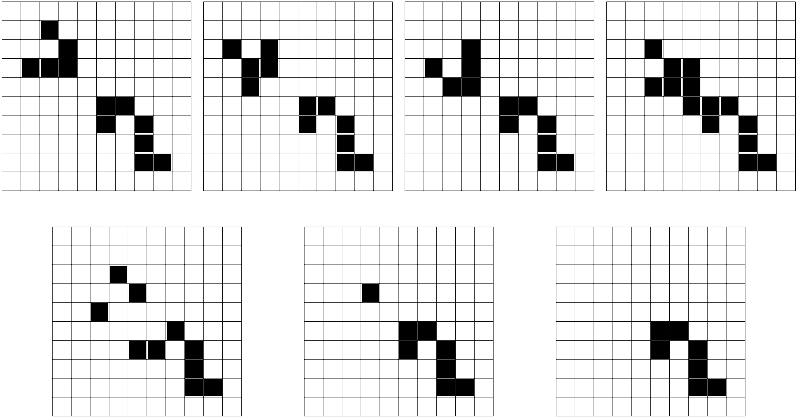
Theorem GoL is **universal for boolean circuits**.

The construction uses **gliders** as signals.

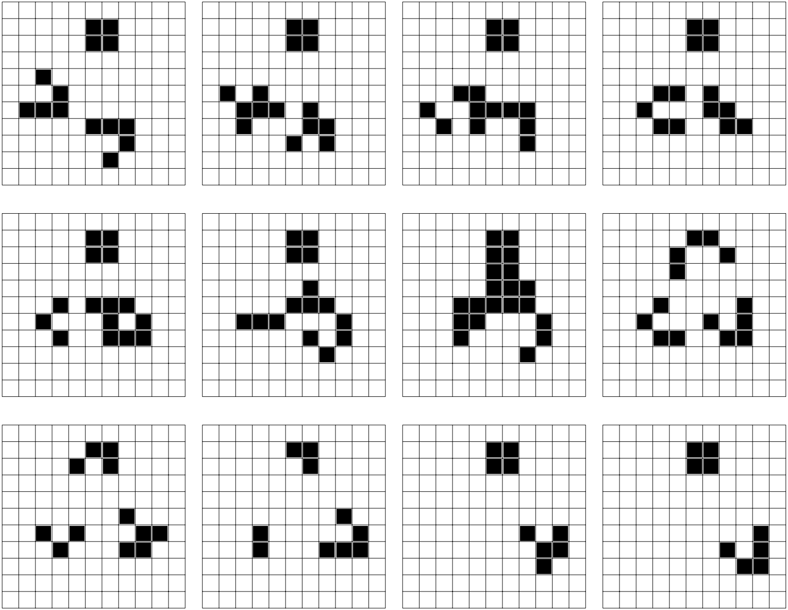


(Conway *et al.*, Winning Ways Vol. 2., 1971)

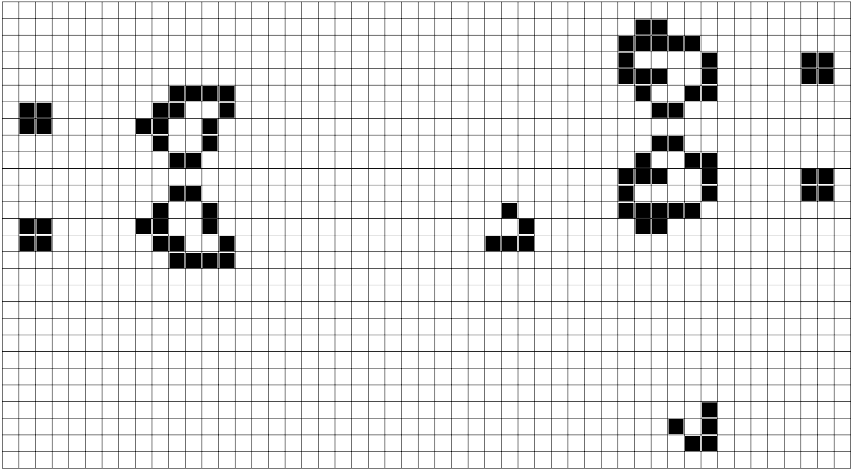
GoL: Eater



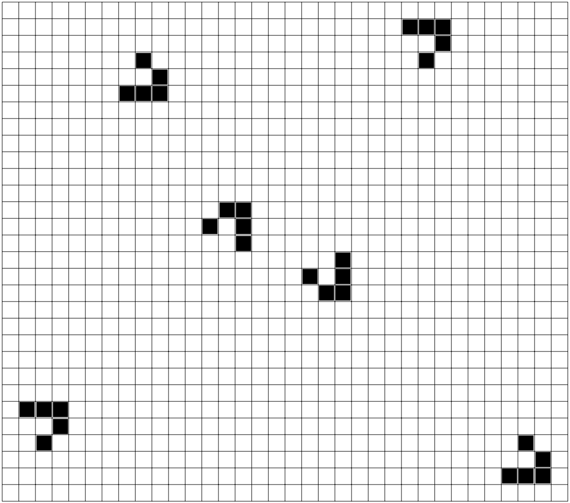
GoL: Duplicator



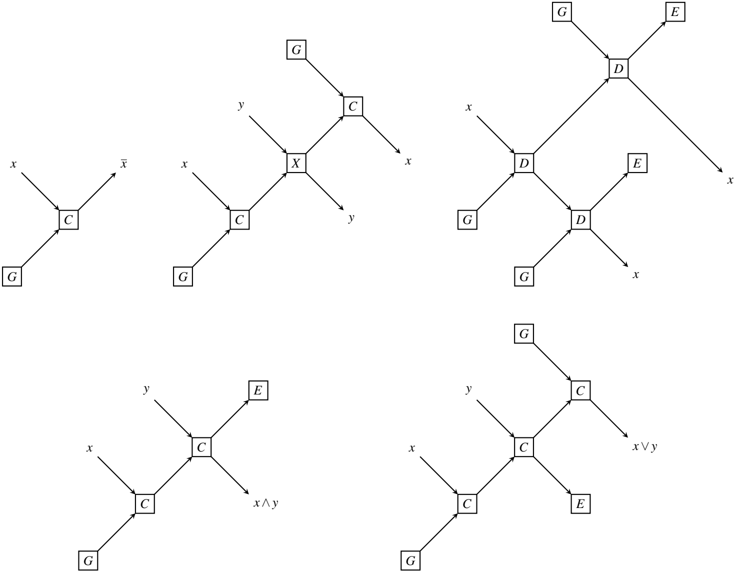
GoL: Gosper's p46 Gun



GoL: Xing



GoL: Combining



Universality in 1D

Remark Boolean circuits are **less intuitive** to simulate in 1D, but it is easy to simulate **sequential models of computation** like Turing machines.

(A. R. Smith III, Simple computation-universal cellular spaces, 1971)

1971	Smith III	18
1987	Albert & Culik II	14
1990	Lindgren & Nordhal	7
2004	Cook	2

A cellular automaton is **Turing-universality** if

Universality in 1D

Remark Boolean circuits are **less intuitive** to simulate in 1D, but it is easy to simulate **sequential models of computation** like Turing machines.

(A. R. Smith III, Simple computation-universal cellular spaces, 1971)

1971	Smith III	18
1987	Albert & Culik II	14
1990	Lindgren & Nordhal	7
2004	Cook	2

A cellular automaton is **Turing-universality** if... What exactly is the **formal definition**?

Universality in 1D

Remark Boolean circuits are **less intuitive** to simulate in 1D, but it is easy to simulate **sequential models of computation** like Turing machines.

(A. R. Smith III, Simple computation-universal cellular spaces, 1971)

1971	Smith III	18
1987	Albert & Culik II	14
1990	Lindgren & Nordhal	7
2004	Cook	2

A cellular automaton is **Turing-universality** if... What exactly is the **formal definition**? What is a **non universal** CA?

Universality in 1D

Remark Boolean circuits are **less intuitive** to simulate in 1D, but it is easy to simulate **sequential models of computation** like Turing machines.

(A. R. Smith III, Simple computation-universal cellular spaces, 1971)

1971	Smith III	18
1987	Albert & Culik II	14
1990	Lindgren & Nordhal	7
2004	Cook	2

A cellular automaton is **Turing-universality** if... What exactly is the **formal definition**? What is a **non universal** CA?

A consensual yet **formal definition** is **unknown** and seems difficult to achieve. (Durand & Roka, 1999)

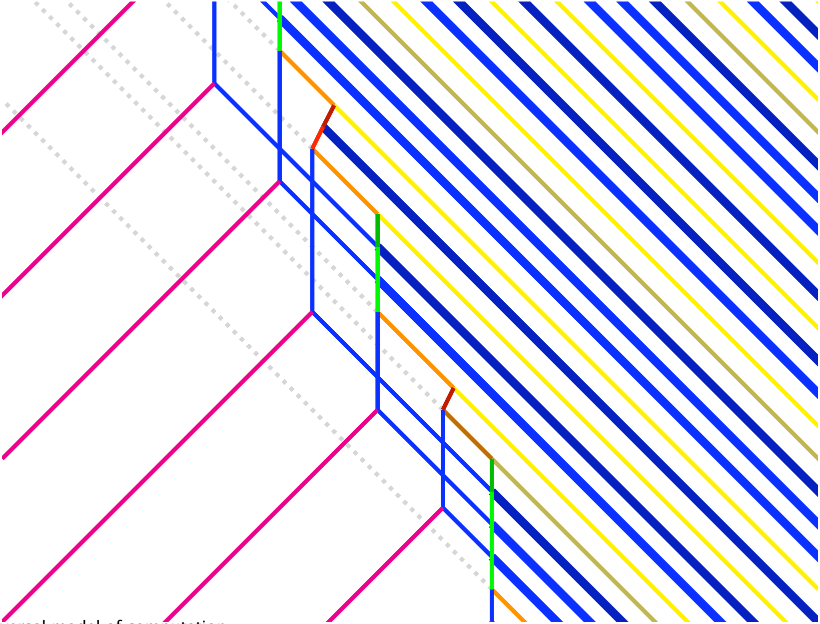
TM *à la* Smith III

B	B	(q_0, a)	a	B
B	B	a	(q_1, B)	B
B	B	(q_1, b)	B	B
B	(q_0, B)	b	B	B
B	B	(q_0, a)	B	B

TM *à la* Lindgren & Nordhal

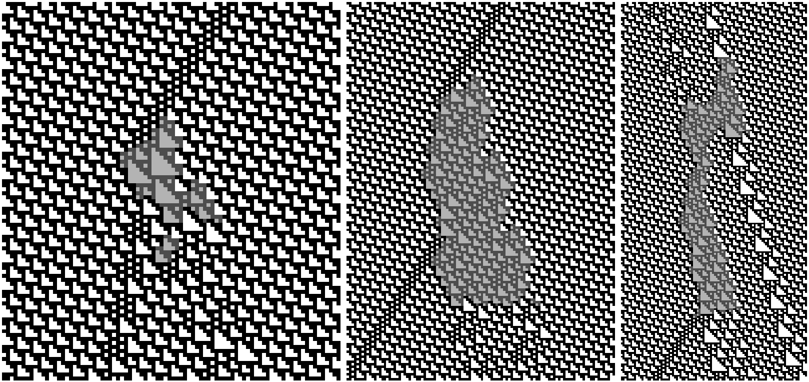
B	$\bullet$	B	$\bullet$	a	q_0	$\bullet$	a	$\bullet$	B
B	$\leftrightarrow$	B	$\leftrightarrow$	a	$\leftrightarrow$	q_0	a	$\leftrightarrow$	B
B	$\bullet$	B	$\bullet$	a	$\bullet$	q_1	B	$\bullet$	B
B	$\leftrightarrow$	B	$\leftrightarrow$	a	q_1	$\leftrightarrow$	B	$\leftrightarrow$	B
B	$\bullet$	B	$\bullet$	q_1	b	$\bullet$	B	$\bullet$	B
B	$\leftrightarrow$	B	q_1	$\leftrightarrow$	b	$\leftrightarrow$	B	$\leftrightarrow$	B
B	$\bullet$	B	q_0	$\bullet$	b	$\bullet$	B	$\bullet$	B
B	$\leftrightarrow$	B	$\leftrightarrow$	q_0	b	$\leftrightarrow$	B	$\leftrightarrow$	B
B	$\bullet$	B	$\bullet$	q_0	a	$\bullet$	B	$\bullet$	B

Universality of Rule 110 *à la* Cook



Cook: Details...

Uses **huge** particles and collisions...



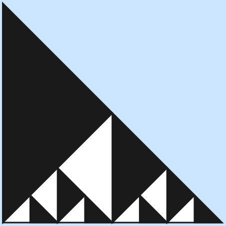
Theorem Rule 110 is **Turing-universal**.

Part I
Computing
inside the cellular space

1. Cellular automata

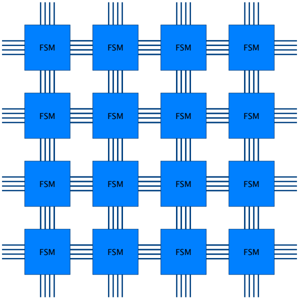
2. A universal model of computation

3. A model of parallel computation



Another path to universality

Remark Boolean circuits are not sequential and can also simulate **parallel models of computation**.



This leads to a stronger notion of **intrinsic universality** on CA, the ability to simulate any CA.

Bulking classifications

Idea define a **quasi-order** on cellular automata, **equivalence classes** capturing behaviors.

Definition A CA $\mathcal{A}$ is **algorithmically simpler** than a CA $\mathcal{B}$ if all the space-time diagrams of $\mathcal{A}$ are space-time diagrams of $\mathcal{B}$ *(up to uniform state renaming)*.

Formally, $\mathcal{A} \subseteq \mathcal{B}$ if there exists $\varphi : S_{\mathcal{A}} \to S_{\mathcal{B}}$ injective such that $\overline{\varphi} \circ G_{\mathcal{A}} = G_{\mathcal{B}} \circ \overline{\varphi}$.

That is, the following diagram commutes:

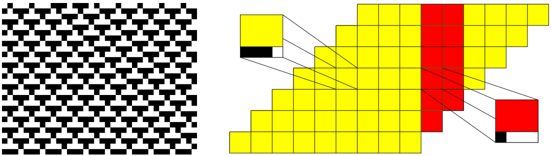
$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "G_{\mathcal{A}}"'] & \overline{\varphi}(C) \arrow[d, "G_{\mathcal{B}}"] \\ G_{\mathcal{A}}(C) \arrow[r, "\varphi"'] & \overline{\varphi}(G_{\mathcal{A}}(C)) \end{tikzcd}$$

Bulking quasi-order

Quotient the set of CA by **discrete affine transformations**, the only geometrical transformations preserving CA.

The $\langle m, n, k\rangle$ transformation of $\mathcal{A}$ satisfies:

$$G_{\mathcal{A}^{\langle m,n,k\rangle}} = \sigma^k \circ o^m \circ G_{\mathcal{A}}^n \circ o^{-m} \ .$$

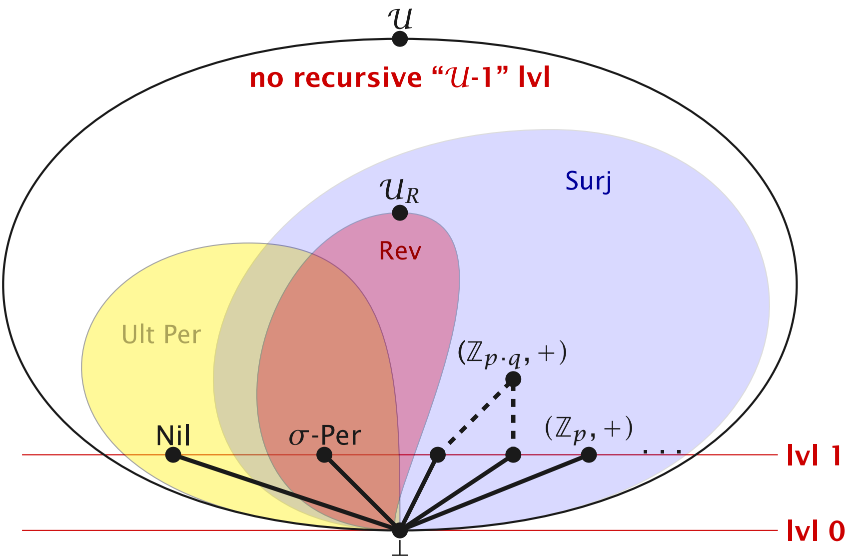


$\mathcal{A}$ $\qquad$ $\mathcal{A}^{\langle 4,4,1\rangle}$

Definition The **bulking quasi-order** is defined by $\mathcal{A} \leqslant \mathcal{B}$ if there exists $\langle m, n, k\rangle$ and $\langle m', n', k'\rangle$ such that

$$\mathcal{A}^{\langle m,n,k\rangle} \subseteq \mathcal{B}^{\langle m',n',k'\rangle} \ .$$

The big picture



Intrinsic universality

Definition A CA $\mathcal{U}$ is **intrinsically universal** if it is maximal for $\leqslant$, *i.e.* for all CA $\mathcal{A}$, there exists α such that $\mathcal{A} \subseteq \mathcal{U}^{\alpha}$.

Theorem There exists **Turing universal** CA that are not intrinsically universal.

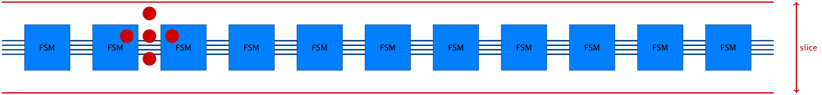
where Turing universality is obtained in a very classical way to ensure compatibility with your own definition.

Theorem **Boolean circuit universal** 2D CA are also **intrinsically universal**.

(Delorme et al., 2011)

Using boolean circuits

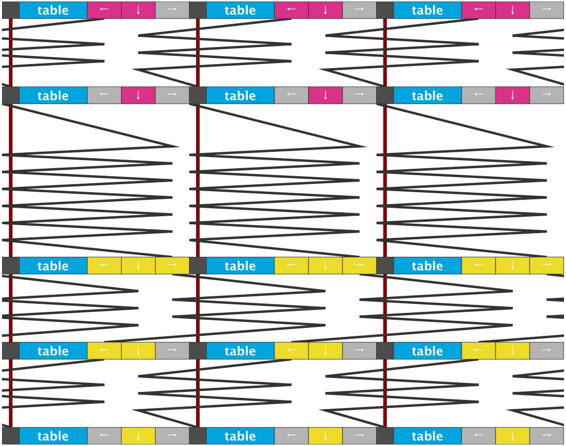
Every **2D intrinsically universal** CA can be converted to a **1D intrinsically universal** CA **[Banks 1970]**.



Cut **slices** of a periodic configuration, catenate them **horizontally**, use the **adequate neighborhood**.

The neighborhood can be transformed into radius 1 at the cost of **increase of the number of states**.

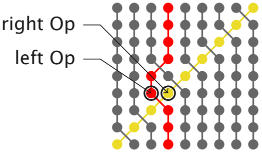
Using highly parallel Turing machines

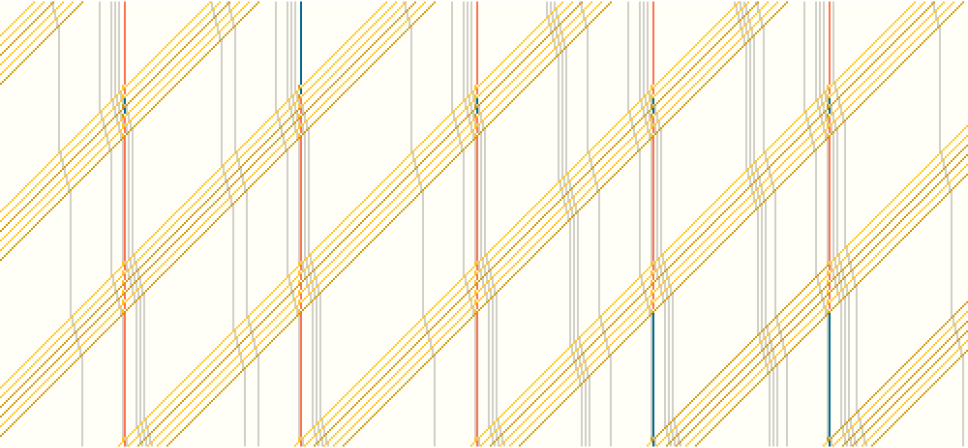


Use one **Turing-like head** per macro-cell, the **moving sequence** being **independent** of the computation.

More intricate: 6 states

A **6 states** intrinsically universal CA of radius 1 embedding **boolean circuits** into the line.





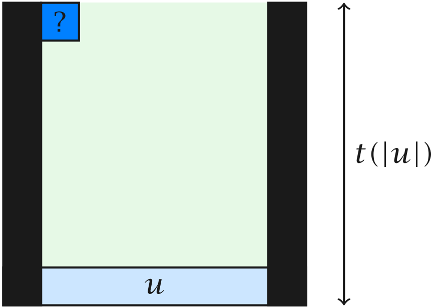
Parallel language recognition

In the **70s** CA have been studied as **a model of massive parallelism**, in particular as **language recognizers**.

A CA (S, N, f) **recognizes** a language $L \subseteq \Sigma^*$ in time $t(n)$ with border state $\#$ and accepting states $Y \subseteq S$ if for each $u \in \Sigma^*$, starting from the configuration ${}^{\omega}\#u\#^{\omega}$, at time $t(|u|)$ the state of cell 0 is in Y if and only if $u \in L$.

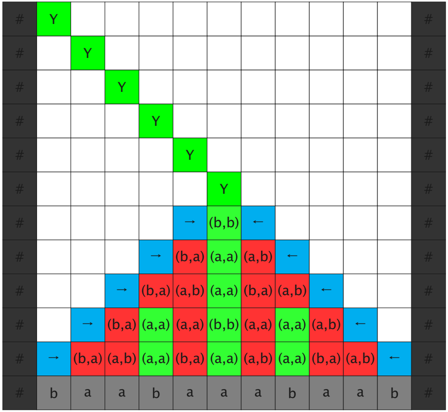
Real time: $t(n) = n$

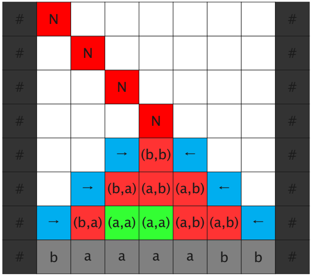
Linear time: $t(n) = \alpha n$



Example: palindromes

Recognizing **palindromes** in real time with 16 states.





Real time vs Linear time

Theorem Every language recognized in time $t(n) = n + T(n)$ is recognized in time $t(n) = n + T(n)/k$ for all k. The cost of **acceleration** is paid using more states.

Same techniques as for Turing machine **acceleration**.

Theorem[Ibarra] Every language recognized in **linear time** is recognized in **real time** if and only if the class of real time langages is **closed by mirror image**.

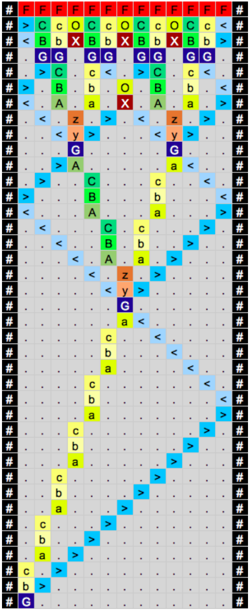
Open Pb Does **Real time** = **Linear time**?

Firing Squad Synchronization Pb

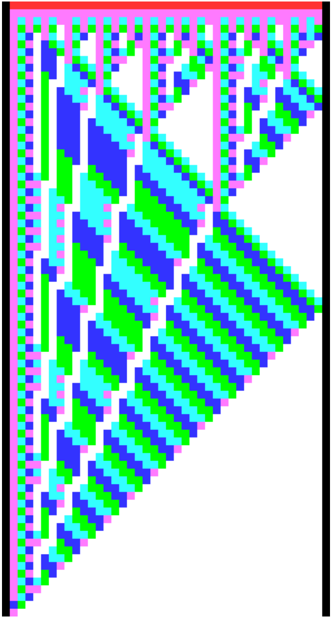
CA can also solve **purely parallel tasks**.

Definition A CA solves the **FSSP** if for all $n > 0$ starting from $\#GB^{n-1}\#$ it eventually enters $\#F^n\#$ and the fire state F never appears before.

Theorem[Minksy] A **CA solves FFSP** in time $3n - 1$ with 15+1 states.



Mazoyer's solution



Remark No CA can solve the FSSP in time less than $2n - 2$.

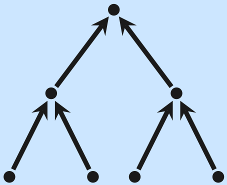
Theorem[Mazoyer 1984] A **CA solves FSSP** in **optimal time** with 6+1 states.

Part II

Computing properties of CA

Part II
Computing properties of CA

4. Discrete dynamical systems



5. Immediate properties

6. Dynamical properties

Cellular automata

Definition A **CA** is a tuple (d, S, r, f) where S is a **finite set of states**, $r \in \mathbb{N}$ is the **neighborhood radius** and $f : S^{(2r+1)^d} \to S$ is the **local rule** of the cellular automaton.

A **configuration** $c \in S^{\mathbb{Z}^d}$ is a coloring of $\mathbb{Z}^d$ by S.

The **global map** $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ applies f uniformly and locally:

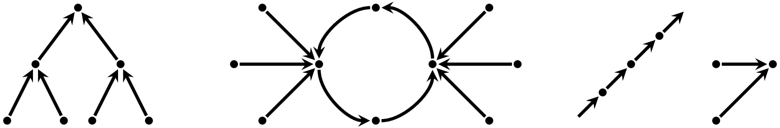
$$\forall c \in S^{\mathbb{Z}}, \forall z \in \mathbb{Z}^d, \qquad F(c)(z) = f(c(z-r), \ldots, c(z+r)).$$

A **space-time diagram** $\Delta \in S^{\mathbb{Z}^d \times \mathbb{N}}$ satisfies, for all $t \in \mathbb{N}$,

$$\Delta(t+1) = F(\Delta(t)).$$

Discrete dynamical systems

Definition A **DDS** is a pair (X, F) where X is a topological space and $F : X \to X$ is a continuous map.



Definition The **orbit** of $x \in X$ is the sequence $(F^n(x))$ obtained by iterating F.

In this tutorial, $X = S^{\mathbb{Z}}$ is endowed with the **Cantor topology** (product of the discrete topology on S), and F is a continuous map **invariant by translation**.

Topology

Definition A **topological space** is a pair $(E, \mathcal{O})$ where $\mathcal{O} \subseteq \mathcal{P}(E)$ is the set of **open** subsets satisfying:

- $\mathcal{O}$ contains both $\varnothing$ and E;
- $\mathcal{O}$ is closed under union;
- $\mathcal{O}$ is closed under finite intersection.

S is endowed with the **discrete topology**: $\mathcal{O} = \mathcal{P}(S)$.

$S^{\mathbb{Z}^d}$ is endowed with the **Cantor topology**: the product topology of the discrete topology.

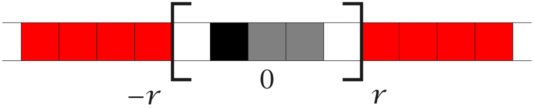
$$\mathcal{O} = \left\{ \prod X_i \,\middle|\, X_i \subseteq S \wedge \mathrm{Card}(\{i | X_i \neq S\}) < \omega \right\}$$

Cantor topology is **metric** and **compact**.

Cylinders

Definition The cylinder $[m] \subseteq S^{\mathbb{Z}^d}$ with radius $r \geqslant -1$ generated by the pattern $m \in S^{[-r,r]^d}$ is

$$[m] = \left\{c \in S^{\mathbb{Z}^d} \,\middle|\, \forall p \in \mathbb{Z}^d, ||p||_\infty \leqslant r \Rightarrow c(p) = m(p)\right\}$$



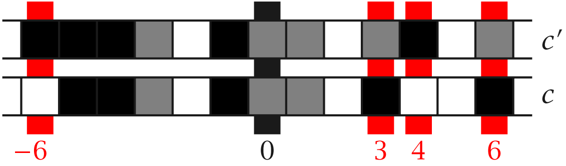
Proposition Cylinders are a countable **clopen generating set**.

Notation $[m] \prec [m']$ means $[m]$ is a **sub-cylinder** of $[m']$, *i.e.* $[m'] \subset [m]$.

Metric

Proposition Cantor topology is **metric**

$$\forall c, c' \in S^{\mathbb{Z}^d}, \quad \delta(c, c') = 2^{-\min\left\{\|p\|_\infty \,\middle|\, c_p \neq c'_p\right\}}$$



$$\delta(c, c') = 1/8$$

Remark Open balls of δ exactly correspond to cylinders.

Compact

Proposition Every sequence of configurations $(c_i) \in {S^{\mathbb{Z}^d}}^{\mathbb{N}}$ admits a converging subsequence.

Proof by **extraction**:

By recurrence, let $\left(c_i^0\right) = (c_i)$.

It is alway possible to find:

- a cylinder $[m_n]$ of radius n and
- an infinite subsequence $\left(c_i^{n+1}\right)$ de $\left(c_{i+1}^{n}\right)$

such that for all $i \in \mathbb{N}$, $c_i^{n+1} \in [m_n]$.

By construction $[m_{n+1}] \subset [m_n]$ and $\left(c_0^{i+1}\right)$ is a converging subsequence of (c_i) (to $\bigcap [m_i]$): $\delta(c_0^{n+1}, c_0^{n+2}) \leqslant 2^{-n}$.

König trees

Remark Cantor topology is essentially **combinatorial**.

Remark Main properties can be obtained using **extraction**.

König's Lemma Every infinite tree with finite branching admis an infinite branch.

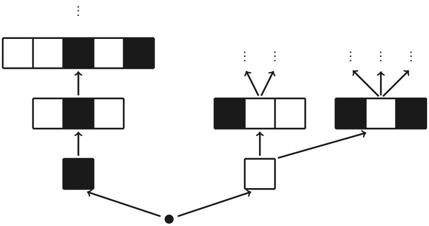
Definition The **König tree** $\mathcal{A}_C$ of a set of configurations $C \subseteq S^{\mathbb{Z}^d}$ is the tree (V_C, E_C) where

$$\begin{aligned} V_C &= \{[m] | C \cap [m] \neq \emptyset\} \\ E_C &= \left\{([m],[m']) \,\middle|\, [m] \prec [m'] \wedge \mathrm{r}([m']) = \mathrm{r}([m]) + 1\right\} \end{aligned}$$

The root of the tree is the cylinder $[\,] = S^{\mathbb{Z}^d}$ of radius -1.

Toppings

The König tree of a **non empty** set of configurations is an **infinite tree** with finite branching.



To each infinite branch $([m_i])$ is associated a **unique configuration** $\bigcap[m_i]$.

Definition The **topping** $\overline{\mathcal{A}_C}$ of a König tree is the set of configurations associated to its infinite branches.

König topology

The **König topology** is defined by its closed sets: toppings of König trees.

The **complementary of a closed set** is the **union of cylinders** that are not nodes of the tree.

Theorem **Cantor** and **König** topologies are the **same**.

Most **topological concepts** can be explained using **trees**:

- dense sets;
- closed sets with non empty interior;
- compacity;
- Baire's theorem.

Continuity

Proposition **clopen** sets are finite unions of cylinders.

Definition A mapping $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is **local** in $p \in \mathbb{Z}^d$ if there exists a radius r such that:

$$\forall c, c' \in S^{\mathbb{Z}^d}, \quad \left[c_{|r}\right] = \left[c'_{|r}\right] \Rightarrow G(c)_p = G(c')_p \quad .$$

Proposition A mapping $G : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ is **continuous** if and only if it is **local in every point**.

Curtis-Hedlund-Lyndon Theorem

Definition The **translation** $\sigma_k : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ with vector $k \in \mathbb{Z}^d$ satisfies:

$$\forall c \in S^{\mathbb{Z}^d}, \forall p \in \mathbb{Z}^d, \quad \sigma_k(c)_p = c_{p-k} \quad .$$

Theorem[Hedlund 1969] Continuous mapping commuting with translations are exactly global maps of CA.

Thus **CA** can be given by there **global map**.

Remark CA have a **dual nature**: discrete dynamical systems with a description as finite automata.

Symbolic dynamics

A central object in **symbolic dynamics** is **subshift**.

Definition A **subshift** of $S^{\mathbb{Z}^d}$ is a set of configurations both **closed** and **invariant by translation**.

Ex $\dots abaababaaa \dots$

$$X = \left\{ c \in \{a,b\}^{\mathbb{Z}} \,\middle|\, \forall p \in \mathbb{Z}, c_p = b \Rightarrow c_{p+1} = a \right\}$$

Remark **Subshifts** are also very natural when studying **CA**.

Langage of a subshift

Definition The **language** $L(X)$ of a **subshift** X is the set of finite patterns appearing in X.

Proposition A **subshift** is characterized by its **language**.

$$\overline{L} = \left\{ c \in S^{\mathbb{Z}^d} \,\middle|\, \forall r \geqslant 0, \forall m \in S^{[-r,r]^d}, m \prec c \Rightarrow m \in L \right\}$$

Warning It might be that $L(\overline{L}) \neq L$.

Forbidden words

Proposition A subshift is characterized by the set of its **forbidden words**: the complementary of its language.

Proposition Subshifts are in bijection with **minimal sets of forbidden words** (for set inclusion).

Ex $X = S_{\{bb\}}$

SFT, tilings, soficity

Definition A **subshift of finite type** (**SFT**) is defined by a finite set of forbidden words.

Proposition The **set of SFT** is invariant by **CA preimage**.

Remark **SFT** correspond to **tilings**: colorings with local uniform constraints.

Definition A **sofic subshift** is the image of a SFT by a CA.

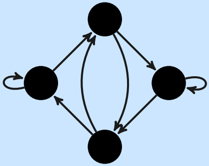
Proposition **1D sofic subshifts** are subshifts that admit a **regular language** of forbidden words.

Part II
Computing properties of CA

4. Discrete dynamical systems

5. Immediate properties

6. Dynamical properties



Configurations

Definition A configuration $c \in S^{\mathbb{Z}^d}$ is **periodic**, with period $(p_i) \in {\mathbb{N}^*}^d$, if

$$\forall p \in \mathbb{Z}^d, \forall (k_i) \in \mathbb{Z}^d, \quad c(p) = c(p + (k_1 p_1, \ldots, k_d p_d)) \quad .$$

Notation. G_p denotes G restricted to periodic configurations.

Definition A configuration $c \in S^{\mathbb{Z}^d}$ is **s-finite** if s is quiescent ($f(s, \ldots, s) = s$) and

$$\mathrm{Card}\left(\left\{p \in \mathbb{Z}^d \,\middle|\, c(p) \neq s\right\}\right) < \omega \quad .$$

Notation. G_f denotes G restricted to s-finite configurations.

Immediate properties

Definition A CA $G : C \to C$ is:

- **injective** if $\forall x, y \in C, F(x) \neq F(y)$;
- **surjective** if $\forall x \in C, F^{-1}(x) \neq \emptyset$;
- **bijective** if both injective and surjective.

Definition A bijective CA G is **reversible** if there exists a CA H such that $H = G^{-1}$.

Corollary Every **bijective** CA is **reversible**.

Garden of Eden and orphans

Definition A configuration of a CA is a **garden of Eden** if it has no preimage.

Proposition A CA is **surjective** if and only if it has **no garden of Eden**.

Definition Given a CA, a pattern $m \in S^{[-r,r]^d}$ is an **orphan** if m has no preimage.

Proposition A CA is **surjective** if and only if it has **no orphan**.

Moore-Myhill

Theorem[Moore 1962] G surjective $\Rightarrow$ G_f injective.

Theorem[Myhill 1963] G_f injective $\Rightarrow$ G surjective.

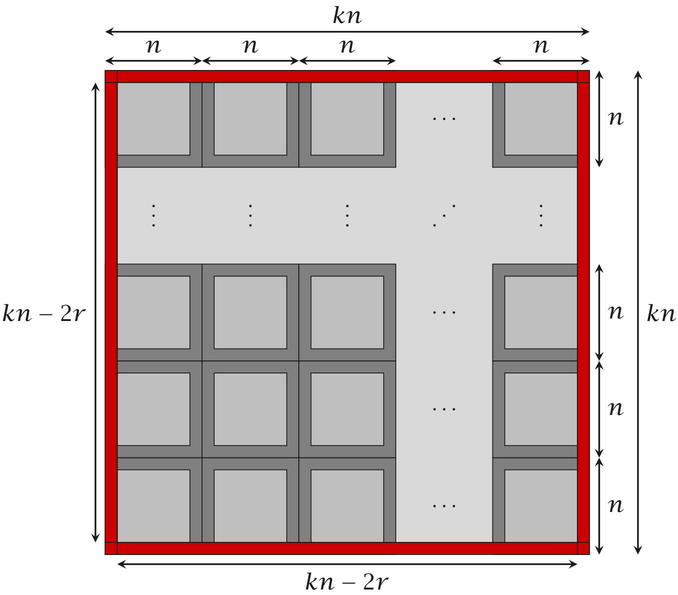
Corollary G injective $\Rightarrow$ G bijective.

Let's prove it!

Lemma For all $d, n, s, r \in \mathbb{N}$, there exists k big enough so that

$$\left(s^{n^d} - 1\right)^{k^d} < s^{(kn-2r)^d}$$

Key picture



G surjective $\Rightarrow G_f$ injective

Let G be a CA with radius r that is not injective on s-finite configurations. There exists two patterns p_1 and p_2 of size n^d bordered by s on a width r with a same image. Replacing p_1 by p_2 in any configuration does not change its image.

Consider square patterns of side kn. Their images are patterns of side $kn - 2r$. There exists $s^{(kn-2r)^d}$ possible images for at most $(s^{n^d} - 1)^{k^d}$ preimages. By previous lemma, there is an orphan thus G is not surjective.

G_f injective $\Rightarrow$ G surjective

Let G be a non surjective CA with radius r. It admits an orphan of size n^d.

Consider the set of s-finite configurations the non-quiescent pattern of which is of side $kn - 2r$. There are $s^{(kn-2r)^d}$ such patterns. Their s-finite images have patterns of side kn. At most $(s^{n^d} - 1)^{k^d}$ of them are not orphans. Thus two of the configurations have a same image, G is not injective on finite configurations.

The big picture

$$
\begin{array}{ccccc}
G \text{ bijective} & \Longleftrightarrow & G \text{ réversible} & \Longleftrightarrow & G \text{ injective} \\
\Downarrow \; \not\Uparrow & & & & \Downarrow \; \Uparrow^{\triangle} \\
G_f \text{ bijective} & \underset{\square\Longleftarrow}{\not\Longrightarrow} & G_p \text{ injective} & \Longleftrightarrow & G_p \text{ bijective} \\
\Updownarrow & & & & \Downarrow \; \not\Uparrow \\
G_f \text{ surjective} & & \overset{\Longrightarrow^{\square}}{\not\Longleftarrow} & & G_p \text{ surjective} \\
\Downarrow \; \not\Uparrow & & & & \Downarrow \; \Uparrow^{\square} \\
G_f \text{ injective} & \Longleftrightarrow & \Lambda_G = S^{\mathbb{Z}^d} & \Longleftrightarrow & G \text{ surjective}
\end{array}
$$

$\triangle$ means true for $d = 1$, false for $d \geqslant 2$

$\square$ means true for $d = 1$, **open** for $d \geqslant 2$

Examples

Proposition G_p injective $\Rightarrow$ G_p bijective.

Key CA preserve periods.

Proposition G_f surjective $\Rightarrow$ G_f injective.

Key Finite configurations are dense + Moore-Myhill.

Counter-examples

Proposition $\exists G$ surjective $\not\Rightarrow G_f$ surjective.

XOR rule: $S = \mathbb{Z}_2$, $f(x, y) = x + y \pmod 2$.

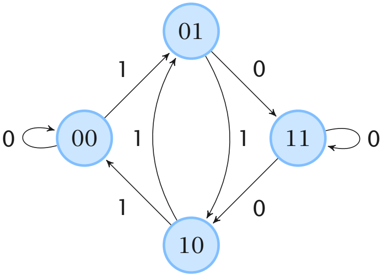
Proposition $\exists G$, G_f surjective $\not\Rightarrow G$ injective.

controlled-XOR rule: $S = \{0, 1\} \times \mathbb{Z}_2$,
$f((1, x), (\_, y)) = (1, x + y \pmod 2))$ and
$f((0, x), \_) = (0, x)$.

De Bruijn Graph

Definition The **De Bruijn graph** of a 1D CA (S, r, f) is the labelled graph (V, E) where:

- $V = S^{2r}$;
- $(u, s, v) \in E$ if $f(s_0, \ldots, s_{2r}) = s$ where $u = (s_0, \ldots, s_{2r-1})$ and $v = (s_1, \ldots, s_{2r})$.



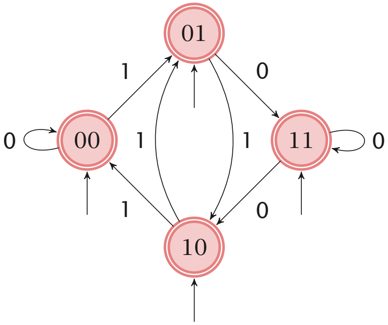
A convenient tool to decide properties of 1D CA.

Deciding surjectivity in 1D

Remark A CA G is surjective if and only if $G\left(S^{\mathbb{Z}^d}\right) = S^{\mathbb{Z}^d}$.

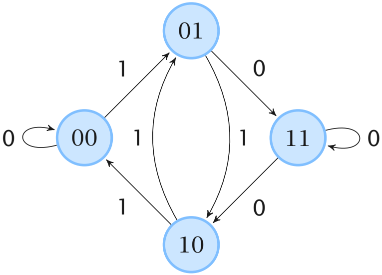
Lemma A 1D CA G is surjective iff $L\left(G\left(S^{\mathbb{Z}}\right)\right) = S^*$.

Theorem[Amoroso, Patt 1972] Surjectivity 1D is decidable.



Deciding injectivity in 1D

A 1D CA is **not injective** iff its De Bruijn graph contains **two distinct paths with the same biinfinite word**.



If such a pair of paths exist, there exists one with **ultimately periodic** paths.

Theorem[Amoroso, Patt 1972] Injectivity is decidable in 1D.

Part II
Computing properties of CA

4. Discrete dynamical systems

5. Immediate properties

6. Dynamical properties

Symbolic dynamical systems

Definition A **SDS** is a SDD (X, F) where X is a **subshift**.

$x \in X$ is **periodic** with period n if $F^n(x) = x$.

$x \in X$ is **ultimately periodic** with transitory m if $F^m(x)$ is periodic.

Definition $Y \subseteq X$ is **invariant** if $F(Y) \subseteq Y$.

Definition $Y \subseteq X$ is **strongly invariant** if $F(Y) = Y$.

Attractors

Definition The **limit set** of an invariant clopen $V \subseteq X$ is

$$\Lambda_F(V) = \bigcap_{n \in \mathbb{N}} F^n(V)$$

Definition An **attractor** is the limit set of a non-empty invariant clopen.

Definition The **bassin** of an attractor Y is

$$\mathcal{B}_F(Y) = \left\{ x \in X \,\middle|\, \lim_{n \to \infty} \delta(F^n(x), Y) = 0 \right\}$$

Definition A **minimal attractor** is an attractor with no strict subset which is also an attractor.

Limit set

Definition The **limit set** of a CA $\left(S^{\mathbb{Z}^d}, F\right)$ is the set of configurations than can appear at all time:

$$\Lambda_F = \bigcap_{n \in \mathbb{N}} F^n\left(S^{\mathbb{Z}^d}\right) \quad .$$

Proposition The **limit set** is a **non empty subshift**.

$F^n\left(S^{\mathbb{Z}^d}\right)$ is a non empty subshift and $F^{n+1}\left(S^{\mathbb{Z}^d}\right) \subseteq F^n\left(S^{\mathbb{Z}^d}\right)$.

Proposition The **limit set** is the **maximal attractor**.

Biinfinite space-time diagrams

Definition A **biinfinite space-time diagram** $\Delta \in S^{\mathbb{Z}^{d+1}}$ satisfies:

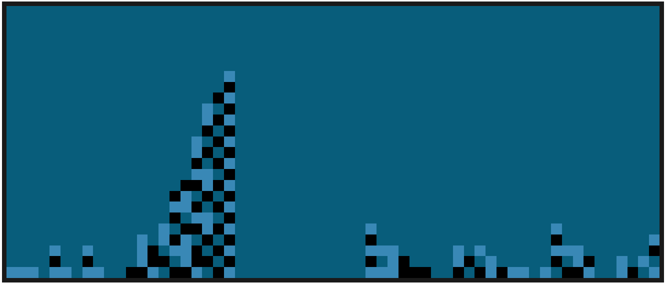
$$\forall t \in \mathbb{Z}, \quad \Delta(t+1) = F(\Delta(t)) \quad .$$

Proposition The **limit set** is the set of configurations of **biinfinite space-time diagrams**.

Every element x of the limit set admits an infinite chain of predecessors $x = x_0$, $x_0 = F(x_1)$, $\dots$, $x_n = F(x_{n+1})$, $\dots$

Nilpotency

Definition A CA with quiescent state q is **nilpotent** if every configuration converges in finite time to the q-monochromatic configuration.

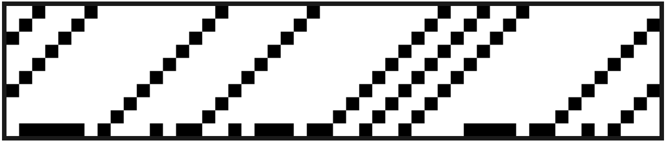


Proposition A CA is **nilpotent** if and only if its limit set is a **singleton**.

SFT example

Let F be the 1D CA with radius 1 and local rule

$$f(x,y,z) = \begin{cases} 1 & \text{si } (x,y,z) = (1,0,0) \\ 0 & \text{sinon} \end{cases} .$$

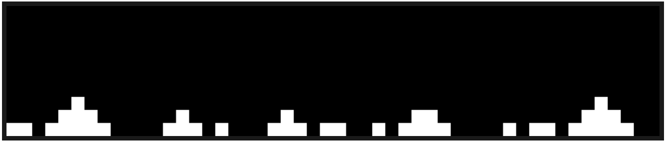


$$\Lambda_F = F\left(S^{\mathbb{Z}}\right) = S_{\{11,101\}}$$

Max example

Let F be the 1D CA with radius 1 and local rule

$$f(x, y, z) = \max(x, y, z) \quad .$$



$$\Lambda_F = {}^{\omega}0^{\omega} + {}^{\omega}1^{\omega} + {}^{\omega}01^{\omega} + {}^{\omega}10^{\omega} + {}^{\omega}10^{*}1^{\omega}$$

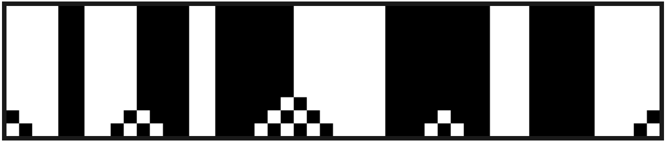
$$\Lambda_F = S_{\{01^n0 \mid n \in \mathbb{N}\}}$$

Λ_F is **countable** and **not SFT**.

Majority example

Let F be the 1D CA with radius 1 and local rule

$$f(x, y, z) = \mathrm{maj}(x, y, z) \quad .$$



Exercise What is Λ_F?

Hint Consider $01 \cdot 00^+ \, (11^+00^+)^* \, 00^+ \cdot 10$.

Language

Proposition For every CA, $L(\Lambda_F) = \bigcap_{n\in\mathbb{N}} L\left(F^n\left(S^{\mathbb{Z}^d}\right)\right)$.

Corollary If Λ_F is a **SFT** then $\exists n\, \Lambda_F = F^n\left(S^{\mathbb{Z}^d}\right)$.

Consider the first **time of appearance** of each **minimal forbidden word**.

Proposition If $\Lambda_F = F^n\left(S^{\mathbb{Z}^d}\right)$ then Λ_F is **sofic**.

If F is a CA, so is F^n.

Cardinality

Proposition[CPY89] If Λ_F contains **two distinct elements** then it contains a **non spatially periodic** element.

Corollary A **limit set** is either a **singleton** either an **infinite** set.

Recursivity

Proposition $L(\Lambda_F)$ is **co-recursively enumerable**.

Orphans of F^n can be tested thus enumerated.

Proposition[Hurd90] For every co-recursively enumerable language $L \subseteq \Sigma^*$ there exists a CA F, a rational language $R \subseteq S^*$ and a morphism $\varphi : S^* \to \Sigma^*$ such that

$$\varphi(L(\Lambda_F) \cap R) = L \quad .$$

Corollary There exists CA with **non recursive limit sets**.

Non rational context free

Proposition[Hurd87] There exists a 1D CA whose limit set has a **non rational context free** language.

Exercise Build one!

Hint Consider bouncing particles and walls.

Context sensitive non context free

Proposition[Hurd87] There exists a 1D CA whose limit set has a **non context free context sensitive** language.

Exercise Build one!

Hint Complexify previous example.

Non recursive 2D

Proposition There exists a 2D CA whose limit set has a **non recursive** language.

Exercise Build one!

Hint Consider space-time diagrams of Turing machines.

Non recursive 1D

Proposition[CPY89] There exists a 1D CA whose limit set has a **non recursive** language.

Exercise Build one!

Hint Consider a CA that can simulate every CA.

Surjectivity and injectivity

Notation F_Λ is the restriction of F to Λ_F.

Proposition F_Λ is **surjective**.

Proposition If $\forall n\, \Lambda_F \neq F^n\left(S^{\mathbb{Z}^d}\right)$ then

$$\forall n \exists c \quad \forall i < n\, F^i(c) \notin \Lambda_F \wedge F^n(c) \in \Lambda_F \quad .$$

Proposition[Taati 2008] If F_Λ is **injective** then $\exists n\, \Lambda_F = F^n\left(S^{\mathbb{Z}^d}\right)$.

Bijectivity

Proposition[Taati 2008] If a CA F is injective on its limit set, the limit set is a **SFT** and there exists a CA G with the same limit set and such that $G_\Lambda = F_\Lambda^{-1}$.

Let H be a CA such that $H_\Lambda = F_\Lambda^{-1}$ and let $G = H^{n+1} \circ F^n$. It holds

$$\Lambda_F = \left\{c \,\middle|\, G^n(F^n(x)) = x\right\} \quad .$$

Stability

A CA is **stable** if its limit set is obtained in finite time, otherwise it is **unstable**.

A subshift is **stable** if it is the limit set of a stable CA.

A subshift is **unstable** if it is the limit set of an unstable CA.

Open Pb Can a subshift be both **stable** and **unstable** ?

Part III

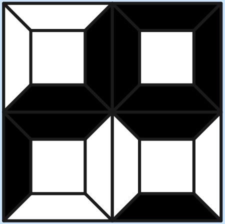
Computation and reduction: undecidability results

Part III
Computation and reduction: undecidability results

7. Tilings

8. Undecidability results in 2D

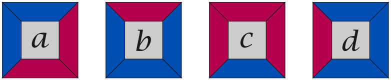
9. Undecidability results in 1D

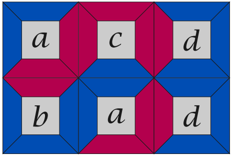


The Domino Problem (DP)

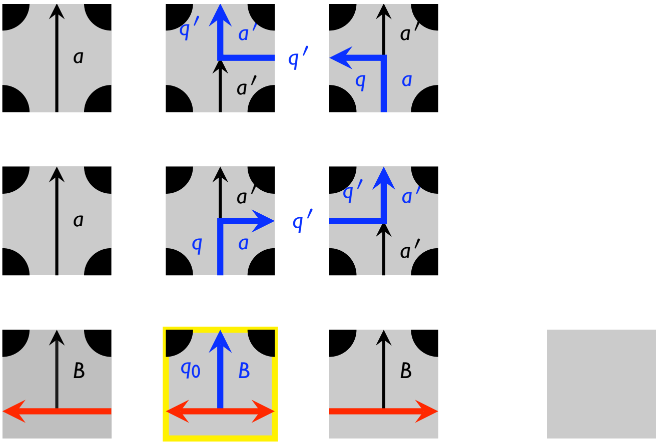
"Assume we are ***given a finite set of square plates*** *of the same size with* ***edges colored****, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are* ***not permitted to rotate or reflect a plate****. The question is to find an effective procedure by which we can* ***decide****, for each given finite set of plates,* ***whether we can cover up the whole plane*** *(or, equivalently, an infinite quadrant thereof)* ***with copies of the plates subject to the restriction that adjoining edges must have the same color****."*

(Wang, 1961)

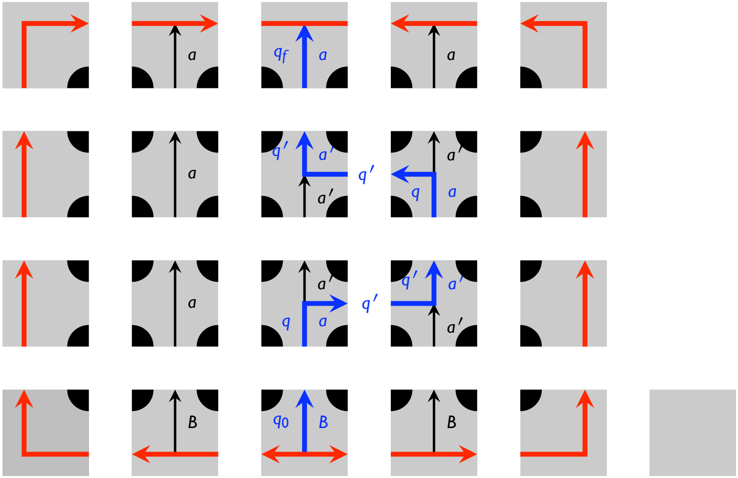




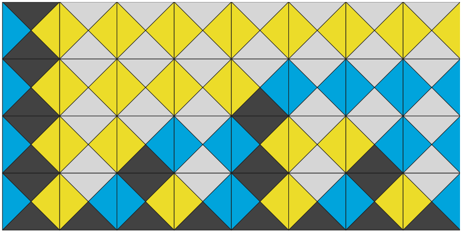
Tiling with a fixed tile

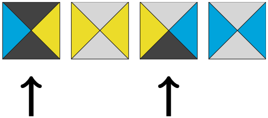


Finite tiling



Tiling with diagonal constraint





Undecidability of DP

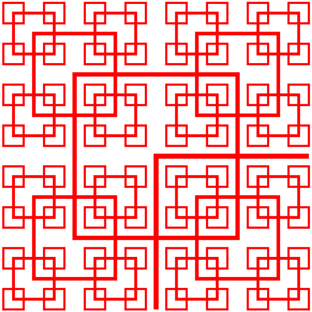
Theorem[Berger64] **DP** is **recursively undecidable**.

Remark To prove it one needs **aperiodic** tile sets.

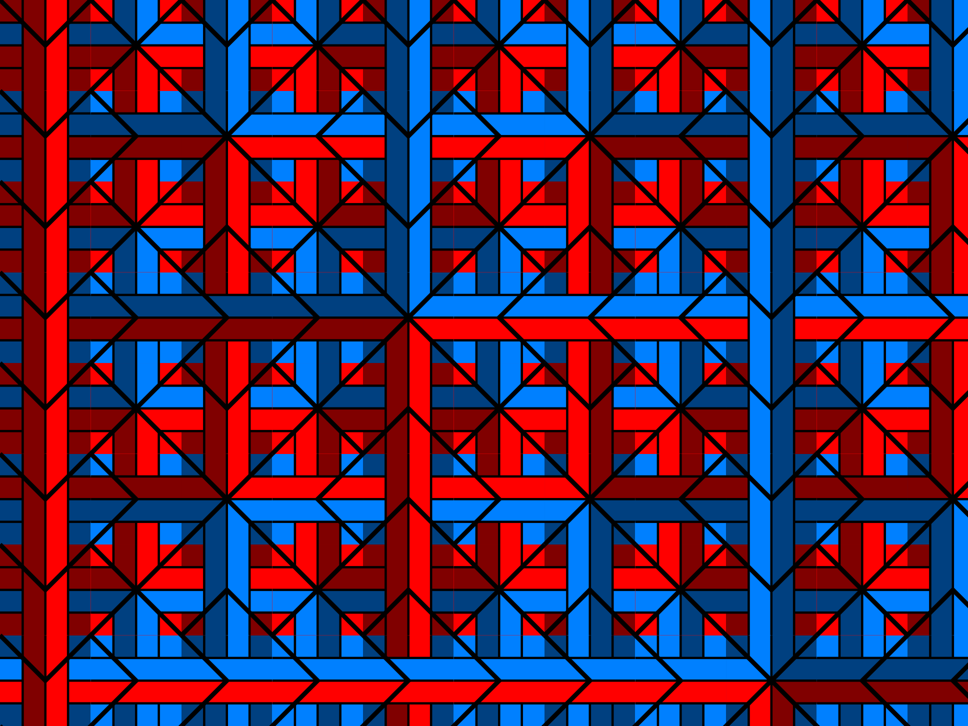
Idea of the proof

Enforce an (aperiodic) **self-similar structure** using local rules.

Insert a **Turing machine** computation **everywhere** using the structure.



Remark Plenty of different proofs!

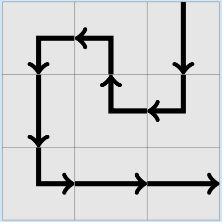


Part III
Computation and reduction: undecidability results

7. Tilings

8. Undecidability results in 2D

9. Undecidability results in 1D



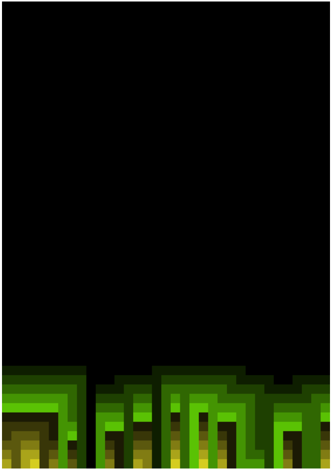
The nilpotency problem (Nil)

Definition A DDS is **nilpotent** if
$\exists z \in X,\ \forall x \in X,\ \exists n \in \mathbb{N},\ F^n(x) = z.$

Given a recursive encoding of the DDS, can we **decide** nilpotency?

A DDS is **uniformly nilpotent** if
$\exists z \in X,\ \exists n \in \mathbb{N},\ \forall x \in X,\ F^n(x) = z.$

Given a recursive encoding of the DDS, can we **bound recursively** n?



Nilpotency and limit set

Definition The limit set of a CA F is the non-empty subshift

$$\Lambda_F = \bigcap_{n \in \mathbb{N}} F^n \left(S^{\mathbb{Z}} \right)$$

Remark Λ_F is the set of configurations appearing in biinfinite space-time diagrams $\Delta \in S^{\mathbb{Z} \times \mathbb{Z}}$ such that $\forall t \in \mathbb{Z}, \Delta(t+1) = F(\Delta(t))$.

Lemma A CA is nilpotent iff its limit set is a singleton.

2D Nilpotency

2D Nilpotency
Input: a CA (S, N, f).
Question: Is F nilpotent?

Theorem[CPY89] Nilpotency is **undecidable** in 2D.

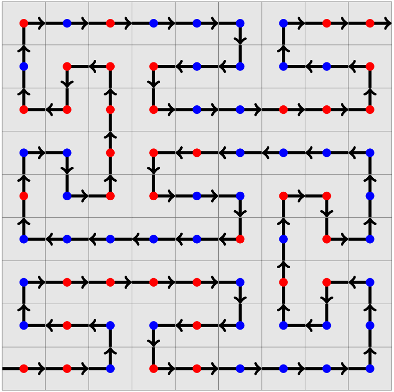
Prove than $\overline{\mathbf{DP}} \leqslant_m \mathbf{Nil2D}$.

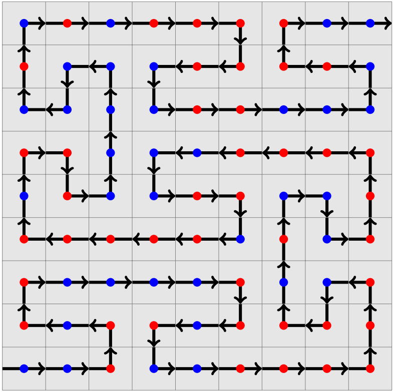
Given a set of Wang tiles τ, build a CA with alphabet $\tau \cup \{\perp\}$ where $\perp$ is a spreading tiling error state.

Surjectivity/Injectivity 2D

Theorem[Kari 1990] Both injectivity and surjectivity are **undecidable** in 2D.

For surjectivity, using Moore-Myhill, prove that injectivity on finite is undecidable in 2D.



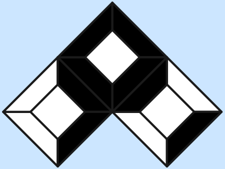


Part III
Computation and reduction: undecidability results

7. Tilings

8. Undecidability results in 2D

9. Undecidability results in 1D

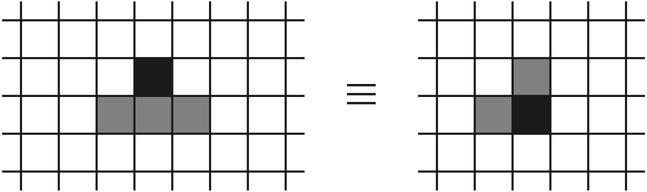


Nilpotency 1D

A state $\perp \in S$ is **spreading** if $f(N) = \perp$ when $\perp \in N$.

A CA with a spreading state $\perp$ is not nilpotent iff it admits a biinfinite space-time diagram without $\perp$.

A tiling problem Find a coloring $\Delta \in (S \setminus \{\perp\})^{\mathbb{Z}^2}$ satisfying the tiling constraints given by f.



Theorem[Kari92] **NW-DP** $\leqslant_m$ **Nil**

Revisiting DP

Theorem[Kari92] **NW-DP** is **recursively undecidable**.

Remark Reprove of undecidability of **DP** with the additionnal determinism constraint!

Corollary **Nil** is **recursively undecidable**.

More on limit sets

Theorem[Kari 1994] The set of CA whose limit sets satisfy a non trivial property is never recursive.

Theorem[Guillon Richard 2010] Still true with a fixed alphabet!

Definition A CA F is **weakly nilpotent** if

$$\forall c \forall p \exists t_0 \forall t > t_0 \quad F^t(c)_p = q \quad .$$

Theorem[Guillon Richard 2008] A CA is **weakly nilpotent** if and only if it is **nilpotent**.

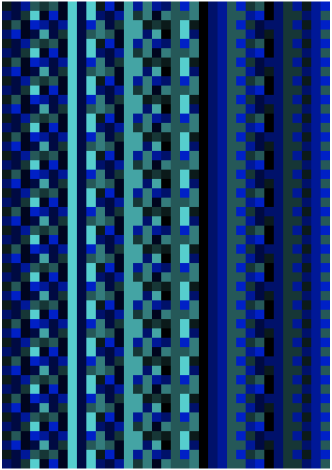
The periodicity problem (Per)

Definition A DDS is **periodic** if
$\forall x \in X, \exists n \in \mathbb{N}, F^n(x) = x$.

Given a recursive encoding of the DDS, can we **decide** periodicity?

A DDS is **uniformly periodic** if
$\exists n \in \mathbb{N}, \forall x \in X, F^n(x) = x$.

Given a recursive encoding of the DDS, can we **bound recursively** n?



Undecidability results

Theorem Both **Nil** and **Per** are **recursively undecidable**.

The proofs inject **computation** into **dynamics**.

Undecidability is not necessarily a negative result:
it is a **hint of complexity**.

Remark Due to **universe configurations** both nilpotency and periodicity are uniform.

The bounds grow **faster than any recursive function**: there exists simple nilpotent or periodic CA with huge bounds.

The Immortality Problem (IP)

"(T_2) *To find an effective method, which for every Turing-machine* M *decides whether or not, for all tapes* I *(finite and infinite) and all states* B, M *will eventually halt if started in state* B *on tape* I" *(Büchi, 1962)*

Theorem[Hooper66] **IP** is **recursively undecidable**.

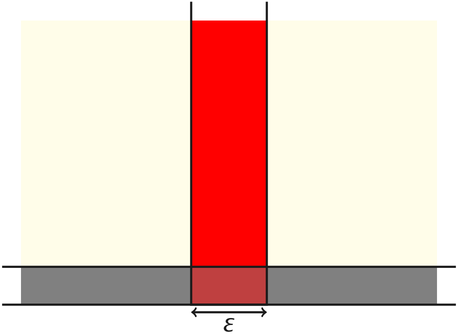
Theorem[KO2008] **R-IP** $\leqslant_m$ **TM-Per** $\leqslant_m$ **Per**

Theorem[KO2008] **R-IP** is **recursively undecidable**.

Open Problem

Definition A CA F is **positively expansive** if

$$\exists \varepsilon > 0,\ \forall x \neq y,\ \exists n \geqslant 0,\ d\left(F^n(x), F^n(y)\right) \geqslant \varepsilon$$



Question Is positive expansivity **decidable**?

Table of Contents

Part I

1. Cellular automata
2. A universal model of computation
3. A model of parallel computation

Part II

4. Discrete dynamical systems
5. Immediate properties
6. Dynamical properties

Part III

7. Tilings
8. Undecidability results in 2D
9. Undecidability results in 1D