Two-States Bilinear Intrinsically Universal Cellular Automata*

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Abstract. Linear cellular automata have been studied in details using algebraic techniques [3]. The generalization to families of polynomial cellular automata seems natural. The following step of complexity consists of bilinear cellular automata which study has begun with the work of Bartlett and Garzon [2]. Thanks to bulking techniques [5], two-states bilinear intrinsically universal cellular automata are constructed. This result answers a question from Bartlett and Garzon [2] of 1995.

A cellular automaton consists of a regular network, for example a line of cells, carrying finite values that are updated synchronously on discrete time steps by applying uniformly a local rule. Despite their apparent simplicity, cellular automata exhibit varied, sometimes complex, behaviors.

The properties of linear algebraic objects are easier to describe than the properties of general objects. In the case of cellular automata, the study of linear cellular automata has begun with the work of Martin *et al.* [3]. They showed that linear cellular automata are really simple and they completely described their behavior. In 1995, Bartlett and Garzon [2] studied bilinear cellular automata. They proved that a particular sub-family of bilinear cellular automata, cellular automata over $\mathbb{Z}_p^p$ with p prime, is as complex as the whole family of cellular automata. The question remained open whether bilinear cellular automata over $\mathbb{Z}_m$ were as complex as the whole family of cellular automata for small values of m. Moreover, the result from Bartlett and Garzon [2] was given thanks to a notion of simulation and π-universality that were not formally defined.

In the spirit of Mazoyer and Rapaport [4], we have introduced [5] a tool, called geometrical bulking, to classify and prove properties on cellular automata. This tool is based on a notion of simulation and a notion of space-time diagrams rescaling. Within this scope, we formalized the notion of intrinsic universality implicitly introduced by Albert and Čulik II [1], which corresponds to the notion of π-universality in the paper of Bartlett and Garzon [2].

In the present paper, we close an open question from Bartlett and Garzon [2] by constructing a two-states bilinear intrinsically universal cellular automaton.

* a longer version of this paper is available from the author, see [6]

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1 Cellular Automata and Geometrical Bulking

In the following, we only consider one-dimensional cellular automata, that is straight lines of cells. We briefly recall some necessary definitions and theorems about cellular automata and geometrical bulking. The interested reader is invited to consult the longer version of this paper [6].

Definition 1. *A* cellular automaton $\mathcal{A}$ *is a triple* $(S, \{n_1, \ldots, n_d\}, \delta)$ *such that* S *is a finite set of states,* N *is a finite ordered set of* d *integers called the neighborhood of* $\mathcal{A}$ *and* δ *is the local transition function of* $\mathcal{A}$ *which maps* S^d *to* S.

A configuration $\mathcal{C}$ of a cellular automaton $\mathcal{A}$ maps $\mathbb{Z}$ to the states set of $\mathcal{A}$. The state of the i-th cell of $\mathcal{C}$ is denoted as $\mathcal{C}_i$. The local transition function δ of $\mathcal{A}$ is naturally extended to a global transition function $G_{\mathcal{A}}$ which maps a configuration $\mathcal{C}$ of $\mathcal{A}$ to a configuration $\mathcal{C}'$ of $\mathcal{A}$ satisfying, for each cell i, the equation $\mathcal{C}'_i = \delta\left(\mathcal{C}_{i+n_1}, \ldots, \mathcal{C}_{i+n_d}\right)$.

A sub-automaton of a cellular automaton corresponds to a stable restriction on the states set. A cellular automaton is a sub-automaton of another cellular automaton if (up to a renaming of states) the space-time diagrams of the first one are space-time diagrams of the second one. To compare cellular automata, we introduce a notion of space-time diagrams rescaling. To formalize this idea, we introduce the following notations:

σ^k. Let S be a finite state set and k be an integer. The shift σ^k is the bijective map from $S^{\mathbb{Z}}$ onto $S^{\mathbb{Z}}$ which maps a configuration $\mathcal{C}$ to the configuration $\mathcal{C}'$ such that, for each cell i, the equation $\mathcal{C}'_{i+k} = \mathcal{C}_i$ is satisfied.

o^m. Let S be a finite state set and m be a strictly positive integer. The packing map o^m is the bijective map from $S^{\mathbb{Z}}$ onto $(S^m)^{\mathbb{Z}}$ which maps a configuration $\mathcal{C}$ to the configuration $\mathcal{C}'$ such that, for each cell i, the equation $\mathcal{C}'_i = (\mathcal{C}_{mi}, \ldots, \mathcal{C}_{mi+m-1})$ is satisfied.

Definition 2. *Let* $\mathcal{A}$ *be a cellular automaton with states set* S. *A* $\langle m,n,k\rangle$-rescaling *of* $\mathcal{A}$ *is a cellular automaton* $\mathcal{A}^{\langle m,n,k\rangle}$ *with states set* S^m *and global transition function* $G_{\mathcal{A}}^{\langle m,n,k\rangle} = \sigma^k \circ o^m \circ G_{\mathcal{A}}^n \circ o^{-m}$.

Definition 3. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two cellular automata. Then* $\mathcal{B}$ simulates $\mathcal{A}$ *if there exists a rescaling of* $\mathcal{A}$ *which is a sub-automaton of a rescaling of* $\mathcal{B}$.

This relation of simulation has good properties. In particular, in [5], we proved that the relation of simulation is a quasi-order with a maximal induced equivalence class exactly corresponding to the set of intrinsically universal cellular automata in the sense of Albert and Čulik II [1]. As any cellular automaton can be simulated by a one-way cellular automaton, that is a cellular automaton with neighborhood $\{-1, 0\}$, there exist intrinsically universal one-way cellular automata. Therefore, to prove that a particular family of cellular automata contains an intrinsically universal cellular automaton, it is sufficient to prove that any one-way cellular automaton can be simulated by a cellular automaton from the family. The details and the formal definitions of intrinsically universal cellular automata are presented in the longer version [6].

2 Two-States Bilinear Cellular Automata

Bilinear cellular automata are polynomial cellular automata which polynomial is a bilinear functional. Bartlett and Garzon [2] proved the universality of this family of cellular automata in the special case where the states set is of the kind $\mathbb{Z}_p^p$ with p prime. We prove that the family of bilinear cellular automata with states set $\mathbb{Z}_2$ is universal, answering their question concerning bilinear cellular automata with states set $\mathbb{Z}_m$ for small values of m.

Definition 4. *A* bilinear cellular automaton *is a polynomial cellular automaton of degree* 2*, that is, a cellular automaton which states set is a finite commutative ring and which local transition function can be represented as a polynomial only consisting of quadratic monomials:* $\delta(s_1, \ldots, s_d) = \sum_{i=1}^{d} \sum_{j=1}^{d} b_{i,j} s_i s_j$.

Theorem 1. *Each one-way cellular automaton can be simulated by a two-states bilinear cellular automaton.*

Proof. Let $\mathcal{A}$ be a one-way cellular automaton $(S, \{-1, 0\}, \delta)$. Let n be the cardinal of S. Up to a renaming of states, we can assume that $S = \{0, 1, \ldots, n-1\}$. We construct a two-states bilinear cellular automaton $(\mathbb{Z}_2, \{-r, \ldots, r\}, P)$ which simulates $\mathcal{A}$. The basic idea of the construction is to represent a cell of a configuration of $\mathcal{A}$ by a block of cells all but one in the state 0. The position of the cell with value 1 determines the state for the cell of the configuration of $\mathcal{A}$. To encode the transition $\delta(i, j) = k$, we build a monomial $s_p s_q$ where p is the distance from the position of k to the position of i and q the distance from the position of k to the position of j as represented on Fig. 1. To avoid multiplying the monomial by $(1 - s_l)$ for every l between p and q, we must be sure that all these cells can only be 0. Eventually the mapping from (i, j, k) to (p, q) must be injective to avoid any "misinterpretation" of a monomial.

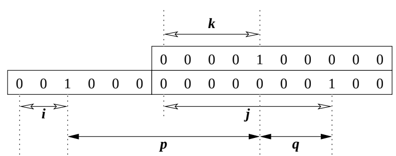

Fig. 1. the idea of a cell encoding

A complete and motivated construction can be found in the longer version of this paper [6]. Here, we only provide the technical part of the proof. We choose to discriminate between the encoding of cells of a configuration of $\mathcal{A}$ thanks to the parity of its position into the configuration. An even cell with value i is encoded as a block of cells of size $18n^2$ with value 1 at cell $(6n+2)i$. An odd cell with value j is encoded as a block of cells of size $18n^2 + 1$ with value 1 at cell $6nj$.

First, we show that the distance between two cells with value 1 permits to know whether it correspond to two next encoded cells. The maximal distance between two cells with value 1 corresponding to two next encoded cells is

$18n^2+1+(6n+2)n=24n^2+2n+1$. The minimal distance between two cells with value 1 corresponding to two encoded cells at distance three or more is $18n^2-(6n+2)n+18n^2+1=30n^2-2n+1$. As $n \geqslant 1$, it is clear that two cells with value 1 correspond to two next encoded cells if and only if their distance is less or equal to $24n^2+2n+1$.

Second, we show that, when building a monomial $s_p s_q$ encoding the transition $\delta(i,j)=k$, the mapping from (i,j,k) to (p,q) is injective. As it is straightforward to compute, if the position of k is odd, $p=18n^2-(6n+2)i+6nk$ and $q=6nj-6nk$; if the position of k is even, $p=18n^2+1-6ni+(6n+2)k$ and $q=(6n+2)j-(6n+2)k$. Being given p and q, computing p modulo 2 gives the parity of the position of k. Then, depending of this parity, p modulo $6n$ permits to obtain i or k. From p, we then deduce both i and k. Finally, we compute j from q. The mapping is injective.

From the above computations, we obtain the following polynomial.

$$P(s_{-r},\ldots,s_r)=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \begin{array}{l} s_{18n^2-(6n+2)i+6n\delta(i,j)}s_{6nj-6n\delta(i,j)} \\ +s_{18n^2+1-6ni+(6n+2)\delta(i,j)}s_{(6n+2)j-(6n+2)\delta_(i,j)} \end{array}$$

The parameter r can be chosen as $36n^2+1$. ■

3 Conclusion and Open Problems

In this paper, we have proven, thanks to geometrical bulking, the existence of two-states bilinear intrinsically universal cellular automata, drastically decreasing the previous known number of states (using 2 states instead of 211^{211} states in the paper of Bartlett and Garzon [2]). Our result naturally extends to higher dimensions. The difference between linear and non-linear cellular automata is worth studying. To continue the study of bilinear cellular automata, one has to find a bound on the neighborhood size for intrinsical universality (our best today estimation is a radius of 1297 cells).

References

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