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Rapport de Recherche



# Solving First-Order Constraints in the Theory of the Evaluated Trees

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Rapport N$^o$ **2006-05**

**Abstract**

We present in this paper a general algorithm for solving first-order constraints in the theory $T$ of the evaluated trees which is a combination of the theory of finite or infinite trees and the theory of the rational numbers with addition and subtraction and a linear dense order relation. The algorithm is given in the form of 28 rewriting rules. It transforms a first-order formula $\varphi$ - which can possibly contain free variables - into a disjunction $\phi$ of solved formulas which is equivalent in $T$, without new free variables and such that $\phi$ is either the formula *true* or the formula *false* or a formula having at least one free variable and being equivalent neither to *true* nor to *false* in $T$. In particular if $\varphi$ does not contain free variables then $\phi$ is either the formula *true* or *false*. If $\phi$ has free variables then the solutions on free variables are expressed in an explicit way and $\phi$ can be directly transformed into a boolean combination of quantified conjunctions of atomic formulas which do not accept elimination of quantifiers. The correctness of our algorithm is another proof of the completeness of this theory.

# 1 Introduction

The algebra of finite or infinite trees plays a fundamental role in computer science: it is a model for data structures, program schemes and program executions. As early as 1976, G. Huet proposed an algorithm for unifying infinite terms, that is solving equations in that algebra [12]. B. Courcelle has studied the properties of infinite trees in the scope of recursive program schemes [8]. A. Colmerauer has described the execution of Prolog II, III and IV programs in terms of solving equations and disequations in that algebra [4, 3, 1]. The unification of finite terms, i.e. the resolution of conjunctions of equations in the theory of finite trees has first been studied by A. Robinson [18]. Some better algorithms with better complexities has been proposed after by M.S. Paterson and M.N.Wegman [16] and A. Martelli and U. Montanari [15]. The resolution of conjunctions of equations in the theory of infinite trees has been studied by G. Huet [12], by A. Colmerauer [5, 4] and by J. Jaffar [13]. The resolution of conjunctions of equations and disequations in the theory of possibly infinite trees has been studied by A. Colmerauer [4] and H.J. Bürckert [2]. An incremental algorithm for solving conjunctions of equations and disequations on rational trees has been proposed after by V.Ramachandran and P. Van Hentenryck [17]. On the other hand, there exists an algorithm for elimination of quantifications which transforms a first-order formula into a boolean combination of simple constraints. We can refer to the work of M.J. Maher [14] and H. Comon [7].

M.J. Maher has axiomatized all the cases by complete first-order theories with infinite set of function symbols [14]. We have then extended this theory by giving a complete first-order axiomatization of the evaluated trees [10] which are combination of finite or infinite trees with construction operations and the rational numbers with addition and subtraction and a linear dense order relation. This theory $T$ reflects essentially to Prolog III and IV which have been modeled by Alain Colmerauer using combination of trees, rational numbers, booleans and intervals [3, 1]. In this paper we give a general algorithm for solving the most general first-order constraints in $T$, i.e. in all models of $T$. Our aim is not only to decide the validity of the propositions (sentences) i.e. formulas without free variables, but to be able to express solutions of constraints in $T$, which can possibly contain free variables, in a simple and explicit way as it has done in one of our previous works [9] for the theories of finite trees and possibly infinite trees. By solving a constraint $\varphi$ in $T$ we mean to transform the logical formula $\varphi$, which can possibly contain free variables, into a disjunction $\phi$ of solved formulas, without new free variables, which is equivalent to $\varphi$ in $T$ and such that $\phi$ is either the formula *true*, or the formula *false* or has at least one free variable and is equivalent neither to *true* nor to *false* in $T$. In particular if $\varphi$ is a proposition, then $\phi$ is either *true* or *false*. We would be able also to check formulas which contain free variables but are always true or false in $T$. In this case $\phi$ is either *true* or *false*. We show also that if $\phi$ has at least one free variable then $\phi$ can be directly be transformed into a boolean combination of quantified conjunctions of atomic formulas which do not accept elimination of quantifiers. The correctness of our algorithm is another proof of the completeness of this theory.

Our algorithm is not simply a combination of an algorithm over trees with one over rational numbers, but a powerful mechanism to solve mixed constraints. This algorithm is able to solve any first order constraint containing untyped variables or typed variables and presents the solutions of the free variables in a clear and explicit way. One of the major difficulty in this work resides in the fact that (1) the theory of trees does not accept full elimination of quantifiers, (2) every algorithm deciding propositions in the theory of finite or infinite trees has a non-elementary complexity [19] and (3) the function symbols $+$ and $-$ of $T$ have two different behaviors whether they are applied on trees or rational numbers. For example $+(1,1)$ is the rational number 2, while $+(1,f_0)$ is the tree whose root is labeled $+$ and whose children are 1 and the tree's constant $f_0$. The result of the completeness of this theory given in [10] is not enough to induce solved formulas over trees and rational numbers expressing solutions of hard problems, such as: planning, wining-strategies in multiplayers games,...etc. Indeed, our goal in these kinds of problems is not to know only if there exists a winning strategy but to express this winning strategy in the form of a first-order formula whose free variables have clear and explicit solutions. While in [10] the proof of the completeness handles only propositions, in this paper, our algorithm handles general first order formulas with free variables and typed or untyped variables. It includes full systems of typing deduction and constraints simplification and propagation. The expressiveness and clearness of the solutions of the free variables in the final solved formula are our main goal in this paper.

The paper is organized in four sections followed by a conclusion. This introduction is the first section. In Section 2 we present the theory of the evaluated trees and introduce an example of constraints in this theory. In Section 3, we define the notions of basic formulas, blocks and solved blocks in $T$ which are particular conjunctions of atomic formulas. We end this section by showing that every quantified solved block can be decomposed in three embedded sequences of quantifications having particular properties which enable us to eliminate some quantifiers. In Section 4, we present the working formulas, the general solved formulas and the algorithm of constraint solving in $T$. The algorithm is given in the form of 28 rewriting rules and transforms an initial working formula of depth $d$ to a final working formula of depth less than or equal to three. The main idea behind this algorithm consists in (1) a top-down simplification and propagation of constraints, in each level, quantified blocks are solved locally, are decomposed and propagated to the embedded sub-formulas, and inconsistent sub-formulas are removed (2) a bottom-up elimination of quantifiers and working formulas' depth decrease using distribution. The disjunction $\phi$ of general solved formulas extracted from the final working formula is either the formula *false* or *true* or a formula having at least one free variable and is equivalent neither to *false* nor to *true* in $T$. We show also that every general solved formula is equivalent in $T$ to a boolean combination of quantified conjunctions of basic formulas which do not accept elimination of quantifiers. We end this section by giving an example of constraint solving in $T$. The algorithm represented by a set of rewriting rules, the working formulas and the general solved formulas are our main contribution in this paper.

# 2 Theory $T$ of evaluated trees

## 2.1 Preliminaries

Let $F$ be an infinite set of *function symbols* containing the symbols $+$, $-$, 0 and 1. To each element of $F$ is associated a non-negative integer, its *arity*. The arities of $+$, $-$, 0 and 1 are respectively 2, 1, 0 and 0. Let $R = \{<, num, tree\}$ be the set of relation symbols, of respective arities 2, 1 and 1. Let $V$ be an infinite countable set of *variables*. A *term* is an expression of the form $x$ or $ft_1 \dots t_n$ where $n \geq 0$, $f$ an $n$-ary symbol in $F$ and the $t_i$'s are shorter terms. A *formula* is an expression of the forms:
$s=t$, $rt_1..t_n$, *true*, *false*, $\neg(\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$, $\exists x \varphi$, $\forall x \varphi$, where $x \in V$, $s$, $t$ and the $t_i$'s are terms, $r$ is an $n$-ary relation symbol in $R$ and $\varphi$ and $\psi$ are shorter formulas. The first four forms are called atomic. An occurrence of a variable $x$ in a formula is *bound* if it occurs in a sub-formula of the form $(\exists x \varphi)$ or $(\forall x \varphi)$. It is *free* otherwise. The *free variables* of a formula are those which have at least a free occurrence in the formula. For each formula $\varphi$, we denote by $var(\varphi)$ the set of all free variables of $\varphi$. Let $\bar{x} = x_1 \dots x_n$ and $\bar{y} = y_1 \dots y_n$ be two vectors of variables of the same length. The empty vector is denoted by $\varepsilon$. Let $\varphi$ and $\varphi(\bar{x})$ be formulas. We write

$$\begin{array}{ll}
\exists \bar{x}\, \varphi & \text{for } \exists x_1 ... \exists x_n\, \varphi, \\
\forall \bar{x}\, \varphi & \text{for } \forall x_1 ... \forall x_n\, \varphi, \\
\exists ? \bar{x}\, \varphi(\bar{x}) & \text{for } \forall \bar{x} \forall \bar{y}\, \varphi(\bar{x}) \wedge \varphi(\bar{y}) \rightarrow \bigwedge_{i \in \{1,\dots,n\}} x_i = y_i, \\
\exists ! \bar{x}\, \varphi & \text{for } (\exists \bar{x}\, \varphi) \wedge (\exists ? \bar{x}\, \varphi).
\end{array}$$

## 2.2 Axiomatization of $T$

Let $a$ be a positive integer and let $t_1, ..., t_n$ be terms. Let us denote by:

- $t_1 < t_2$, the term $< t_1 t_2$,
- $t_1 + t_2$, the term $+t_1 t_2$,
- $t_1 + t_2 + t_3$, the term $+t_1(+t_2 t_3)$,
- $0t_1$, the term 0,
- $at_1$, the term $\underbrace{t_1 + \cdots + t_1}_{a}$,
- $-at_1$, the term $\underbrace{(-t_1) + \cdots + (-t_1)}_{a}$.

The theory $T$ of evaluated trees is the set of first-order propositions of the following forms:

| | |
|---|---|
| 1 | $\forall\bar{x}\forall\bar{y}\,((\mathit{tree}\, f\bar{x}) \wedge (\mathit{tree}\, f\bar{y}) \wedge f\bar{x} = f\bar{y}) \rightarrow \bigwedge_i x_i = y_i,$ |
| 2 | $\forall\bar{x}\forall\bar{y}\, f\bar{x} = g\bar{y} \rightarrow \mathit{num}\, f\bar{x} \wedge \mathit{num}\, g\bar{y},$ |
| 3 | $\forall\bar{x}\forall\bar{y}\,((\bigwedge_{i\in I} \mathit{num}\, x_i) \wedge (\bigwedge_{j\in J} \mathit{tree}\, y_j)) \rightarrow (\exists!\bar{z}\, \bigwedge_{k\in K}(\mathit{tree}\, z_k \wedge z_k = t_k(\bar{x},\bar{y},\bar{z}))),$ |
| 4 | $\forall x\forall y\, x < y \rightarrow (\mathit{num}\, x \wedge \mathit{num}\, y),$ |
| 5 | $\forall x\forall y\, \mathit{num}\, x + y \leftrightarrow \mathit{num}\, x \wedge \mathit{num}\, y,$ |
| 6 | $\forall x\, \mathit{num}\, -x \leftrightarrow \mathit{num}\, x,$ |
| 7 | $\forall\bar{x}\, \mathit{tree}\ h\bar{x},$ |
| 8 | $\forall x\forall y\,(\mathit{num}\, x \wedge \mathit{num}\, y) \rightarrow x + y = y + x,$ |
| 9 | $\forall x\forall y\forall z\,(\mathit{num}\, x \wedge \mathit{num}\, y \wedge \mathit{num}\, z) \rightarrow x + (y + z) = (x + y) + z,$ |
| 10 | $\forall x\, \mathit{num}\, x \rightarrow x + 0 = x,$ |
| 11 | $\forall x\, \mathit{num}\, x \rightarrow x + (-x) = 0,$ |
| $12_n$ | $\forall x\, \mathit{num}\, x \rightarrow (nx = 0 \rightarrow x = 0),$ |
| $13_n$ | $\forall x\, \mathit{num}\, x \rightarrow \exists!y\, \mathit{num}\, y \wedge ny = x,$ |
| 14 | $\forall x\, \mathit{num}\, x \rightarrow \neg x < x,$ |
| 15 | $\forall x\forall y\forall z\, \mathit{num}\, x \wedge \mathit{num}\, y \wedge \mathit{num}\, z \rightarrow ((x < y \wedge y < z) \rightarrow x < z),$ |
| 16 | $\forall x\forall y\,(\mathit{num}\, x \wedge \mathit{num}\, y) \rightarrow (x < y \vee x = y \vee y < x),$ |
| 17 | $\forall x\forall y\,(\mathit{num}\, x \wedge \mathit{num}\, y) \rightarrow (x < y \rightarrow (\exists z\, \mathit{num}\, z \wedge x < z \wedge z < y)),$ |
| 18 | $\forall x\, \mathit{num}\, x\ \rightarrow (\exists y\, \mathit{num}\, y \wedge x < y),$ |
| 19 | $\forall x\, \mathit{num}\, x\ \rightarrow (\exists y\, \mathit{num}\, y \wedge y < x),$ |
| 20 | $\forall x\, \forall y\, \forall z\,(\mathit{num}\, x \wedge \mathit{num}\, y \wedge\ \mathit{num}\ z) \rightarrow (x < y \rightarrow (x + z < y + z)),$ |
| 21 | $\forall x\,(\neg \mathit{num}\, x) \leftrightarrow \mathit{tree}\, x$ |
| 22 | $0 < 1,$ |

where $n$ is a non-null integer, $f$ and $g$ are two distinct function symbols taken from $F$, $h \in F - \{+,-,0,1\}$, $x$, $y$, $z$ are variables, $\bar{x}$ is a vector of variables $x_i$, $\bar{y}$ is a vector of variables $y_i$, $\bar{z}$ is a vector of distinct variables $z_i$, $I$ and $J$ are finite possibly empty sets, and where $t_k(\bar{x},\bar{y},\bar{z})$ is a term which begins by a function symbol $f_k$ element of $F - \{0,1\}$ followed by variables taken from $\bar{x}$ or $\bar{y}$ or $\bar{z}$, moreover, if $f_k \in \{+,-\}$ then $t_k(\bar{x},\bar{y},\bar{z})$ contains at least one variable taken from $\bar{y}$ or $\bar{z}$. The axiom 3 shows that all models of $T$ contain infinite trees. In fact we have $T \models \exists!z\, z = fz \wedge \mathit{tree}\, z$ for $I = J = \emptyset$. In this case, the tree $z$ is an infinite tree of the form $f(f(f(...)))$. Note that we have not $T \models \forall x \mathit{num}\, x \rightarrow (\exists!z\, z = x + x \wedge \mathit{tree}\, z)$, since we have $T \models \mathit{num}\, x \leftrightarrow \mathit{num}\,(x + x)$ according to axiom 5 which contradicts $\mathit{tree}\, z$ and $z = x + x$. This is why we have a condition if $f_k$ belongs to $\{+,-\}$.

This theory has been introduced in [10], where we have proved its completeness. This theory has as model (possibly) infinite trees whose nodes are labelled by $\boldsymbol{Q}\cup F$ such that each subtree labelled by $\boldsymbol{Q}\cup\{+,-\}$ is evaluated in $\boldsymbol{Q}$ and reduced to a leaf labeled by an element of $\boldsymbol{Q}$.

Let us now introduce an example of *constraints* in $T$. Let us consider the following two-player game: An ordered pair $(n,m)$ of non-negative rational numbers is given and one after another each player subtracts 1 or 2 from $n$ or $m$ but keeping $n$ and $m$ non-negative. The first player who cannot play any more has lost.

Suppose that it is the turn of player $A$ to play. A position $(n,m)$ is called *k-winning* if, no matter the way the other player $B$ plays, it is always possible for $A$ to win, after having made at most $k$ moves. The constraint defined in [6] expressing that a position $x$ is $k$-winning is:

$$winning_k(x) \leftrightarrow \left[\begin{array}{c} \exists y move(x,y) \wedge \neg(\exists x move(y,x) \wedge \\ \neg(\exists y move(x,y) \wedge \neg(\exists x move(y,x) \wedge \neg(\ldots \wedge \\ \neg(\exists y move(x,y) \wedge \neg(\exists x move(y,x) \wedge \neg(false \underbrace{))\ldots)}_{2k} \end{array}\right]$$

Each position $(n,m)$ is represented by $c(i,j)$ with $c$ a function symbol of arity 2 and $i,j \in \boldsymbol{Q}$. The constraint $move(x,y)$ is defined by

$$\left[\begin{array}{l} (\exists i \exists j\, x = c(i,j) \wedge y = c(i-1,j) \wedge i > 1 \wedge j > 0) \vee \\ (\exists i \exists j\, x = c(i,j) \wedge y = c(i-2,j) \wedge i > 2 \wedge j > 0) \vee \\ (\exists i \exists j\, x = c(i,j) \wedge y = c(i,j-1) \wedge i > 0 \wedge j > 1) \vee \\ (\exists i \exists j\, x = c(i,j) \wedge y = c(i,j-2) \wedge i > 0 \wedge j > 2) \vee \\ (\neg(\exists i \exists j\, x = c(i,j) \wedge num\, i \wedge num\, j) \wedge x = y) \end{array}\right]$$

By replacing the definition of $move$ in the constraint $winning_k(x)$, we have a first-order constraint with one free variable $x$ in the theory $T$ of evaluated trees. Solving this constraint means finding the positions $x$ which are $k$-winning.

# 3 Block and quantified block in $T$

We will now present structured formulas called blocks and show some of their properties. Essentially a block is a conjunction of atomic formulas where all the variables are well typed and which gives enough informations to be locally solved. We will also define a mechanism to decompose each quantified block in three quantified blocks having interesting properties that will help us solving first order constraints on quantified blocks.

## 3.1 Basic formulas and block in $T$

Suppose that the variables of $V$ are ordered by a linear strict and dense order relation without endpoints, denoted by "$\succ$". For each formula $\varphi$, the bounded variables are renamed such that for each sub-formula of $\varphi$ we have $x \succ y$ for each bounded variable $x$ and each free variable $y$. We denote by $\Sigma_{i=1}^n t_i$ the term $\overline{t_1 + \ldots + t_n} + 0$ with $\overline{t_1 + \ldots + t_n}$ the term $t_1 + \ldots + t_n$ where all the terms 0 have been removed.

Let $f \in F - \{0,1\}$, $a_0 \in \boldsymbol{Z}$ and $a_i \in \boldsymbol{Z}$. We call *leader* of the equation $x_0 = fx_1...x_n$ or $x_0 = x_1$ the variable $x_0$. We call *leader* of the formula $\Sigma_{i=1}^n a_i x_i = a_0 1$ the greatest variable $x_k$ (in the order $\succ$) such that $a_k \neq 0$.

Let $f \in F$, $a_0 \in \boldsymbol{Z}$ and $a_i \in \boldsymbol{Z}$. We call *basic formula* every conjunction $\alpha$ of formulas of the form:

- $true$, $false$, $num\, x$, $tree\, x$,
- $x = y$, $x = fy_1...y_n$, $\Sigma_{i=1}^n a_i x_i = a_0 1$, $\Sigma_{i=1}^n a_i x_i < a_0 1$.

The formulas $num\, x$ and $tree\, x$ are called *typing constraints*. The formulas $x = y$, $x = fy_1...y_n$, $\Sigma_{i=1}^n a_i x_i = a_0 1$ are called *equations*. The formula $\Sigma_{i=1}^n a_i x_i < a_0 1$ is called *inequation*. Let $\alpha$ be a basic formula:

(1) We say that "*$num\, x$ is a consequence* of $\alpha$" iff $\alpha$ contains at least one of the following sub-formulas: $num\, x$, $x = y \wedge num\, y$, $y = x \wedge num\, y$, $x =$

$-y \wedge num\, y$, $y = -x \wedge num\, y$, $z = y + x \wedge num\, z$, $z = x + y \wedge num\, z$, $x = y + z \wedge num\, z \wedge num\, y$, $x = 0$, $x = 1$, $\Sigma_i a_i x_i = a_0 1$ or $\Sigma_i a_i x_i < a_0 1$ and $x$ is one of the $x_i$'s.

(2) We say that "*tree* $x$ *is a consequence* of $\alpha$" iff $\alpha$ contains at least one of the following sub-formulas: $tree\, x$, $x = y \wedge tree\, y$, $y = x \wedge tree\, y$, $x = -y \wedge tree\, y$, $y = -x \wedge tree\, y$, $x = y + z \wedge tree\, z$, $x = z + y \wedge tree\, z$, $y = x + z \wedge tree\, y \wedge num\, z$, $y = z + x \wedge tree\, y \wedge num\, z$, $x = hy_1...y_n$, with $h \in F - \{+, -, 0, 1\}$.

(3) We call *tree-section* of $\alpha$ the conjunction $\alpha_t$ of the sub-formulas of $\alpha$ of the form:

- $true$, $tree\, x$,
- $x = y$ or $x = fy_1...y_n$, with $f \in F - \{0, 1\}$ and where $x$ is such that $tree\, x$ is a sub-formula of $\alpha$.

This tree-section $\alpha_t$ is called *formatted* iff all the left-hand sides of the equations of $\alpha_t$ are distinct and for each equation $x = y$ of $\alpha_t$ we have $x \succ y$.

(4) We call *numeric-section* of $\alpha$ the conjunction $\alpha_n$ of sub-fomulas of $\alpha$ of the form:

- $true$, $false$, $\Sigma_{i=1}^n a_i x_i = a_0 1$, $\Sigma_{i=1}^n a_i x_i < a_0 1$, $num\, x$,
- $x = y$, $x = -y$, $x = y + z$, where $x$ is such that $num\, x$ is a sub-formula of $\alpha$.

This numeric-section $\alpha_n$ is called *consistent* iff $T \models \exists \bar{x}\, \alpha_n$ with $\bar{x} = var(\alpha_n)$ and *formatted* iff

- $\alpha_n$ does not contain sub-formulas of the form $x = y$, $x = -y$, $x = y + z$, $0 = a_0 1$, $0 < a_0 1$, with $a_0 \in \boldsymbol{Z}$
- $\alpha_n$ is consistent and each leader of the equations of $\alpha_n$ has one occurrence in only one the equations of $\alpha_n$ and no occurrence in the inequations of $\alpha_n$.

(5) The variable $u$ is called *reachable* in $\exists \bar{x} \alpha$ if $u$ is a free variable in $\exists \bar{x} \alpha$ or $\alpha$ has a sub-formula of the form $y = t(u) \wedge tree\, y$ with $t(u)$ a term containing $u$ and $y$ a reachable variable. In the last case, the equation $y = t(u)$ is also called reachable in $\exists \bar{x} \alpha$.

**Example 3.1.1** *In the formula $\exists xyz\, w = fxy \wedge z = v \wedge tree\, w$, the variables $w, v, x, y$ are reachable because $w, v$ are free and $x$ and $y$ occur in the sub-formula $w = fxy \wedge tree\, w$. The variable $z$ is not reachable and since $z$ is bound and $v$ is free, they must be such that $z \succ v$. The equation $w = fxy$ is reachable while the equation $z = v$ is not.*

We call *block* every basic formulas $\alpha$ such that for each variable $x$ in $\alpha$ either $num\, x$ or $tree\, x$ is a sub-formula of $\alpha$ and $\alpha$ does not contain sub-formulas of the form:

- $x = 0 \wedge tree\, x$, $x = 1 \wedge tree\, x$,

- $x = y \wedge num\, x \wedge tree\, y$, $x = y \wedge tree\, x \wedge num\, y$,
- $x = -y \wedge tree\, x \wedge num\, y$, $x = -y \wedge num\, x \wedge tree\, y$
- $x = y + z \wedge num\, x \wedge tree\, y$, $x = y + z \wedge num\, x \wedge tree\, z$, $x = h\bar{y} \wedge num\, x$,
- $x = y + z \wedge tree\, x \wedge num\, y \wedge num\, z$,
- $\Sigma_{i=1}^{n} a_i x_i = a_0 1 \wedge tree\, x_k$, $\Sigma_{i=1}^{n} a_i x_i < a_0 1 \wedge tree\, x_k$

with $h \in F - \{+, -, 0, 1\}$, $k \in \{1, ..., n\}$, $a_0 \in \boldsymbol{Z}$ and $a_i \in \boldsymbol{Z}$.

Since each variable $x$ in a block is typed i.e. occurs in a sub-formula of the form $num\, x$ or $tree\, x$, every block $\alpha$ can be divided into two disjoint sections: a tree-section and a numeric-section.

A block $\alpha$ without equations is called *relation block*. A block $\alpha$ without inequations and where each variable has an occurrence in at least one of the equations of $\alpha$ is called *equation* block. A block $\alpha$ is called *solved* iff its tree-section and numerical-section are formatted.

### 3.2 Decomposition of quantified solved blocks

Let $\psi$ be a formula. Let $\bar{x}$ be a vector of variables and $\alpha$ a solved block such that for all unreachable quantified variable $u$ in $\exists\bar{x}\alpha$ and all reachable quantified variable $v$ in $\exists\bar{x}\alpha$ we have $u \succ v$. We call *decomposition* of the formula $\exists\bar{x}\alpha \wedge \psi$ the formula

$$\exists\bar{x}^1\, \alpha^1 \wedge (\exists\bar{x}^2\, \alpha^2 \wedge (\exists\bar{x}^3\, \alpha^3 \wedge \psi))), \tag{1}$$

obtained as follows : Let $X$ be the set of the variables in $\bar{x}$. Let us decompose the set $X$ into two disjoint subsets: $X_r$ (the set of the elements of $X$ which are reachable in $\exists\bar{x}\alpha$) and $X_u$. Let *Lead* be the set of the leaders of the equations of $\alpha$. We have:
– $\bar{x}^1$ is a vector of the variables of $X_r$.
– $\bar{x}^2$ is a vector of the variables of $X_u - Lead$.
– $\bar{x}^3$ is a vector of the variables of $X_u \cap Lead$.
– $\alpha^1$ is of the form $\alpha_1^1 \wedge \alpha_2^1$ where $\alpha_1^1$ is the conjunction of all the equations in $\exists\bar{x}\alpha$ whose leader is reachable, $\alpha_2^1$ is the conjunction of all the typing constraints of $\alpha$ which concern variables of $var(\alpha_1^1)$.
– $\alpha^2$ is of the form $\alpha_1^2 \wedge \alpha_2^2$ where $\alpha_1^2$ is the conjunction of all the inequations of $\alpha$ and $\alpha_2^2$ is the conjunction of all the typing constraints of $\alpha$ which do not concern variables of $\bar{x}^3$.
– $\alpha^3$ is of the form $\alpha_1^3 \wedge \alpha_2^3$ where $\alpha_1^3$ is the conjunction of the other equations and $\alpha_2^3$ is the conjunction of all the typing constraints of $\alpha$ which concern the variables of $var(\alpha_1^3)$. The restriction on the order $\succ$ of the quantified unreachable and reachable variables is due to an aim to get as leader of the equations of the numeric section of $\alpha$ unreachable variables. If one quantified leader is reachable then we deduce that all the quantified variables of this equation are reachable. This condition will help us for the algorithm of resolution given at Section 4. The intuitions behind this decomposition come from an aim to decompose a quantified solved block into three embedded sections each one having

particular properties that enable us either to remove quantifiers or make special distributions in $\psi$ and reduce the size of the formula $\exists\bar{x}\alpha \wedge \psi$.

Let $A$ be the set of the solved blocks. Let $A^1$ be the set of the formulas of the form $\exists\bar{x}^1\alpha^1$, where $\alpha^1$ is a solved equation block and all the variables of $\bar{x}^1$ are reachable in $\exists\bar{x}^1\alpha^1$. Let $A^2$ be the set of solved relation blocks.

**Property 3.2.1** *For all decomposed formula of the form (1) we have :* $\exists\bar{x}^1\alpha^1 \in A^1$, $\alpha^2 \in A^2$, $\alpha^3 \in A$ *and* $T \models \forall\bar{x}^2\,\alpha^2 \rightarrow \exists!\bar{x}^3\alpha^3$.

**Example 3.2.2** *Let* $v, w, x, y, z$ *variables such that* $w \succ y \succ z \succ x \succ v$*. Let us decompose the formula*

$$\exists wxyz \left[\begin{array}{l} v = fvx \wedge w + 2x + (-2)z = 1 \wedge y + 3z = 0 \wedge \\ z < 1 \wedge 3z + 2x < 0 \wedge \\ tree\, v \wedge num\, w \wedge num\, x \wedge num\, y \wedge num\, z \end{array}\right] \tag{2}$$

*The reachable variables in the formula (2) are* $v$ *and* $x$*. We have* $X_r = \{x, v\}$, $X_u = \{w, y, z\}$ *and* $Lead = \{v, w, y\}$*. Since* $w \succ y \succ z \succ x$ *then the formula (2) is equivalent in* $T$ *to the decomposed formula*

$$\left[\begin{array}{l} \exists x\, v = fvx \wedge tree\, v \wedge num\, x \wedge \\ (\exists z\, z < 1 \wedge 3z + 2x < 0 \wedge num\, z \wedge num x \wedge tree\, v \wedge \\ (\exists wy\, w{+}2x{+}(-2)z{=}1 \wedge y{+}3z{=}0 \wedge num\, w \wedge num\, x \wedge num\, y \wedge num\, z)) \end{array}\right]$$

*Note that the elements of* $A^1$ *does not accept elimination of quantifiers, this is due to the fact that all the variables of* $\bar{x}^1$ *are reachable in* $\exists\bar{x}^1\,\alpha^1$*. Indeed in the formula* $\exists x\, v = fvx$ *the quantification* $\exists x$ *can not be eliminated in* $T$*.*

In all what follows we will use the notations $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$, $\alpha^1$, $\alpha^2$,$\alpha^3$ to refer to the decomposition of the formula $\exists\bar{x}\alpha$.

# 4 Solving first-order constraints in $T$

## 4.1 Working and general solved formulas

**Definition 4.1.1** *A normalized formula* $\varphi$ *of depth* $d \geq 1$ *is a formula of the form*

$$\neg(\exists\bar{x}\,\alpha \wedge \textstyle\bigwedge_{i\in I}\varphi_i), \tag{3}$$

*with* $I$ *a finite (possibly empty) set,* $\alpha$ *a basic formula and the* $\varphi_i$ *normalized formulas of depth* $d_i$ *and* $d = 1{+}\max\{0, d_1, ..., d_n\}$*. Formulas of depth 1 (*$I = \emptyset$*) are of the form* $\neg(\exists\bar{x}\,\alpha)$ *with* $\alpha$ *a basic formula.*

**Property 4.1.2** *Every formula is equivalent in* $T$ *to a normalized formula.*

**Definition 4.1.3** *A working formula is a normalized formula in which all the occurrences of* $\neg$ *are of the form* $\neg^k$ *with* $k \in \{0, ..., 9\}$ *and such that each occurrence of a sub-formula of the form*

$$\phi = \neg^k(\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \textstyle\bigwedge_{i\in I}\varphi_i), \tag{4}$$

*has* $\alpha^p = true$ *if* $k = 0$ *and* **satisfies the first** $k$ **conditions** *of the following condition list if* $k > 0$. *Here* $\alpha^p$ *is a solved block and is called propagated constraint section,* $\alpha^c$ *is a basic formula and is called core constraint section, the* $\varphi_i$ *are working formulas, and in the conditions:* $\beta^p \wedge \beta^c$ *is the conjunction of the equations and relations of the immediate top-working formula* $\psi$ *of* $\phi$ *if it exists. i.e.* $\psi = \neg^k(\exists \bar{y} \beta^c \wedge \beta^p \wedge \phi \wedge \bigwedge_{j\in J} \phi_j)$ *with* $\phi_j$ *working formulas.*

1. *if* $\psi$ *exists then* $T \models \alpha^p \wedge \alpha^c \to \beta^p \wedge \beta^c$, *and the tree-sections of* $\alpha^p$ *and* $\beta^c \wedge \beta^p$ *have the same set of left-hand side of equations,*
2. *the tree-section of* $\alpha^p \wedge \alpha^c$ *is formatted and the formula* $\alpha^p \wedge \alpha^c$ *does not contain* $tree\, x \wedge num\, x$ *for any variable* $x$,
3. $\alpha^p \wedge \alpha^c$ *is a block,*
4. *the numeric-section of* $\alpha^p \wedge \alpha^c$ *is consistent, and we have* $u \succ v$ *for* $u$ *any unreachable variable in* $\bar{x}$ *and* $v$ *any reachable variable in* $\bar{x}$,
5. $\alpha^p \wedge \alpha^c$ *is a solved block,*
6. $\alpha^p$ *is the formula* $\beta^c \wedge \beta^p$ *if* $\psi$ *exists, and is the formula true otherwise. The formula* $\alpha^c$ *is a solved block and for each relation* $num\, x$ *(or* $tree\, x$*) in* $\alpha^p$, *if* $x$ *does not occur in an equation or inequation of* $\alpha^c$ *then* $num\, x$ *(resp.* $tree\, x$*) does not occur in* $\alpha^c$,
7. $(\exists \bar{x}\, \alpha^c)$ *is decomposable into* $(\exists \bar{x}^1\, \alpha^{c1} \wedge (\exists \bar{x}^2\, \alpha^{c2} \wedge (\exists \varepsilon\, true)))$,
8. $(\exists \bar{x}\, \alpha^c)$ *is decomposable into* $(\exists \bar{x}^1\, \alpha^{c1} \wedge (\exists \varepsilon\, \alpha^{c2} \wedge (\exists \varepsilon\, true)))$,
9. $(\exists \bar{x}\, \alpha^c)$ *is decomposable into* $(\exists \bar{x}^1\, \alpha^{c1} \wedge (\exists \varepsilon\, true \wedge (\exists \varepsilon\, true)))$.

The use of $k$ aims to be able to control the execution of our rewriting rules on working formulas. We strongly insist in the fact that $\neg^k$ does not mean that the normalized formula satisfies only the $k^{th}$ condition but all the conditions $i$ with $1 \leq i \leq k$. We call *initial* working formula a working formula of the form

$$\neg^6(\exists \varepsilon\, true \wedge \bigwedge_{i\in I} \varphi_i)$$

with $\varphi_i$ working formulas where all negation symbols $\neg^k$ have $k = 0$ and all propagated constraint sections are *true*. We call *final* working formula a formula of the form

$$\neg^7(\exists \varepsilon\, true \wedge \bigwedge_{i\in I} \neg^8(\exists \bar{x}_i\, \alpha_i^c \wedge \alpha_i^p \wedge \bigwedge_{j\in J_i} \neg^9(\exists \bar{y}_{ij}\, \beta_{ij}^c \wedge \beta_{ij}^p))), \tag{5}$$

where the $\beta_{ij}^c$ are different from *true*.

**Definition 4.1.4** *A* general solved formula *is a formula of the form*

$$\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2 \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}_i^1\, \beta_i^1), \tag{6}$$

*where* $\exists \bar{x}^1\, \alpha^1 \in A^1$, $\alpha^2 \in A^2$, $\exists \bar{y}_i^1\, \beta_i^1 \in A^1$, *all the* $\alpha^1 \wedge \alpha^2 \wedge \beta_i^1$ *are solved blocks and all the* $\beta_i^1$ *are different from true.*

According to the properties of $\neg^8$ and $\neg^9$, in the final working formula (5), $\alpha_i^p = true$ and $\beta_{ij}^p = \alpha_i^p \wedge \alpha_i^c$. Thus the formula (5) is equivalent in $T$ to the following disjunction of general solved formulas

$$\bigvee_{i \in I} (\exists \bar{x}_i \, \alpha_i^c \wedge \bigwedge_{j \in J_i} \neg(\exists \bar{y}_{ij} \, \beta_{ij}^c)) \tag{7}$$

**Property 4.1.5** *Let $\varphi$ be a general solved formula of the form (6). If $\varphi$ has no free variables then $\varphi$ is the formula true, otherwise neither $T \models \varphi$ nor $T \models \neg\varphi$.*

**Property 4.1.6** *Every general solved formula is equivalent in $T$ to a boolean combination of formulas of the form $\exists \bar{x}^1 \alpha^1 \wedge \alpha^2$, with $\exists \bar{x}^1 \alpha^1 \in A^1$ and $\alpha^2 \in A^2$, which do not accept elimination of quantifiers.*

## 4.2 Main idea

The general algorithm for solving first-order constraints in $T$ uses a system of rewriting rules. The main idea is to transform an initial working formula of depth $d$ to a final working formula of depth less than or equal to three. The transformation is done in two steps:

(1) The first step is a top-down simplification and propagation. In each sub-working formula, $\alpha^c \wedge \alpha^p$ is transformed to a solved block, then $\exists \bar{x} \alpha^c$ is decomposed into three parts as in subsection 3.2. The third part is eliminated and added to the core-constraint section of the immediate sub-working formulas using a special property of the quantifier $\exists !$. The constraints of the two other parts in $\alpha^p$ are propagated to the propagated-constraint section of the immediate sub-working formulas. In this step, the rules 1 to 24 are applied and transform the initial working formula to a working formula where each negation symbol is of the form $\neg^7$.

(2) The second step is a bottom-up simplification and elimination of quantifiers. This step is done by the rules 25 to 28. In each sub-working formula of depth one or two, the rule 25 eliminates quantified variables of the second part of the decomposition (the third one had been already removed in the first step). The rule 26 eliminates the constraints of the second part in the deepest level. Each sub-working formula of depth 3 is transformed step by step to a conjunction of working formulas of depth 2 by the rule 28 using a property of the quantifier $\exists ?$. The transformations in this step can create new sub-working formulas where the first step needs to be done. At the end of the transformation, we obtain a final working formula of depth less than or equal to 3.

## 4.3 Rewriting rules

We present in Figure 1 the rewriting rules which transform an initial working formula to a final working formula, which is equivalent in $T$. To apply the rule $p_1 \Longrightarrow p_2$ to the working formula $p$ means to replace in $p$, a sub-formula $p_1$ by the formula $p_2$, by considering that the connector $\wedge$ is associative and commutative. In all these rules, $\alpha$ are basic formulas, $\varphi$ and $\psi$ are conjunctions of working formulas.

Figure 1: The rewriting rules

| | | | |
|---|---|---|---|
| 1 | $\neg^1(\exists\bar{u}\, \mathit{num}\, x \wedge \mathit{tree}\, x \wedge \alpha \wedge \varphi)$ | $\Longrightarrow$ | $\mathit{true}$ |
| 2 | $\neg^1(\exists\bar{u}\, x = f\bar{y} \wedge x = g\bar{z} \wedge \mathit{tree}\, x \wedge \alpha \wedge \varphi)$ | $\Longrightarrow$ | $\mathit{true}$ |
| 3 | $\neg^1(\exists\bar{u}\, x = x \wedge \alpha \wedge \varphi)$ | $\Longrightarrow$ | $\neg^1(\exists\bar{u}\, \alpha \wedge \varphi)$ |
| 4 | $\neg^1(\exists\bar{u}\, y = x \wedge \mathit{tree}\, x \wedge \alpha \wedge \varphi)$ | $\Longrightarrow$ | $\neg^1(\exists\bar{u}\, x = y \wedge \mathit{tree}\, x \wedge \alpha \wedge \varphi)$ |
| 5 | $\neg^1\left[\begin{array}{l}\exists\bar{u}\, x = fy_1...y_n \wedge x = fz_1...z_n \wedge \\ \mathit{tree}\, x \wedge \alpha \wedge \varphi\end{array}\right]$ | $\Longrightarrow$ | $\neg^1\left[\begin{array}{l}\exists\bar{u}\, x = fy_1...y_n \wedge \bigwedge_i y_i = z_i \wedge \\ \mathit{tree}\, x \wedge \alpha \wedge \varphi\end{array}\right]$ |
| 6 | $\neg^1\left[\begin{array}{l}\exists\bar{u}\, x = y \wedge x = fz_1...z_n \wedge \\ \mathit{tree}\, x \wedge \mathit{tree}\, y \wedge \alpha \wedge \varphi\end{array}\right]$ | $\Longrightarrow$ | $\neg^1\left[\begin{array}{l}\exists\bar{u}\, x = y \wedge y = fz_1...z_n \wedge \\ \mathit{tree}\, x \wedge \mathit{tree}\, y \wedge \alpha \wedge \varphi\end{array}\right]$ |
| 7 | $\neg^1(\exists\bar{u}\, x = y \wedge x = z \wedge \mathit{tree}\, x \wedge \alpha \wedge \varphi)$ | $\Longrightarrow$ | $\neg^1(\exists\bar{u}\, x = y \wedge y = z \wedge \mathit{tree}\, x \wedge \alpha \wedge \varphi)$ |
| 8 | $\neg^4(\exists\bar{u}\, 0 = 01 \wedge \alpha \wedge \varphi)$ | $\Longrightarrow$ | $\neg^4(\exists\bar{u}\, \alpha \wedge \varphi)$ |
| 9 | $\neg^4(\exists\bar{u}\, 0 < a_0 1 \wedge \alpha \wedge \varphi)$ | $\Longrightarrow$ | $\neg^4(\exists\bar{u}\, \alpha \wedge \varphi)$ |
| 10 | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, x = y \wedge \\ \mathit{num}\, x \wedge \mathit{num}\, y \wedge \alpha \wedge \varphi\end{array}\right]$ | $\Longrightarrow$ | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, x + (-1y) = 0 \wedge \\ \mathit{num}\, x \wedge \mathit{num}\, y \wedge \alpha \wedge \varphi\end{array}\right]$ |
| 11 | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, x = -y \wedge \\ \mathit{num}\, x \wedge \mathit{num}\, y \wedge \alpha \wedge \varphi\end{array}\right]$ | $\Longrightarrow$ | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, x + y = 0 \wedge \\ \mathit{num}\, x \wedge \mathit{num}\, y \wedge \alpha \wedge \varphi\end{array}\right]$ |
| 12 | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, x = y + z \wedge \mathit{num}\, x \wedge \\ \mathit{num}\, y \wedge \mathit{num}\, z \wedge \alpha \wedge \varphi\end{array}\right]$ | $\Longrightarrow$ | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, x + (-1y) + (-1z) = 0 \wedge \\ \mathit{num}\, x \wedge \mathit{num}\, y \wedge \mathit{num}\, z \wedge \alpha \wedge \varphi\end{array}\right]$ |
| 13 | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, \Sigma_{i=1}^n a_i x_i = a_0 1 \wedge \\ \Sigma_{i=1}^n b_i x_i = b_0 1 \wedge \\ \alpha \wedge \varphi\end{array}\right]$ | $\Longrightarrow$ | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, \Sigma_{i=1}^n a_i x_i = a_0 1 \wedge \\ \Sigma_{i=1}^n (b_k a_i - a_k b_i) x_i = (b_k a_0 - a_k b_0) 1 \wedge \\ \alpha \wedge \varphi\end{array}\right]$ |
| 14 | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, \Sigma_{i=1}^n a_i x_i = a_0 1 \wedge \\ \Sigma_{i=1}^n b_i x_i < b_0 1 \wedge \\ \alpha \wedge \varphi\end{array}\right]$ | $\Longrightarrow$ | $\neg^4\left[\begin{array}{l}\exists\bar{u}\, \Sigma_{i=1}^n a_i x_i = a_0 1 \wedge \\ \Sigma_{i=1}^n \lambda (b_k a_i - a_k b_i) x_i < (b_k a_0 - a_k b_0) 1 \wedge \\ \alpha \wedge \varphi\end{array}\right]$ |
| 15 | $\neg^1(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ | $\Longrightarrow$ | $\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ |
| 16 | $\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ | $\Longrightarrow$ | $\neg^1(\exists\bar{x}\, \mathit{num}\, z \wedge \alpha^c \wedge \alpha^p \wedge \varphi)$ |
| 17 | $\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ | $\Longrightarrow$ | $\neg^1(\exists\bar{x}\, \mathit{tree}\, z \wedge \alpha^c \wedge \alpha^p \wedge \varphi)$ |
| 18 | $\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ | $\Longrightarrow$ | $\left[\begin{array}{l}\neg^1(\exists\bar{x}\, \mathit{num}\, z \wedge \alpha^c \wedge \alpha^p \wedge \varphi) \wedge \\ \neg^1(\exists\bar{x}\, \mathit{tree}\, z \wedge \alpha^c \wedge \alpha^p \wedge \varphi)\end{array}\right]$ |
| 19 | $\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ | $\Longrightarrow$ | $\neg^3(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ |
| 20 | $\neg^3(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ | $\Longrightarrow$ | $\mathit{true}$ |
| 21 | $\neg^3(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ | $\Longrightarrow$ | $\neg^4(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ |
| 22 | $\neg^4(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ | $\Longrightarrow$ | $\neg^5(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$ |
| 23 | $\neg^7\left[\begin{array}{l}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^5(\exists\bar{y}\, \beta^c \wedge \beta^p \wedge \psi)\end{array}\right]$ | $\Longrightarrow$ | $\neg^7\left[\begin{array}{l}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^6(\exists\bar{y}\, \gamma^c \wedge \gamma^p \wedge \psi)\end{array}\right]$ |
| 24 | $\neg^6\left[\begin{array}{l}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \\ \bigwedge_i \neg^0(\exists\bar{y}_i \beta_i^c \wedge \beta_i^p \wedge \varphi_i)\end{array}\right]$ | $\Longrightarrow$ | $\neg^7\left[\begin{array}{l}\exists\bar{x}^1\bar{x}^2\, \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p \wedge \\ \bigwedge_i \neg^1(\exists\bar{y}_i\bar{x}^3 \gamma_i^c \wedge \gamma_i^p \wedge \varphi_i)\end{array}\right]$ |
| 25 | $\neg^7\left[\begin{array}{l}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \\ \bigwedge_{i\in I} \neg^9(\exists\bar{y}_i \beta_i^c \wedge \beta_i^p)\end{array}\right]$ | $\Longrightarrow$ | $\neg^8\left[\begin{array}{l}\exists\bar{x}^1 \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p \wedge \\ \bigwedge_{i\in I'} \neg^9(\exists\bar{y}_i \beta_i^c \wedge \beta_i^{p*})\end{array}\right]$ |
| 26 | $\neg^7\left[\begin{array}{l}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8(\exists\bar{y} \beta^c \wedge \beta^p)\end{array}\right]$ | $\Longrightarrow$ | $\left[\begin{array}{l}\neg^7(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists\bar{y}\beta^{c1} \wedge \beta^p)) \wedge \\ \bigwedge_{i\in I} \neg^1(\exists\bar{x}\bar{y}\beta^p \wedge \beta^{c1} \wedge \beta_i^{c2*} \wedge \varphi_0)\end{array}\right]$ |
| 27 | $\neg^7\left[\begin{array}{l}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^9(\exists\varepsilon \mathit{true} \wedge \beta^p)\end{array}\right]$ | $\Longrightarrow$ | $\mathit{true}$ |
| 28 | $\neg^7\left[\begin{array}{l}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8\left[\begin{array}{l}\exists\bar{y}\, \beta^c \wedge \beta^p \wedge \\ \bigwedge_{i\in I} \neg^9(\exists\bar{z}_i\, \gamma_i^c \wedge \gamma_i^p)\end{array}\right]\end{array}\right]$ | $\Longrightarrow$ | $\left[\begin{array}{l}\neg^7(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^8(\exists\bar{y}\, \beta^c \wedge \beta^p)) \wedge \\ \bigwedge_{i\in I} \neg^6(\exists\bar{x}\bar{y}\bar{z}_i\, \delta_i^c \wedge \delta_i^p \wedge \varphi_0)\end{array}\right]$ |

In the rules 1 to 14, the equations and relations in $\alpha^c$ and $\alpha^p$ are mixed by considering the connector $\wedge$ associative and commutative. In these rules, except the rule 6, all modifications in the right hand side are done in $\alpha^c$, since $\alpha^p$ is a solved block.

In the rule 2, $f$ and $g$ are two distinct function symbols taken from $F$. In the rules 4, 6, 7 $x \succ y$ This condition prevent infinite loop and makes the procedure terminating. In the rule 5, the equation $x = fz_1...z_n$ does not belong to $\alpha^p$. In the rule 6, if the equation $x = fz_1...z_n$ belongs to $\alpha^p$, then $x = y \wedge \mathit{tree}\, y$ is moved to $\alpha^p$. In the rule 7, the equation $x = z$ does not belong to $\alpha^p$.

We recall that the notation 01 in the rule 8 means the term 0. In the rule 9, $a_0 > 0$. In the rules 13 and 14 the variable $x_k$ is the leader of the equation $\Sigma_i a_i x_i = a_0 1$ and $b_k \neq 0$. More over the equation $\Sigma_j b_j x_j = b_0 1$ does not belong to $\alpha^p$. In the rule 14, the relation $\Sigma_j b_j x_j < b_0 1$ does not belong to $\alpha^p$ and $\lambda = 1$ if $a_k > 0$ and $\lambda = -1$ otherwise.

In the rule 15, the tree section of $\alpha^c \wedge \alpha^p$ is formatted and there is no sub-formula in $\alpha^c \wedge \alpha^p$ of the form $\mathit{num}\, x \wedge \mathit{tree}\, x$. In the rule 16 respectively 17, the typing constraint $\mathit{num}\, z$, respectively $\mathit{tree}\, z$ is not in $\alpha^c \wedge \alpha^p$ and is a consequence of $\alpha^c \wedge \alpha^p$. In the rule 18, $z$ does not have typing constraint in $\alpha^c \wedge \alpha^p$ and neither $\mathit{num}\, z$ nor $\mathit{tree}\, z$ is a consequence of $\alpha^c \wedge \alpha^p$.

In the rule 19, $\alpha^c \wedge \alpha^p$ is a block. In the rule 20, the numeric section of $\alpha^c \wedge \alpha^p$ is inconsistent. In the rule 21, the unreachable variables in $\bar{x}$ are renamed if necessary such that $u \succ v$ for each unreachable variable $u$ and each reachable variable $v$ in $\bar{x}$ and the numeric section of $\alpha^c \wedge \alpha^p$ is consistent. The consistency can be verified for example by using the first step of the Simplex. In the rule 22, $\alpha^c \wedge \alpha^p$ is a solved block.

In the rule 23, $\gamma^c$ is obtained from $\beta^c$ as follows: for all variable $x \in \mathit{var}(\beta^c)$, we add all the relations $\mathit{num}\, x$ or $\mathit{tree}\, x$ which are in $\beta^p$ but not in $\beta^c$, and for all the variables $y$ which do not occur in an equation or inequation of $\beta^c$ we remove all relations $\mathit{num}\, y$ or $\mathit{tree}\, y$ which are both in $\beta^c$ and $\beta^p$. The formula $\gamma^p$ is the formula $\alpha^p \wedge \alpha^c$.

In the rule 24, $\exists \bar{x} \alpha^c$ is decomposed to $\exists \bar{x}^1 \alpha^{c1} \wedge (\exists \bar{x}^2 \alpha^{c2} \wedge (\exists \bar{x}^3 \alpha^{c3}))$, $\gamma_i^c = \beta_i^c \wedge \alpha^{c3}$ and $\gamma_i^p = \beta_i^p \wedge \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p$.

The four rules 25, 26, 27 and 28 can not be applied on the occurrence of $\neg^7$ of the first level of the general working formula. In the rule 25, $I'$ is the set of $i \in I$ such that $\beta_i^c$ does not contain occurrences of any variables in $\bar{x}^2$. The formula $\alpha^{c2*}$ is such that $T \models (\exists \bar{x}^2 \alpha^{c2}) \leftrightarrow \alpha^{c2*}$ and is computed using the Fourier quantifier elimination. The propagated-constraint section $\beta_i^{p*} = \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p$.

In the rule 26, $\varphi$ is such that every negation symbol $\neg^k$ has $k \geq 6$, $\varphi_0$ is obtained from $\varphi$ by replacing all occurrences of $\neg^k$ by $\neg^0$ and all propagated-constraint sections by $\mathit{true}$. Let $\beta^2$ the formula obtained from $\beta^{c2}$ by removing the multiple occurrences of typing constraints and for all the variables $y$ which do not occur in an inequation of $\beta^{c2}$ we remove all relation $\mathit{num}\, y$ or $\mathit{tree}\, y$ which are both in $\beta^{c1}$ and $\beta^{c2}$. If $\beta^2$ is the formula $\mathit{true}$ then $I = \emptyset$, otherwise the $\beta_i^{c2*}$ with $i \in I$ are obtained from $\beta^2$ as follows: Since $\beta^2 \in A^2$ then it is of the form

$$\begin{bmatrix} (\bigwedge_{\ell \in L} \mathit{num}\, z_\ell) \wedge (\bigwedge_{k \in K} \mathit{tree}\, v_k) \wedge \\ ((\bigwedge_{j \in J} \sum_{i=1}^n a_{ij} x_i < a_{0j}) \wedge \bigwedge_{m=1}^n \mathit{num}\, x_m) \end{bmatrix},$$

thus $\neg\beta^2$ is of the form

$$\begin{bmatrix} (\bigvee_{\ell\in L} \mathit{tree}\ z_\ell) \vee (\bigvee_{k\in K} \mathit{num}\, v_k) \vee (\bigvee_{m=1}^n \mathit{tree}\, x_m) \vee \\ \bigvee_{j\in J}((\sum_{i=1}^n a_{ij}x_i = a_{0j}1 \wedge \bigwedge_{m=1}^n \mathit{num}\, x_m) \vee \\ (\sum_{i=1}^n (-a_{ij})x_i < (-a_{0j})1 \wedge \bigwedge_{m=1}^n \mathit{num}\, x_m)) \end{bmatrix}$$

Each element of this disjunction is a block and represents a formula $\beta_i^{c2*}$. Of course we have $T \models (\neg\beta^2) \leftrightarrow \bigvee_i \beta_i^{c2*}$.

In the rule 28, $I \neq \emptyset$, $\varphi$ is such that every negation symbol $\neg^k$ has $k \geq 6$, $\varphi_0$ is obtained from $\varphi$ by replacing all occurrences of $\neg^k$ by $\neg^0$ and all propagated-constraint sections by *true*. Moreover $\delta_i^p = \alpha^p$ and $\delta_i^c = \gamma_i^c \wedge \beta^c \wedge \alpha^c$.

**Property 4.3.1** *Every repeated application of the precedent rewriting rules on an inital working formula terminates and produces a final working formula equivalent in $T$ and without new free variables.*

## 4.4 Algorithm for solving constraints in $T$

Solving the constraint $\varphi$ in $T$ is made as follows:

1. Transform $\varphi$ to a normalized formula, then to an initial working formula $\phi$, which is equivalent to $\varphi$ in $T$.

2. Transform $\phi$ to a final working formula $\psi$ using the rewriting rules defined in the subsection 4.3.

3. Extract from $\psi$ the disjunction of general solved formulas, equivalent to $\psi$ in $T$. If the disjunction contains the general solved formula *true*, then it is reduced to *true*.

**Example 4.4.1** *Let $\varphi$ be the following constraint having $i, j$ as free variables: $\exists x\, x = fij \wedge i > 0 \wedge \mathit{tree}\, x \wedge \mathit{num}\, i \wedge \mathit{num}\, j \wedge \neg(\exists k\, j = 2k \wedge \mathit{num}\, k)$. We can see that $\mathit{num}\, j \wedge \neg(\exists k\, j = 2k \wedge \mathit{num}\, k)$ is always false in $T$ since for every variable $j$, there exists a unique variable $k$ such that $j = 2k$ (axiom $13_n$). Let us transform $\varphi$ to an initial working formula (the propagated-constraint sections are underlined):*

$$\neg^6\neg^0(\exists x\, x = fij \wedge i > 0 \wedge \mathit{tree}\, x \wedge \mathit{num}\, j \wedge \underline{\mathit{true}} \wedge \neg^0(\exists k\, j = 2k \wedge \mathit{num}\, k \wedge \underline{\mathit{true}}))$$

*After having applied the rules 24, 15, 16, 15, 19, 21, 22, 23 in this order, we obtain:*

$$\neg^7\neg^6(\exists x\, x = fij \wedge i > 0 \wedge \mathit{tree}\, x \wedge \mathit{num}\, i \wedge \mathit{num}\, j \wedge \underline{\mathit{true}} \wedge \neg^0(\exists k\, j = 2k \wedge \mathit{num}\, k \wedge \underline{\mathit{true}}))$$

*The rule 24 being applied changes the formula to:*

$$\neg^7\neg^7 \begin{bmatrix} i > 0 \wedge \mathit{num}\, i \wedge \mathit{num}\, j \wedge \underline{\mathit{true}} \wedge \\ \neg^1(\exists xk\, x = fij \wedge j = 2k \wedge \mathit{num}\, k \wedge \mathit{tree}\, x \wedge \underline{i > 0 \wedge \mathit{num}\, i \wedge \mathit{num}\, j}) \end{bmatrix}$$

*After having applied on the sub-working formula $\neg^1(...)$ the rule 15, 19, 21, 12, 22, 23*

$$\neg^7\neg^7 \begin{bmatrix} i > 0 \wedge \mathit{num}\, i \wedge \mathit{num}\, j \wedge \underline{\mathit{true}} \wedge \\ \neg^6(\exists xk\, x = fij \wedge j - 2k = 0 \wedge \mathit{num}\, k \wedge \mathit{tree}\, x \wedge \underline{i > 0 \wedge \mathit{num}\, i \wedge \mathit{num}\, j}) \end{bmatrix}$$

*The rule 24 is applied then we obtain:*

$$\neg^7\neg^7(i > 0 \wedge \mathit{num}\, i \wedge \mathit{num}\, j \wedge \underline{\mathit{true}} \wedge \neg^7(\mathit{true} \wedge \underline{i > 0 \wedge \mathit{num}\, i \wedge \mathit{num}\, j}))$$

*The rules 25, 26 being applied in this order, giving:*

$$\neg^7\neg^7(i > 0 \wedge num\, i \wedge num\, j \wedge \underline{true} \wedge \neg^9(true \wedge \underline{i > 0 \wedge num\, i \wedge num\, j}))$$

*Finally, by application of the rule 27, we obtain the final working formula* $\neg^7 true$*, which is equivalent to the empty disjunction of general solved formulas, i.e. false.*

Since $T$ has at least a model [10] and according to Property 4.1.5 and Property 4.3.1 we obtain the following corollary, which is another proof of the completeness of the theory $T$:

**Corollary 4.4.2** *Every formula is equivalent in $T$ either to true or to false or to a disjunction of general solved formulas, which has at least one free variable and which is equivalent neither to true nor to false in $T$.*

# 5 Conclusion

We have given in this paper a general algorithm for solving the most general first-order constraints in the theory of the evaluated trees. The algorithm is given in the form of 28 rewriting rules and its correctness is another proof of the completeness of $T$. The algorithm transforms a formula $\varphi$ which can contain free variables to a disjunction of solved formulas $\phi$ equivalent in $T$, without new free variables and such that $\phi$ is either the formula *true* or *false* or has at least a free variable and is equivalent neither to *true* nor to *false* in $T$. Our aim in this work was not only to decide proposition i.e. to decide if a formula without free variables is true or false in $T$ but to simplify a general constraint with free variables and to present its solutions in a clear and explicit way. This algorithm is also able to detect formulas having free variables but being always false or true in $T$, by giving at the end *false* or *true*.

S. Vorobyov [19] has shown that the problem of deciding if a proposition is true or not in the trees theory is non-elementary, i.e. the complexity of all algorithm which solves it cannot be bounded by a tower of powers of $2's$ (with a top down evaluation) with a fixed height. A. Colmerauer and B. Dao [6] have also given a proof of non-elementary complexity of solving constraints in the trees theory. Thus our algorithm must not escape this kind of complexity in the worst case. This is why we have used two strategies in the algorithm: a top down propagation of constraints and a bottom-up elimination of quantifiers and distribution. This technic can detect quickly (using propagation and local solving) sub-formulas which are equivalent to false and prevents us from solving a big sub-working formula which contradicts its immediate embedding working formula. We have programmed a similar algorithm only on the theory of finite or infinite trees and in spite of the high complexity we can solve formulas on two partners games having 160 nested quantifiers [9].

Actually we plan with Thom Fruehwirth [11] to add to CHR a mechanism to treat the normalized formulas which will enable us to implement easily our algorithm in CHR. We try also to find an automatic axiomatization for a combination of any theory $T$ with the theory of finite or infinite trees.

**Acknowledgements** We thank Alain Colmerauer for our many discussions

and his help in this work. We dedicate to him this paper with our best wishes for a speedy recovery.

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# 6 Appendix

In what follows we use the abbreviation wnfv for *"without new free variables"*. By saying a formula $\varphi$ is equivalent to a wnfv formula $\psi$ in $T$ we mean $T \models \varphi \leftrightarrow \psi$ and $\psi$ does not contain other free variables than those of $\varphi$.

## 6.1 Preliminaries

We present here new properties that will be used to show the properties given in this paper. Some of these new properties are already defined and proved in [10].

**Property 6.1.1** *[10] Let $T$ a theory. If $T \models \psi \rightarrow (\exists! \bar{x}\varphi)$ then*

$$T \models (\psi \wedge (\exists \bar{x}\varphi \wedge \neg\phi)) \leftrightarrow (\psi \wedge \neg(\exists \bar{x}\varphi \wedge \phi))$$

**Property 6.1.2** *[10] Let $T$ a theory. If $T \models \exists ? \bar{y}\phi$ and if each variable of $\bar{y}$ does not have free occurrences in $\varphi$ then*

$$T \models (\exists \bar{x}\varphi \wedge \neg(\exists \bar{y}\phi \wedge \psi)) \leftrightarrow (\exists \bar{x}\varphi \wedge \neg(\exists \bar{y}\phi)) \vee (\exists \bar{x}\bar{y}\varphi \wedge \phi \wedge \neg\psi)$$

We call *instantiation* of a formula $\varphi$ by individuals of $D_M$ the obtained formula from $\varphi$ in which for each free variable $x$ in $\varphi$, we replace each free occurrence of $x$ by the same individual $i$ of $D_M$.

**Definition 6.1.3 (Quantifier zero-infinite[10] )** *Let $M$ be a model. Let $\varphi_i$ and $\varphi(\bar{x})$ be $M$-formulas and let $\Psi(u)$ be a set of formulas of the form $\exists \bar{x} u = f\bar{x} \wedge tree\, u$, with $f \in F - \{0, 1\}$. We write*

$$M \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x), \tag{8}$$

*iff for any instantiation $\exists x\, \varphi'(x)$ of $\exists x\, \varphi(x)$ by individuals of $M$ one of the following properties holds:*

- *the set of the individuals $i$ of $M$ such that $M \models \varphi'(i)$, is empty,*

- *for all finite sub-set $\{\psi_1(u), .., \psi_n(u)\}$ of elements of $\Psi(u)$, the set of the individuals $i$ of $M$ such that*
  *$M \models \varphi'(i) \wedge \bigwedge_{j \in \{1,\dots,n\}} \neg\psi_j(i)$ is infinite.*

*Let $T$ be a theory. We write $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$, iff for each model $M$ of $T$ we have (8).*

**Property 6.1.4** *[10] If $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$ and if for each $\varphi_i$, one at least of the following properties holds:*

- *$T \models \exists ?x\, \varphi_i$,*

- *there exists $\psi_i(u) \in \Psi(u)$ such that $T \models \forall x\, \varphi_i \rightarrow \psi_i(x)$,*

*then*

$$T \models (\exists x\, \varphi(x) \wedge \textstyle\bigwedge_{i \in I} \neg\varphi_i) \leftrightarrow (\exists x\, \varphi(x))$$

**Proof L** et $M$ be an any model of $T$. Let $\exists x\, \varphi'(x)$ an instantiation of $\exists x\, \varphi(x)$ by individuals of $M$. Let us show that if the conditions of the Property 6.1.4 hold then

$$M \models (\exists x\, \varphi'(x) \wedge \textstyle\bigwedge_{j \in J} \neg\varphi_j(x)) \leftrightarrow (\exists x\, \varphi'(x)). \tag{9}$$

Let $J'$ the set of the $j \in J$ such that $M \models \exists ?x\, \varphi_j(x)$ and let $m$ be the cardinality of $J'$. Since for all $j \in J'$, $M \models \exists ?x\, \varphi_j(x)$, then it is enough to have at least $m+1$ individuals of $M$ in order to have $M \models \exists x\, \bigwedge_{j \in J'} \neg\varphi_j(x)$. On the other hand, Since $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$ and according to Definition 6.1.3 of the infinite quantifier two cases arise:

(1) Either $M \models \neg(\exists x\, \varphi'(x))$, thus $M \models \neg(\exists \bar{x}\, \varphi'(x) \wedge \bigwedge_{j \in J} \neg\varphi_j(x))$ and thus the equivalence (9) is true in $M$.

(2) Or, for all finite sub-set $\{\psi_1(u), ..., \psi_n(u)\}$ of $\Psi(u)$, the set of the individuals $i$ of $M$ such that $M \models \varphi'(i) \wedge \bigwedge_{j=1}^{n} \neg\psi_j(i)$ is infinite. Thus, since for all $j \in J - J'$ we have, $M \models \forall x\, \varphi_j(x) \rightarrow \psi_j(x)$, then there exists an infinite set $\xi$ of individuals $i$ of $M$ such that $M \models \varphi'(i) \wedge \bigwedge_{j \in J-J'} \neg\varphi_j(i)$. By removing from this infinite set $\xi$ the $m$ preceding individuals we get another infinite sub-set of $\xi$ such that for all $i \in \xi$ we have $M \models \varphi'(i) \wedge \bigwedge_{j \in J'} \neg\varphi_j(i) \wedge \bigwedge_{k \in J-J'} \neg\varphi_k(i)$ and thus $M \models \exists x\, \varphi'(x) \wedge \bigwedge_{j \in J} \neg\varphi_j(x)$. Since We have $M \models \exists x\, \varphi'(x) \wedge \bigwedge_{j \in J} \neg\varphi_j(x)$, then we get $M \models \exists x\, \varphi'(x)$ and thus the equivalence (9) is true in $M$.

**Property 6.1.5** *[10] Let $\alpha$ be a basic formula. If all the variables of $\bar{x}$ are reachable in $\exists \bar{x}\, \alpha$ then $T \models \exists ?\bar{x}\alpha$.*

This property is a direct consequence of the axioms 1 and 2 of $T$.

**Property 6.1.6**

$$T \models \Sigma_{i=1}^{k} a_i x_i = a_0 \wedge \bigwedge_{i=1}^{k} num\, x_i \leftrightarrow a_k x_k = \Sigma_{i=1}^{k} (-a_i) x_i + a_0 \wedge \bigwedge_{i=1}^{k} num\, x_i.$$

This property uses the axioms 5,6,8...13 of $T$.

**Property 6.1.7** *[10] if $\exists \bar{x}^1 \alpha^1 \in A^1$ then $T \models \exists ?\bar{x}^1\, \alpha^1$ and for each free variable $y$ in $\exists \bar{x}^1 \alpha^1$, at least one of the following properties holds:*

- *$T \models \exists ? y \bar{x}^1\, \alpha^1$,*
- *there exists $\psi(u) \in \Psi(u)$ such that $T \models \forall y\, (\exists \bar{x}^1\, \alpha^1) \rightarrow \psi(y)$,*

**Property 6.1.8** *[10] if $\alpha^2 \in A^2$ then for each $x^2$, $T \models \exists_{o\,\infty}^{\Psi(u)} x^2\, \alpha^2$*

**Property 6.1.9** *[10] if $\alpha^2 \in A^2$ then for each $x^2$, the formula $\exists x^2 \alpha^2$ is equivalent in $T$ to a formula which belongs to $A^2$,*

This property uses the Fourier elimination of quantifiers. Recall that formulas of $A^2$ are solved and contains only inequations and typing constraints.

**Property 6.1.10** *Let $\alpha$ a solved block and $\bar{x}$ the vector of the variables of $\alpha$. We have $T \models \exists \bar{x} \alpha$.*

**Proof L** et $M$ a model of $T$ let us show that $M \models \exists \bar{x} \alpha$. Since $\alpha$ is a solved block then the numeric section is consistent, all the leaders of the equations of $\alpha$ do not occur in the inequations of $\alpha$ and all the leaders of the equations of $\alpha$ are distinct and have one occurrence in only one equation. Let us transform the equations of the numeric section by moving to the right hand sides all the terms containing variables that are not leaders of equations (see Property 6.1.6). We get a conjunction of equations with distinct left hand sides that do not occur in other right hand side of equations of the numeric section or in inequations of $\alpha$ (see definition of solved block). Since the inequations are consistent then there exists an instantiation of the variables of the inequations of $\alpha$ that makes true these inequations. Thus, using this instantiation and for all instantiation of the variables of the right hand sides in the equations of the numeric sections there exists a value for the leaders of these equations(axiom 13). Using this instantiation and for all instantiation of the variables of the right hand sides of the equations of the tree section and which are not leader, there exists a value for the leaders (axiom 3). Thus there exists an instantiation of $\alpha$ that makes true $\alpha$ i.e. $M \models \exists \bar{x} \alpha$.

**Property 6.1.11** *Let $\exists \bar{x}^1 \alpha^1$ a formula without free variables that belongs to $A^1$. We have $\bar{x}^1 = \varepsilon$ and $\alpha^1 = true$.*

**Proof S** ince the formula $\exists \bar{x}^1 \alpha^1$ has no free variables then there is no reachable variables in $\exists \bar{x}^1 \alpha^1$ and thus using the definition of the set $A^1$ we get $\bar{x}^1 = \varepsilon$. Thus the formula $\exists \bar{x}^1 \alpha^1$ is equivalent in $T$ to $\alpha^1$ which has no free variables. According to the definition of $A^1$, $\alpha^1$ is a solved block and since it does not contain free variables then it is reduced to the formula *true*. It can not be reduced to *false* because it is a solved block thus consistent. It can not be reduced also to an equation or inequation without variables according to the definition of solved block and more exactly formatted.

**Property 6.1.12** *Let $\alpha$ be a solved equation block. Let $\bar{x}$ be the vector of the leaders of the equations of $\alpha$. Let $\alpha^*$ be the conjunction of the typing constraints of $\alpha$ which concern elements that do not belong to $\bar{x}$. We have*

$$T \models \alpha^* \rightarrow \exists! \bar{x}\, \alpha$$

**Proof R** ecall that we write $T \models \exists! \bar{x}\alpha$ if for every model $M$ and every instantiation $\alpha'$ of $\alpha$ by individuals of $M$ we have $M \models \exists! \bar{x}\alpha$.

Using this recall, let us prove now Property 6.1.12. This property is a consequence of the axioms 3 and $13_n$ of $T$. In fact, since the tree-section of $\alpha$ has distinct leaders i.e. distinct left-hand sides and since the equations of the numeric section of $\alpha$ have distinct leaders which have one occurrence in only one equation of numeric section of $\alpha$, then by moving in the numeric section the terms containing variables that are not leader in the right hand sides (see Property 6.1.6), we get a system with distinct left hand sides which do not occur in a right hand side of the equations of the numerical section of $\alpha$. Thus, for any instantiation of the free variables which are in the right hand sides of the numerical section by an any numeric value, we get a unique value for all the numerical leaders of the numeric equations of $\alpha$ (using axioms $5, 6, 13$). Thus for every unique value of these variables and every instantiation of the other variables in the right hand sides of the equations of the tree section of $\alpha$ (by values which respect the typing constraints of $\alpha$), there exists a unique solution for the right hand sides of the equations of the tree section (axiom 3). Note that each time we instantiate by value which respect the typing constraints; this is why we need the implication of $\alpha^*$. Let us show this in the following example: Let $x, y, z, v, w$ be variables such that $x \succ y \succ z \succ v \succ w$ we have

$$T \models num\, w \rightarrow \exists! vxzy \begin{bmatrix} x = fxyw \wedge y = x \wedge \\ 2z + 2w = 1 \wedge 3v + w = 0 \wedge \\ tree\, x \wedge num\, v \wedge num\, w \wedge \\ tree\, y \wedge num\, z \end{bmatrix}$$

Note that this property can be written in the following form by moving the leaders in the left-hand sides of the equations and the others in the right hands sides and by doing the needed changes in signs we get:

$$T \models num\, w \rightarrow \exists! vxzy \begin{bmatrix} x = fxyw \wedge y = x \wedge \\ 2z = 1 + (-2)w \wedge 3v = -w \wedge \\ tree\, x \wedge num\, v \wedge num\, w \wedge \\ tree\, y \wedge num\, z \end{bmatrix}$$

For each instantiation of $w$ by numerical value (and not with tree value) there exists one and only value for $v$ and $z$ (axioms $5, 6, 13$) and thus for $x$ too (axiom 3). The unique solutions of $v$ and $z$ are due to the fact that according to our definition of solved block, the leaders have one occurrence in only one equation of $\alpha$. If $w$ is instantiated by tree value (for example $fg0$) the formula will be false. This s why it is an implication by the typing constraints of the variables which are not leaders. In fact, we have

$$T \not\models \exists! vxzy \begin{bmatrix} x = fxyw \wedge y = x \wedge \\ z + 2w = 1 \wedge v + w = 0 \wedge \\ \mathit{tree}\, x \wedge \mathit{num}\, v \wedge \mathit{num}\, w \wedge \\ \mathit{tree}\, y \wedge \mathit{num}\, z \end{bmatrix}$$

because the unique value of $vxzy$ must be for all instantiation of $w$ and here if $w$ is typed as tree then the numerical equation will be false and then all the formula will be false. Thus we have just the implication of the unique solution.

**Property 6.1.13** *Let $\varphi$ a working formula of the form*

$$\neg^k(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_{i \in I} \phi_i)$$

*with $6 \leq k \leq 9$ and $\phi_i$ working formulas. We have*

$$T \models \neg(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_{i \in I} \phi_i^*) \leftrightarrow \neg(\alpha^p \wedge (\exists \bar{x}\, \alpha^c \wedge \bigwedge_{i \in I} \phi_i^*))$$

*with $\phi_i^*$ the normalized formula obtained from $\phi_i$ by replacing all $\neg^k$ by $\neg$.*

**Proof L** et $\psi$ - if it exists - be the immediate top-working formula of $\varphi$. Thus, $\psi$ is of the form

$$\neg^k(\exists \bar{y} \beta^c \wedge \beta^p \wedge \varphi \wedge \bigwedge_{j \in J} \phi'_j)$$

with $\phi'_j$ working formulas. According to Definition 4.1.3, since $k \geq 6$ then the normalized formula satisfies the $k$ first conditions of this definition and thus according to the sixth point of this definition we have two cases:

(1) if $\phi$ does not exists then $\alpha^p$ is the formula *true* according to the sixth condition of Definition 4.1.3. Thus the property is true.

(2) if $\phi$ exists then $\alpha^p = \beta^p \wedge \beta^c$ according to the sixth condition of Definition 4.1.3. Since the variables of $\bar{x}$ can not occur in $\beta^c \wedge \beta^p$, then these variables can not occur in $\alpha^p$, thus we can lift the formula $\alpha^p$ before the quantification $\exists \bar{x}$ and thus the property is true.

**Property 6.1.14** *Let $\alpha$ and $\beta$ two solved blocks having the same numeric-section and typing constraints. Let $\alpha_t$ and $\beta_t$ the respective tree-sections of $\alpha$ and $\beta$. If $T \models \alpha \rightarrow \beta$ and if $\alpha_t$ and $\beta_t$ have the same left hand side variables, then $T \models \alpha \leftrightarrow \beta$.*

**Proof S** ince $\alpha$ and $\beta$ have the same numeric-section and typing constraints, there exists a conjunction $\delta$ such that $\alpha = \delta \wedge \alpha_t$ and $\beta = \delta \wedge \beta_t$.

Let $\bar{x}$ be a vector of the left-hand side variables of $\alpha_t$ (then of $\beta_t$) and $X$ be the set of all variables in $\bar{x}$. Since $\alpha$ and $\beta$ are solved blocks, we have $X \cap \mathit{var}(\delta) = \emptyset$. Let $\gamma$ be the conjunction of the typing constraints of $\alpha$ (then also of $\beta$) which concern variables not belong to $\bar{x}$. Thus $\gamma$ is a sub-formula of $\delta$. By Property 6.1.12, we have $T \models \gamma \rightarrow \exists! \bar{x} \alpha_t$ and $T \models \gamma \rightarrow \exists! \bar{x} \beta_t$. Thus we have $T \models \delta \rightarrow \exists! \bar{x}\, \alpha_t$ and $T \models \delta \rightarrow \exists! \bar{x}\, \beta_t$.

Having $T \models \alpha \rightarrow \beta$, i.e $T \models \forall \bar{y} \forall \bar{x}\, \delta \wedge \alpha_t \rightarrow \delta \wedge \beta_t$, with $\bar{y}\bar{x}$ a vector of all variables in $\alpha \wedge \beta$. The following formulas are all equivalent each to other in $T$:

$$
\begin{array}{lll}
 & \forall \bar{y} \forall \bar{x}\, \delta \wedge \alpha_t \rightarrow \delta \wedge \beta_t & \\
\leftrightarrow & \forall \bar{y} \neg (\exists \bar{x}\, \delta \wedge \alpha_t \wedge \neg(\delta \wedge \beta_t)) & \\
\leftrightarrow & \forall \bar{y} \neg (\delta \wedge (\exists \bar{x}\, \alpha_t \wedge \neg \beta_t)) & \text{since } X \cap var(\delta) = \emptyset \\
\leftrightarrow & \forall \bar{y} \neg (\delta \wedge \neg(\exists \bar{x}\, \alpha_t \wedge \beta_t)) & \text{since } T \models \delta \rightarrow \exists! \bar{x}\, \alpha_t \text{ and by Property 6.1.1} \\
\leftrightarrow & \forall \bar{y} \neg (\delta \wedge \neg(\exists \bar{x}\, \beta_t \wedge \alpha_t)) & \\
\leftrightarrow & \forall \bar{y} \neg (\delta \wedge (\exists \bar{x}\, \beta_t \wedge \neg \alpha_t)) & \text{since } T \models \delta \rightarrow \exists! \bar{x}\, \beta_t \text{ and by Property 6.1.1} \\
\leftrightarrow & \forall \bar{y} \neg (\exists \bar{x}\, \delta \wedge \beta_t \wedge \neg(\delta \wedge \alpha_t)) & \\
\leftrightarrow & \forall \bar{y} \forall \bar{x}\, \delta \wedge \beta_t \rightarrow \delta \wedge \alpha_t &
\end{array}
$$

That means $T \models (\alpha \rightarrow \beta) \leftrightarrow (\beta \rightarrow \alpha)$. Since $T \models \alpha \rightarrow \beta$, we have $T \models \alpha \leftrightarrow \beta$.

## 6.2 The main proofs

**Property 3.2.1** *For all decomposed formula of the form*

$$\exists \bar{x}^1 \alpha^1 \wedge (\exists \bar{x}^2 \alpha^2 \wedge (\exists \bar{x}^3 \alpha^3 \wedge \phi))$$

*we have :* $\exists \bar{x}^1 \alpha^1 \in A^1$*,* $\alpha^2 \in A^2$*,* $\alpha^3 \in A$ *and* $T \models \forall \bar{x}^2\, \alpha^2 \rightarrow \exists! \bar{x}^3 \alpha^3$.

**Proof L** et $\exists \bar{x}^1 \alpha^1 \wedge (\exists \bar{x}^2 \alpha^2 \wedge (\exists \bar{x}^3 \alpha^3 \wedge \phi))$ be the decomposed formula of $\exists \bar{x} \alpha \wedge \phi$. By construction of the set $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$, $\alpha^1$, $\alpha^2$ and $\alpha^3$ using Definition 3.2, it is clear that $\exists \bar{x}^1 \alpha^1 \in A^1$ and $\alpha^2 \in A^2$. Let us show that $\forall \bar{x}^2 \alpha^2 \rightarrow \exists! \bar{x}^3 \alpha^3$. Since $\alpha^2$ contains at least the typing constraints of all the variables of $\alpha$ which are not in $\bar{x}^3$ and since $\bar{x}^3$ contains the leaders of the equations of $\alpha^3$ then using Property 6.1.12 we have $T \models \alpha^2 \rightarrow \exists! \bar{x}^3 \alpha^3$ i.e. $\forall \bar{x}^2\, \alpha^2 \rightarrow \exists! \bar{x}^3 \alpha^3$

**Property 4.1.2** Every formula is equivalent in $T$ to a normalized formula.

**Proof I** t is easy to transform any first-order formula to a normalized formula equivalent in $T$. It is enough[1] for example: (1) to introduce a supplement of equations and existentially quantified variables to transform the conjunctions of equations to conjunctions of flat atomic formulas, (2) to express all the quantifier, constants and logical connectors with $\neg$, $\wedge$ and $\exists$, (3) to remove the double negations i.e. $\neg\neg\varphi$ becomes $\varphi$, (4) to replace each sub-formula of the form $\neg num\, x$ by $tree\, x$ and $\neg tree\, x$ by $num\, x$, (5) if the formula $\varphi$ obtained does not start with $\neg$, we replace it by $\neg\neg\varphi$, (6) to name the quantified variables by distinct names different from the names of the free variables, and as different as possible, (7) to put the quantifier before the conjunction i.e. $\varphi \wedge (\exists \bar{x}\, \psi)$ becomes $(\exists \bar{x}\, \varphi \wedge \psi)$ because the free variables of $\varphi$ are distinct from those of $\bar{x}$. If the

[1] To make the notations easier in these transformations, we will remove the quantification $\exists \bar{x}$ if $\bar{x}$ is the mepty vector and put an empty conjunction rather than the formula *true* i.e. $\neg\neg\exists \bar{x}\, \alpha$ is the normalized formula $\neg(\exists \varepsilon\, true \wedge \neg(\exists \bar{x}\, \alpha))$.

starting formula does not contain the connector $\leftrightarrow$ then this transformation will be linear i.e. there exists a constant $k$ such that $n_2 \leq kn_1$, where $n_1$ is the size of the starting formula and $n_2$ the size of the normalized formula.

**Property 4.1.5** *Let $\varphi$ be a general solved formula of the form. If $\varphi$ has no free variables then $\varphi$ is the formula true, otherwise neither $T \models \varphi$ nor $T \models \neg\varphi$.*

**Proof L** et $\varphi$ be a general solved formula of the form

$$\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2 \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}_i^1\, \beta_i^1), \tag{10}$$

where $\exists \bar{x}^1\, \alpha^1 \in A^1$, $\alpha^2 \in A^2$, $\exists \bar{y}_i^1\, \beta_i^1 \in A^1$, all the $\alpha^1 \wedge \alpha^2 \wedge \beta_i^1$ are solved blocks and all the $\beta_i^1$ are different from *true*. Two cases arise:

Case 1: Let us show that if $\varphi$ has no free variables then $\varphi$ is the formula *true*. Since $\varphi$ has no free variables then $\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2$ has no free variables. Since $\exists \bar{x}^1\, \alpha^1 \in A^1$ and has not free variables then according to Property 6.1.11 the formula (10) is equivalent in $T$ to the following formula without free variables

$$\alpha^2 \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}_i^1\, \beta_i^1), \tag{11}$$

Since $\alpha^2 \in A^2$ and $\alpha^2$ has no free variables then according to the definition of the set $A^2$ we have $\alpha^2 = true$. Thus the precedent formula is equivalent in $T$ to the following formula without free variables

$$\bigwedge_{i \in I} \neg(\exists \bar{y}_i^1\, \beta_i^1), \tag{12}$$

Since $\exists \bar{y}_i^1\, \beta_i^1 \in A^1$ and has not free variables then using Property 6.1.11 we deduce that $\exists \bar{y}_i^1 \beta_i^1 = \exists \varepsilon\, true$. But according to the condition of the formula (10) all the formulas $\beta_i^1$ are different from *true* and thus $I$ must be the empty set. Thus the precedent formula is equivalent to *true* in $T$. (recall that $\bigwedge_{i \in I} \varphi_i = \varphi_1 \wedge ... \wedge \varphi_n \wedge true$ thus $\bigwedge_{i \in \emptyset} \varphi_i = true$).

Case 2: If $\varphi$ has at least a free variables then let us show that there exists a model $M$ of $T$ and two distinct instantiations $\varphi'$ and $\varphi''$ of $\varphi$ by individuals of $M$ such that

$$M \models \neg\varphi' \quad \textit{and} \quad M \models \varphi''$$

Let us choice $M$ be the model of finite or infinite trees labeled by $Q \cup F$ and where all the sub-trees labeled by $Q \cup \{+, -\}$ are evaluated in $Q$ and reduced to a leaf labeled by $Q$. (more details of this model can be found on [10].)

(1) Let us show that $\varphi'$ exists. Let $z$ be a free variable of $\varphi$:

- If $z$ occurs in the formula $\alpha^1 \wedge \alpha^2$ then since $\exists \bar{x}^1\, \alpha^1 \in A^1$ and $\alpha^2 \in A^2$, the formulas $\alpha^1$ and $\alpha^2$ are solved blocks, thus all the variables are typed, thus $num\, z$ or $\neg num\, z$ is a sub formula of $\alpha^1 \wedge \alpha^2$. To make false $\varphi'$ it is enough to instantiate the free variable $z$ by an element of $Q$ if $\neg num\, z$ is a sub formula of $\alpha^1 \wedge \alpha^2$ ; and by $h$ if $num\, z$ is a sub formula of $\alpha^1 \wedge \alpha^2$ with $h \in F - \{0, 1\}$ a 0-ary function symbol i.e. a tree constant. By this instantiation $\varphi'$, we make a contradiction in the typing of $z$, thus $M \models \neg\varphi'$.

- Else, there exists $k \in I$ such that the formula $\exists \bar{y}_k^1\, \beta_k^1$ with $k \in I$ has at least a free variable. Since $\exists \bar{y}_k^1\, \beta_k^1 \in A^1$, then $\beta_k^1$ is a solved block then according to Property 6.1.10 there exists an instantiation $\exists \bar{y}_k^1\, \beta_k'^1$ of the free variables of $\exists \bar{y}_k^1\, \beta_k^1$ such that $M \models \exists \bar{y}_k^1\, \beta_k'^1$ thus $M \models \neg(\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2 \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}_i^1\, \beta_i^1))$, thus $M \models \neg\varphi'$.

(2) Let us show now that there exists $\varphi''$ such that $M \models \varphi''$. The formula $\varphi$ is of the form

$$\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2 \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}_i^1\, \beta_i^1), \tag{13}$$

where $\exists \bar{x}^1\, \alpha^1 \in A^1$, $\alpha^2 \in A^2$, $\exists \bar{y}_i^1\, \beta_i^1 \in A^1$, all the $\alpha^1 \wedge \alpha^2 \wedge \beta_i^1$ are solved blocks and all the $\beta_i^1$ are different from *true*.
Let $\alpha^{2*}$ the formula $\alpha^2$ in which we remove the typing constraints which concern the leaders of the equations of $\alpha^1$. Let us also transform the equations of the numeric section of $\alpha^1$ and $\beta_i^1$ by moving to the right hand side the terms containing variables that are not leaders (see Property 6.1.6). The precedent formula is equivalent in $T$ to

$$\exists \bar{x}^1\, \alpha^1 \wedge \alpha^{2*} \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}_i^1\, \beta_i^1), \tag{14}$$

where the equations of the numeric section of $\alpha^1$ (respectively $\beta_i^1$) have distinct left hand sides that have no occurrences in other right hand sides of equations of the numeric section of $\alpha^1$ (respectively $\beta_i^1$). This is due to the fact that $\exists \bar{x}^1 \alpha^1 \in A^1$ and $\exists \bar{y}_i^1 \beta_i^1 \in A^1$ thus $\alpha^1$ and $\beta_i^1$ are solved. Since $\exists \bar{y}_i^1\, \beta_i^1 \in A^1$ then $\beta_i^1$ is a solved block, thus it is consistent and different from *false*. Moreover since $\beta_i^1$ are different from *true* then each $\beta_i^1$ has at least a variable. According to the definition of $A^1$ all the variable of $\bar{y}_i^1$ are reachable and thus there exists at least a free variable in each $\beta_i^1$ according to the definition of the reachable variables. Since $\alpha^1 \wedge \alpha^{2*} \wedge \beta_i^1$ are solved blocks then they are consistent and thus there exists an instantiation of $\exists \bar{x}^1\, \alpha^1 \wedge \alpha^{2*}$ such that this instantiated formula is true in $M$ (Property 6.1.10), thus according to Property 6.1.8 there exists an infinite instantiations of the variables of $\alpha^{2*}$ which make it true in $M$ (and not zero because there exists at least an instantiation since the blocks are solved). For each value of these instantiations and for all instantiation of right hand sides of the equations of the numeric section of $\alpha^1$, there exists a value for the leaders of these equations because the leaders of the equations of $\alpha^1$ do not occur in $\alpha^{2*}$ ( $\alpha^1 \wedge \alpha^2 \wedge \beta_i^1$ are solved blocks). For each of these values and instantiations of the variables of the equations of the tree section of $\alpha^1$ which are not leaders, there exists a value for the leaders of these equations (axiom 3). Then there exists an infinite instantiations of the free variables of $\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2$ which make the instantiated formula true in $M$. Let us show now that there exists from this infinite instantiation, an instantiation which makes false each formula of the form $\exists \bar{y}_i^1 \beta_i^1$ and thus make true $\varphi''$. In each sub-formula of the form $\exists \bar{y}_i^1 \beta_i^1$ the leaders of the equations of the numeric section of $\beta_i^1$ do not occur in the equations and inequations of $\alpha^1 \wedge \alpha^2$ because $\alpha^1 \wedge \alpha^2 \wedge \beta_i^1$ are solved blocks. Since for each instantiation of the right hand sides of the equations of the numeric section of $\beta_i^1$ there exists a value for the leaders. Thus it is enough to choose a different value to these leaders to make false all the $\exists\bar{y}_i^1\beta_i^1$. This is possible because the domain $M$ is infinite and more exactly $Q$ is infinite. For each instantiation of the variables which are not leaders in the tree section of $\beta_i^1$ there exists a unique value for the leaders thus it is enough to take another value to make false all the $\exists\bar{y}_i^1\beta_i^1$. This is possible because the domain of the trees is infinite and more exactly the set of the function symbols of $F$ is infinite. Thus there exists an instantiation which make true $\exists\bar{x}^1\alpha^1 \wedge \alpha^2$ and false each sub-formula of the form $\exists\bar{y}_i^1\beta_i^1$. Thus this instantiation is the formula $\varphi''$.

**Property 4.1.6** *Every general solved formula is equivalent in $T$ to a boolean combination of formulas of the form $\exists\bar{x}^1\alpha^1 \wedge \alpha^2$, with $\exists\bar{x}^1 \in A^1$ and $\alpha^2 \in A^2$, which do not accept elimination of quantifiers.*

**Proof E** ach general solved formula

$$\bigvee_{i\in I}(\exists\bar{x}_i\alpha_i^c \wedge \bigwedge_{j\in J_i}\neg(\exists\bar{y}_{ij}\beta_{ij}^c)) \tag{15}$$

where the $\beta_{ij}^c$ are different from *true*, is extracted from a final working formula

$$\neg^7(\exists\epsilon true \bigwedge_{i\in I}\neg^8(\exists\bar{x}_i\alpha_i^c \wedge \alpha_i^p \wedge \bigwedge_{j\in J_i}\neg^9(\exists\bar{y}_{ij}\beta_{ij}^c \wedge \beta_{ij}^p))$$

By the conditions of $\neg^8$, we have $\alpha_i^p = true$ and all the variables of $\bar{x}_i$ are reachable in $\exists\bar{x}_i\alpha_i^c$. Also by these conditions, $\exists\bar{x}_i\alpha_i^c$ is decomposed in $T$ into $\exists\bar{x}_i\alpha c1_i \wedge \alpha_i^{c2}$, with $\exists\bar{x}_i\alpha c1_i \in A^1$ and $\alpha_i^{c2} \in A^2$.

By the conditions of $\neg^9$, we have $\beta_{ij}^p = \alpha_i^c \wedge \alpha_i^p = \alpha_i^c$, $\beta_{ij}^c \wedge \beta_{ij}^p$ are solved blocks and $\exists\bar{y}_{ij}\beta_{ij}^c$ is in $A^1$. We deduce then $\beta_{ij}^c \wedge \alpha_i^c$ are solved blocks. Since each variable in $\bar{x}_i$ is reachable in $\exists\bar{x}_i\alpha_i^c$, it remains reachable in $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c \wedge \beta_{ij}^c$. Since each variable $y$ in $\bar{y}_{ij}$ is reachable in $\exists\bar{y}_{ij}\beta_{ij}^c$, there exists two possible cases: (1) $y$ is reachable without using variables in $\bar{x}_i$, in this case, $y$ remains reachable in $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c \wedge \beta_{ij}^c$, and (2) $y$ is reachable using variables in $\bar{x}_i$, in this case, since all variables in $\bar{x}_i$ are reachable in $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c \wedge \beta_{ij}^c$, $y$ is still reachable in this formula.

In consequence, the formulas $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c\wedge\beta_{ij}^c$ can be decomposed into $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^{c1}\wedge \beta_{ij}^c \wedge \alpha_i^{c2}$, with $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^{c1} \wedge \beta_{ij}^c \in A^1$ and $\alpha_i^{c2} \in A^2$.

According to Property 6.1.5, the formula (15) is equivalent in $T$ to the formula

$$\bigvee_{i\in I}((\exists\bar{x}_i\alpha_i^c) \wedge \bigwedge_{j\in J_i}\neg(\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c \wedge \beta_{ij}^c))$$

We have proved that each quantified conjunction is of the form $\exists\bar{x}^1\alpha^1 \wedge \alpha^2$ where $\bar{x}^1\alpha^1 \in A^1$ and $\alpha^2 \in A^2$. This property is then proved.

**Property 4.3.1***Every repeated application of the precedent rewriting rules on an inital working formula terminates and produces a final working formula equivalent to the inital formula in $T$ and without new free variables.*

**Proof T** he proof is divided into three main sub-proofs: (1) every repeated application of the rules on an initial working formula terminates, (2) all the rules are correct, i.e. in each rule, the left-hand side formula is equivalent in $T$ to the right-hand side one and (3) at the end of the rules' application, the eventual formula is in the final working form.

**Every application of the rules terminates** Let us observe that the rules 1 to 7 are applied on a sub-working formula with $\neg^1$ and do not change the negation symbols, as well as the rules 8 to 14 are applied on a sub-working formula with $\neg^4$. Thus we can divide the proof of termination into three parts: (1) every application of the rules 1 to 7 on a sub-working formula with $\neg^1$ terminates, (2) every application of the rules 8 to 14 on a sub-working formula with $\neg^4$ terminates and (3) having considered the terminations of the rules 1 to 7 and 8 to 14, every application of the rules 15 to 28 terminates.

Let us prove that every application of the rules 1 to 7 on a sub-working formula with $\neg^1(\exists\bar{x}\alpha\wedge\varphi)$ terminates. Since the variables of $V$ are ordered by a total strict order $\succ$, we can associate to each variable $x$ an integer $no(x)$, such that $x \succ y$ if and only if $no(x) > no(y)$. We consider three following integers:

- $n_1$ the number of equations of the form $x = fy_1...y_n$ in $\alpha$,
- $n_2$ the sum of $no(x)$ for all occurrences of all variables $x$ in $\alpha$,
- $n_3$ the number of equations of the form $y = x$ with $x \succ y$ in $\alpha$.

We will show that each application of each rule decreases an $n_i$ but keeps all other $n_j$ with $j < i$ unchanged. Indeed, $n_1$ is decreased by the rule 2 and 5, $n_2$ is decreased by the rules 1, 3, 6 and 7, and $n_3$ is decreased by the rule 4. Since $n_1$, $n_2$ and $n_3$ are integers, they cannot be decreased infinitely. Thus any application of the rules 1 to 7 terminates.

Let us prove that every application of the rules 8 to 14 on a sub-working formula with $\neg^4$ terminates. This is quite evident, since the rules 8 to 12 put the equations or inequations into the right form, and the rules 13 and 14 remove a double occurrence of the variable $x_k$ by setting its coefficient to zero.

Having considered the terminations of the rules 1 to 7 and 8 to 14, let us prove now that every application of the rules 15 to 28 terminates. Let us prove that any application of the rules in the first step terminates. This step concerns the rules 15 to 24. We can see that starting with an initial working formula $\neg^6(\exists\varepsilon \mathit{true} \wedge \varphi)$, where $\varphi$ is a conjunction of working formula where all the negation symbols are $\neg^0$, the rule 24 is the only one can be applied. With the application of 24, the first negation symbol is changed to $\neg^7$ and the next level ones are changed to $\neg^1$. On the sub-working formula with $\neg^1$, any application of the rules 1 to 7 terminates, then the rule 15 changes $\neg^1$ to $\neg^2$. We can see that on each sub-working formula $\neg^2$, the rules 16, 17 and 18 can be applied at most once for each variable in $\alpha^c \wedge \alpha^p$. The application of these rules resets a sub-working formula with $\neg^2$ to $\neg^1$, but an application of the rules 1 to 18 cannot be infinite, since no new variable is added. Any application of the rules 19 to 24 terminates also since each rule can be applied only once on each sub-working formula. We then proved that the first step on a sub-working formula always terminates.

In the second step, the rules 25 and 27 can be applied only once on a sub-working formula. The termination of this step thus depends on the rules 26 and 28, which expand the length of working formulas. Concerning the rules 26,

$$\neg^7 \left[ \begin{array}{l} \exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8(\exists\bar{y}\beta^c \wedge \beta^p) \end{array} \right] \Longrightarrow \left[ \begin{array}{l} \neg^7(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists\bar{y}\beta^{c1} \wedge \beta^p)) \wedge \\ \bigwedge_{i\in I} \neg^1(\exists\bar{x}\bar{y}\beta^p \wedge \beta^{c1} \wedge \beta_i^{c2*} \wedge \varphi_0) \end{array} \right]$$

we replace a sub-working formula containing a sequence $\neg^7\neg^8$ by the same working formula but with a sequence $\neg^7\neg^9$ and by $|I|$ working formulas ($|I|$ is finite as shown in the description of the rule). On each working formula of the right-hand side formula, the conjunction $\varphi_0$ contains one less element than the left-hand side formula of the rule. It is true that the working formulas in this $\bigwedge_{i\in I}$ are became $\neg^1$, that is the first step must be redone on these ones, but we have proved that the first step always terminates. Thus on one sub-working formula, the rule 26 cannot be applied infinitely.

With the same reasoning, in the rule 28

$$\neg^7 \left[ \begin{array}{l} \exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8 \left[ \begin{array}{l} \exists\bar{y}\, \beta^c \wedge \beta^p \wedge \\ \bigwedge\limits_{i\in I} \neg^9(\exists\bar{z}_i\, \gamma_i^c \wedge \gamma_i^p) \end{array} \right] \end{array} \right] \Longrightarrow \left[ \begin{array}{l} \neg^7(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^8(\exists\bar{y}\, \beta^c \wedge \beta^p)) \wedge \\ \bigwedge\limits_{i\in I} \neg^6(\exists\bar{x}\bar{y}\bar{z}_i\, \delta_i^c \wedge \delta_i^p \wedge \varphi_0) \end{array} \right]$$

we replace a sub-working formula containing a sequence $\neg^7\neg^8\neg^9$ by the same but with a sequence $\neg^7\neg^8$ and by $|I|$ (finite) sub-working formulas. On each sub-working formula of the right-hand side formula, the conjunction $\varphi_0$ contains one less element than the left-hand side formula of the rule. On the sub-working formulas of $\bigwedge_{i\in I}$, the first step must be redone but it terminates. The rules 28 can be reapplied, but after a while, the left-hand side formula will have the depth less than or equal to 2, so that the 28 cannot be applied anymore.

We can see that even the rules 26 and 28, while being applied on a sub-working formula, expand the length of the formula, but after an finite number of applications, the working formula will have the depth less than or equal to 2. Thus these rules cannot be applied infinitely. We then proved that the second step terminates, thus any application of the rules terminates.

**The rules are correct** The rules 1..12 are evident in $T$ and deduced directly from the axiomatization of $T$. The rule 13 and 14 are the properties of the theory of the rational numbers section of the axiomatization of $T$. These two properties are well known we will not prove its again. The other rules are much less obvious and need some formal proofs.

In the rule 15, since the tree-section of $\alpha^c \wedge \alpha^p$ is formatted and there is no sub-formula in $\alpha^c \wedge \alpha^p$ of the form $\mathit{num}\, x \wedge \mathit{tree}\, x$, the symbol $\neg^1$ can be changed to $\neg^2$.

In the rule 16, since $\mathit{num}\, z$ is a consequence of $\alpha^c \wedge \alpha^p$, we have $\alpha^c \wedge \alpha^p$ is equivalent to $\alpha^c \wedge \alpha^p \wedge \mathit{num}\, z$. This rule is then correct. In the same way, we prove that the rules 17 and 18 are correct.

In the rule 20, Since the numeric-section of $\alpha^c \wedge \alpha^p$ is inconsistent then $\alpha^c \wedge \alpha^p$ is equivalent to *false*, this rule is then correct.

The rules 19, 21 and 22 are correct because their conditions ensure that the number of the first negation symbol can be changed.

In the rule 27

$$\neg^7(\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists\varepsilon\, true \wedge \beta^p)) \Longrightarrow true$$

by the definition of $\neg^9$, $\beta^p$ is the formula $\alpha^c \wedge \alpha^p$. This rule is evidently correct.

*Correction of rule 23:* description:

$$\neg^7\left[\begin{array}{l}\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \varphi\wedge \\ \neg^5(\exists\bar{y}\,\beta^c \wedge \beta^p \wedge \psi)\end{array}\right] \Longrightarrow \neg^7\left[\begin{array}{l}\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \varphi\wedge \\ \neg^6(\exists\bar{y}\,\gamma^c \wedge \gamma^p \wedge \psi)\end{array}\right]$$

$\gamma^c$ is obtained from $\beta^c$ as follows: for all variable $x \in var(\beta^c)$, we add all the relations $num\, x$ or $tree\, x$ which are in $\beta^p$ but not in $\beta^c$, and for all the variables $y$ which do not occur in an equation or inequation of $\beta^c$ we remove all relation $num\, y$ or $tree\, y$ which are both in $\beta^c$ and $\beta^p$. The formula $\gamma^p$ is the formula $\alpha^p \wedge \alpha^c$.

Proof: Since no typing constraint is removed from $\beta^p$, we have $\beta^c \wedge \beta^p$ is equivalent to $\gamma^c \wedge \beta^p$. Let $\beta^p_t$ the tree-section and $\beta^p_n$ the numeric-section of $\beta^p$. Let $\alpha^{cp}_t$ the tree-section and $\alpha^{cp}_n$ the numeric-section of $\alpha^c \wedge \alpha^p$. By the conditions of $\neg^5$, $\alpha^{cp}_t$ and $\beta^p_t$ have the same left hand side variables. Let us observe that by the propagation and since the rule 14 does not change the numeric-section of $\beta^p$, we have $\alpha^{cp}_n = \beta^p_n$. According to the properties of $\neg^5$, we have $T \models \beta^c \wedge \beta^p \rightarrow \alpha^c \wedge \alpha^p$. We have then

$$T \models \gamma^c \wedge \beta^p \rightarrow \gamma^c \wedge \alpha^c \wedge \alpha^p$$

and

$$T \models \gamma^c \wedge \beta^p_t \wedge \beta^p_n \rightarrow \gamma^c \wedge \alpha^{cp}_t \wedge \alpha^{cp}_n$$

and

$$T \models \gamma^c \wedge \beta^p_t \wedge \alpha^{cp}_n \rightarrow \gamma^c \wedge \alpha^{cp}_t \wedge \alpha^{cp}_n$$

Since the tree-section of $\gamma^c \wedge \beta^p_t$ and the tree-section of $\gamma^c \wedge \alpha^{cp}_t$ have the same left hand side variable, according to Property 6.1.14, we have

$$T \models \gamma^c \wedge \beta^p_t \wedge \alpha^{cp}_n \leftrightarrow \gamma^c \wedge \alpha^{cp}_t \wedge \alpha^{cp}_n$$

then

$$T \models \beta^c \wedge \beta^p \leftrightarrow \gamma^c \wedge \alpha^c \wedge \alpha^p$$

Since $\gamma^p = \alpha^c \wedge \alpha^p$, the rule 23's correction is then proved.

*Correction of rule 24:* description:

$$\neg^6\left[\begin{array}{l}\exists\bar{x}\,\alpha^c \wedge \alpha^p\wedge \\ \bigwedge_i \neg^0(\exists\bar{y}_i\beta^c_i \wedge \beta^p_i \wedge \varphi_i)\end{array}\right] \Longrightarrow \neg^7\left[\begin{array}{l}\exists\bar{x}^1\bar{x}^2\,\alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p\wedge \\ \bigwedge_i \neg^1(\exists\bar{y}_i\bar{x}^3\gamma^c_i \wedge \gamma^p_i \wedge \varphi_i)\end{array}\right]$$

with $\gamma^c_i = \beta^c_i \wedge \alpha^{c3}$ and $\gamma^p_i = \beta^p_i \wedge \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p$.

Proof: According to Definition 4.1.3 of the working formula, since we have $\neg^6$ then the condition 6 of Definition 4.1.3 holds then $\beta_i^p = \alpha^c \wedge \alpha^p$. Thus the left-hand side of this rule is equivalent in $T$ to

$$\neg(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_i \neg(\alpha^c \wedge \alpha^p \wedge (\exists \bar{y}_i \beta_i^c \wedge \varphi_i)))$$

Which is equivalent in $T$ after distribution and simplification to

$$\neg(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_i \neg(\exists \bar{y}_i \beta_i^c \wedge \varphi_i))$$

According to Property 6.1.13 the precedent formula is equivalent in $T$ to

$$\neg(\alpha^p \wedge (\exists \bar{x}\, \alpha^c \wedge \bigwedge_i \neg(\exists \bar{y}_i \beta_i^c \wedge \varphi_i)))$$

According to Definition 4.1.3 of the working formula, since we have $\neg^6$ then the conditions 4,5,6 of Definition 4.1.3 hold then the formula $\exists \bar{x} \alpha^c$ can be decomposed in $T$ according to Definition 3.2 and Property 3.2 . Thus the precedent formula is equivalent in $T$ to

$$\neg(\alpha^p \wedge (\exists \bar{x}^1 \alpha^{c1} \wedge (\exists \bar{x}^2 \alpha^{c2} \wedge (\exists \bar{x}^3 \alpha^{c3} \wedge \bigwedge_i \neg(\exists \bar{y}_i \beta_i^c \wedge \varphi_i)))))$$

with $T \models \forall \bar{x}^2 \alpha^{c2} \rightarrow \exists! \bar{x}^3 \alpha^{c3}$. According to Property 6.1.1, the precedent formula is equivalent in $T$ to

$$\neg(\alpha^p \wedge (\exists \bar{x}^1 \alpha^{c1} \wedge (\exists \bar{x}^2 \alpha^{c2} \wedge \bigwedge_i \neg(\exists \bar{x}^3 \bar{y}_i\, \alpha^{c3} \wedge \beta_i^c \wedge \varphi_i))))$$

which is equivalent in $T$ to

$$\neg(\exists \bar{x}^1 \bar{x}^2\, \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p \wedge \bigwedge_i \neg(\exists \bar{x}^3 \bar{y}_i\, \alpha^{c3} \wedge \beta_i^c \wedge \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p \wedge \varphi_i))))$$

Thus this rule is then correct and the $\neg^k$ are correct.

*Correction of rule 25* description:

$$\neg^7 \left[ \begin{array}{l} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \\ \bigwedge_{i \in I} \neg^9 (\exists \bar{y}_i \beta_i^c \wedge \beta_i^p) \end{array} \right] \Longrightarrow \neg^8 \left[ \begin{array}{l} \exists \bar{x}^1 \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p \wedge \\ \bigwedge_{i \in I'} \neg^9 (\exists \bar{y}_i \beta_i^c \wedge \beta_i^{p*}) \end{array} \right]$$

where $I'$ is the set of $i \in I$ such that $\beta_i^c$ does not contain occurrences of any variables in $\bar{x}^2$. The formula $\alpha^{c2*}$ is such that $T \models (\exists \bar{x}^2 \alpha^{c2}) \leftrightarrow \alpha^{c2*}$ and is computed using the Fourier quantifier elimination. The propagated-constraint section $\beta_i^{p*} = \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p$.

Proof: According to Definition 4.1.3 of the working formula, since we have $\neg^9$ then the condition 6 of Definition 4.1.3 holds then $\beta_i^p = \alpha^c \wedge \alpha^p$. Thus the left-hand side of this rule is equivalent in $T$ to

$$\neg(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_i \neg(\alpha^c \wedge \alpha^p \wedge (\exists \bar{y}_i \beta_i^c \wedge \varphi_i)))$$

Which is equivalent in $T$ after distribution and simplification to

$$\neg(\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \bigwedge_i \neg(\exists\bar{y}_i\beta_i^c \wedge \varphi_i))$$

According to Property 6.1.13 the precedent formula is equivalent in $T$ to

$$\neg(\alpha^p \wedge (\exists\bar{x}\,\alpha^c \wedge \bigwedge_i \neg(\exists\bar{y}_i\beta_i^c \wedge \varphi_i)))$$

According to Definition 4.1.3 of the working formula, since we have $\neg^7$ then the conditions 4,5,6,7 of Definition 4.1.3 hold then the formula $\exists\bar{x}\alpha^c$ can be decomposed in $T$ according to Definition 3.2, Property 3.2 and $\exists\bar{x}^3\alpha^{c3} = \exists\varepsilon\, true$. Thus the precedent formula is equivalent in $T$ to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge (\exists\bar{x}^2\alpha^{c2} \wedge \bigwedge_i \neg(\exists\bar{y}_i\beta_i^c \wedge \varphi_i))))$$

Let us denote by $I_1$, the set of the $i \in I$ such that $x_n^2$ does not have free occurrences in the formula $\exists\bar{y}_i^1\beta_i^{c1}$, thus, the preceding formula is equivalent in $T$ to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge \alpha^p \wedge (\exists x_1^2...\exists x_{n-1}^2 \begin{bmatrix} (\bigwedge_{i\in I_1} \neg(\exists\bar{y}_i^1\beta_i^{c1}))\wedge \\ (\exists x_n^2\,\alpha^{c2} \wedge \bigwedge_{i\in I-I_1} \neg(\exists\bar{y}_i^{c1}\beta_i^{c1})) \end{bmatrix}))). \quad (16)$$

According to Property 6.1.4, Property 6.1.7 and Property 6.1.8, the formula (16) is equivalent in $T$ to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^1 \wedge \alpha^p \wedge (\exists x_1^2...\exists x_{n-1}^2 \begin{bmatrix} (\bigwedge_{i\in I_1} \neg(\exists\bar{y}_i^1\beta_i^{c1}))\wedge \\ (\exists x_n^2\,\alpha^{c2}) \end{bmatrix}))). \quad (17)$$

According to Property 6.1.9 there exists $\alpha_n^{c2} \in A^2$ such that $T \models (\exists x_n^2\,\alpha^{c2}) \leftrightarrow \alpha_n^{c2}$ thus the preceding formula is equivalent in $T$ to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge \alpha^p \wedge (\exists x_1^2...\exists x_{n-1}^2((\bigwedge_{i\in I_1} \neg(\exists\bar{y}_i^1\beta_i^{c1})) \wedge \alpha_n^{c2})))). \quad (18)$$

thus to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge \alpha^p \wedge (\exists x_1^2...\exists x_{n-1}^2\,\alpha_n^{c2} \wedge \bigwedge_{i\in I_1} \neg(\exists\bar{y}_i^1\beta_i^{c1})))). \quad (19)$$

By repeating the four preceding steps $(n-1)$ times and by denoting by $I_k$ the set of the $i \in I_{k-1}$ such that $x_{(n-k+1)}^2$ does not have free occurrences in $\exists\bar{y}_i^1\beta_i^{c1}$, the preceding formula is equivalent in $T$ to the following wnfv formula

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge \alpha_1^{c2} \wedge \alpha^p \wedge \bigwedge_{i\in I_n} \neg(\exists\bar{y}_i^1\beta_i^{c1}))),$$

which is equivalent in $T$ to

$$\neg(\exists\bar{x}^1\alpha^{c1} \wedge \alpha_1^{c2} \wedge \alpha^p \wedge \bigwedge_{i\in I_n} \neg(\exists\bar{y}_i^1\beta_i^{c1} \wedge \alpha^{c1} \wedge \alpha_1^{c2} \wedge \alpha^p)),$$

Thus the rule 25 is correct in $T$.

*Correction of rule 26* description:

$$\neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8(\exists \bar{y} \beta^c \wedge \beta^p) \end{bmatrix} \Longrightarrow \begin{bmatrix} \neg^7(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists \bar{y} \beta^{c1} \wedge \beta^p)) \wedge \\ \bigwedge_{i \in I} \neg^1(\exists \bar{x} \bar{y} \beta^p \wedge \beta^{c1} \wedge \beta_i^{c2*} \wedge \varphi_0) \end{bmatrix}$$

$\varphi$ is such that every negation symbol $\neg^k$ has $k \geq 6$, $\varphi_0$ is obtained from $\varphi$ by replacing all occurrences of $\neg^k$ by $\neg^0$ and all propagated-constraint sections by *true*. Let $\beta^2$ the formula obtained from $\beta^{c2}$ by removing the multiple occurrences of typing constraints and for all the variables $y$ which do not occur in an inequation of $\beta^{c2}$ we remove all relation $num\, y$ or $tree\, y$ which are both in $\beta^{c1}$ and $\beta^{c2}$. If $\beta^2$ is the formula *true* then $I = \emptyset$, otherwise the $\beta_i^{c2*}$ with $i \in I$ are obtained from $\beta^2$ as follows: Since $\beta^2 \in A^2$ then it is of the form.

$$\begin{bmatrix} (\bigwedge_{\ell \in L} num\ z_\ell) \wedge (\bigwedge_{k \in K} tree\ v_k) \wedge \\ ((\bigwedge_{j \in J} \sum_{i=1}^n a_{ij} x_i < a_{0j}) \wedge \bigwedge_{m=1}^n num\ x_m) \end{bmatrix},$$

thus $\neg\beta^2$ is of the form

$$\begin{bmatrix} (\bigvee_{\ell \in L} tree\ \ z_\ell) \vee (\bigvee_{k \in K} num\ v_k) \vee (\bigvee_{m=1}^n tree\ x_m) \vee \\ \bigvee_{j \in J}((\sum_{i=1}^n a_{ij} x_i = a_{0j} 1 \wedge \bigwedge_{m=1}^n num\ x_m) \vee \\ (\sum_{i=1}^n (-a_{ij}) x_i < (-a_{0j}) 1 \wedge \bigwedge_{m=1}^n num\ x_m)) \end{bmatrix}$$

Each element of this disjunction is a block and represents a formula $\beta_i^{c2*}$. Of course we have $T \models (\neg\beta^2) \leftrightarrow \bigvee_i \beta_i^{c2*}$.

Proof: By the definition of $\neg^8$, $\exists \bar{y} \beta^c$ is decomposable into $\exists \bar{y} \beta^{c1} \wedge \beta^{c2}$ with $\exists \bar{y} \beta^{c1} \in A^1$. By the description of the rule, let $\beta^2$ be the formula obtained from $\beta^{c2}$ by removing the multiple occurrences of typing constraints and for all the variables $y$ which do not occur in an inequation of $\beta^{c2}$ we remove all relation $num\, y$ or $tree\, y$ which are both in $\beta^{c1}$ and $\beta^{c2}$. The left-hand side formula of the rule is equivalent to the formula

$$\neg(\exists \bar{x} \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists \bar{y} \beta^p \wedge \beta^{c1} \wedge \beta^2))$$

According to the definition of $A^1$, to Property 6.1.5, we have $T \models \exists ? \bar{y} \beta^{c1}$ then $T \models \exists ? \bar{y} \beta^{c1} \wedge \beta^p$. According to Property 6.1.2, the precedent formula is equivalent to

$$\neg(\exists \bar{x} \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists \bar{y} \beta^p \wedge \beta^{c1})) \vee \neg(\exists \bar{x} \bar{y} \beta^p \wedge \beta^{c1} \wedge \neg\beta^2 \wedge \varphi)$$

The formula $\neg\beta^2$ is equivalent to a disjunction $\bigvee_{i \in I} \beta_i^{c2*}$ (see descriptions of this rule). By making a distribution of $\bigvee$ over $\wedge$, the precedent formula is equivalent to

$$\neg(\exists \bar{x} \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists \bar{y} \beta^p \wedge \beta^{c1})) \wedge \bigwedge_{i \in I} \neg(\exists \bar{x} \bar{y} \beta^p \wedge \beta^{c1} \wedge \beta_i^{c2*} \wedge \varphi)$$

The sub-formula $\neg(\exists \bar{y} \beta^p \wedge \beta^{c1})$ verifies the conditions of $\neg^9$. Let us consider the occurrences of $\varphi$ inside $\bigwedge_i$. Since in $\varphi$, each negation symbol $\neg^k$ has $k \geq 6$, by applying Property 6.1.13 from the most nested sub-working formulas of $\varphi$ to $\varphi$ itself, all the propagated-constraint sections can be removed and replaced by *true*. All symbol $\neg^k$ in theses occurrences of $\varphi$ can be then replaced by $\neg^0$. This formula is then the right-hand side formula of the rule. The correction of this rule is proved.

*Correction of rule 28* description:

$$\neg^7 \left[ \begin{array}{l} \exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8 \left[ \begin{array}{l} \exists\bar{y}\, \beta^c \wedge \beta^p \wedge \\ \bigwedge\limits_{i\in I} \neg^9(\exists\bar{z}_i\, \gamma_i^c \wedge \gamma_i^p) \end{array} \right] \end{array} \right] \Longrightarrow \left[ \begin{array}{l} \neg^7(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^8(\exists\bar{y}\, \beta^c \wedge \beta^p)) \wedge \\ \bigwedge\limits_{i\in I} \neg^6(\exists\bar{x}\bar{y}\bar{z}_i\, \delta_i^c \wedge \delta_i^p \wedge \varphi_0) \end{array} \right]$$

$I \neq \emptyset$, $\varphi$ is such that every negation symbol $\neg^k$ has $k \geq 6$, $\varphi_0$ is obtained from $\varphi$ by replacing all occurrences of $\neg^k$ by $\neg^0$ and all propagated-constraint sections by *true*. The formula $\delta_i^p = \alpha^p$, $\delta_i^c = \gamma_i^c \wedge \beta^c \wedge \alpha^c$.

Proof: By the definitions of $\neg^8$ and $\neg^9$, we have $\beta^p = \alpha^c \wedge \alpha^p$ and $\gamma^p = \alpha^c \wedge \alpha^p \wedge \beta^c$. The left hand side formula of the rule is equivalent to the formula

$$\neg(\exists\bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists\bar{y}\beta^c \wedge \bigwedge_{i\in I} \neg(\exists\bar{z}_i\gamma_i^c))) \tag{20}$$

Also by the definition of $\neg^9$, all variables of $\bar{y}$ are reachable in $\exists\bar{y}\beta^c$. According to Property 6.1.5, we have $T \models \exists?\bar{y}\beta^c$. According to Property 6.1.2, the formula (20) is equivalent in $T$ to

$$\neg((\exists\bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists\bar{y}\beta^c)) \vee \bigvee_{i\in I} (\exists\bar{x}\bar{y}\bar{z}_i\gamma_i^c \wedge \beta^c \wedge \alpha^c \wedge \alpha^p \wedge \varphi))$$

This formula is equivalent to the formula

$$\neg(\exists\bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists\bar{y}\beta^c \wedge \alpha^c \wedge \alpha^p)) \wedge \bigwedge_{i\in I} \neg(\exists\bar{x}\bar{y}\bar{z}_i\gamma_i^c \wedge \beta^c \wedge \alpha^c \wedge \alpha^p \wedge \varphi)$$

Having $\beta^p = \alpha^c \wedge \alpha^p$, $\delta_i^p = \alpha^p$ and $\delta_i^c = \gamma_i^c \wedge \beta^c \wedge \alpha^c$, the formula $\neg(\exists\bar{y}\beta^c \wedge \beta^p)$ satisfies the conditions of $\neg^9$. Let us consider the occurrences of $\varphi$ inside $\bigwedge_i$. Since in $\varphi$, each negation symbol $\neg^k$ has $k \geq 6$, by applying Property 6.1.13 from the most nested sub-working formulas of $\varphi$ to $\varphi$ itself, all the propagated-constraint sections can be removed and replaced by *true*. All symbol $\neg^k$ in theses occurrences of $\varphi$ can be then replaced by $\neg^0$. Giving each negation an appropriate number, we obtain the right-hand side formula of the rule. The correction of this rule is proved.

**At the end of the rules' application, the eventual formula is in the final working form** Recall that the application of the rules starts with an initial working formula $\neg^6(\exists\varepsilon\, true \wedge \bigwedge_{i\in I} \varphi_i)$, where in $\varphi_i$ all the negation symbols are $\neg^0$. In this situation, the only rule which can be applied is the rule 24. With this rule, the top-down step of simplification and propagation is started and is done by the rules 1 to 24. This step is done on a sub-working formula independently from the transformation of the other sub-working formulas. At the end of this step on a sub-working formula $\varphi$, the obtained sub-working formula $\varphi'$ is such that all negation symbols are $\neg^7$.

The rule 25

$$\neg^7 \left[ \begin{array}{l} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \\ \bigwedge_{i \in I} \neg^9 (\exists \bar{y}_i \beta_i^c \wedge \beta_i^p) \end{array} \right] \Longrightarrow \neg^8 \left[ \begin{array}{l} \exists \bar{x}^1 \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p \wedge \\ \bigwedge_{i \in I'} \neg^9 (\exists \bar{y}_i \beta_i^c \wedge \beta_i^{p*}) \end{array} \right]$$

with $I = \emptyset$ can be applied on the most embedded sub-formulas of $\varphi'$. It changes the negations $\neg^7$ of this level to $\neg^8$. The rule 26 thus can be applied

$$\neg^7 \left[ \begin{array}{l} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8 (\exists \bar{y} \beta^c \wedge \beta^p) \end{array} \right] \Longrightarrow \left[ \begin{array}{l} \neg^7 (\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9 (\exists \bar{y} \beta^{c1} \wedge \beta^p)) \wedge \\ \bigwedge_{i \in I} \neg^1 (\exists \bar{x} \bar{y} \beta^p \wedge \beta^{c1} \wedge \beta_i^{c2*} \wedge \varphi_0) \end{array} \right]$$

and it changes a sequence $\neg^7 \neg^8$ to $\neg^7 \neg^9$. This rule creates a conjunction of working formulas with $\neg^1$, on which the first step is restarted. Concerning the sequence $\neg^7 \neg^9$, the rule 27 can be applied

$$\neg^7 (\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9 (\exists \varepsilon\, true \wedge \beta^p)) \Longrightarrow true$$

or the rule 25 can be reapplied, in the last case the formula is changed to $\neg^8 \neg^9$. Let us observe that if the most embedded level has the symbol $\neg^9$, then the immediate embedding level must have the symbol $\neg^8$ and the others embedding levels must have the symbol $\neg^7$. If the symbol $\neg^7$ of the embedding formula is not the first symbol of the general working formula, the rule 28 can be applied, which remove one embedded level $\neg^9$

$$\neg^7 \left[ \begin{array}{l} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8 \left[ \begin{array}{l} \exists \bar{y}\, \beta^c \wedge \beta^p \wedge \\ \bigwedge\limits_{i \in I} \neg^9 (\exists \bar{z}_i\, \gamma_i^c \wedge \gamma_i^p) \end{array} \right] \end{array} \right] \Longrightarrow \left[ \begin{array}{l} \neg^7 (\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^8 (\exists \bar{y}\, \beta^c \wedge \beta^p)) \wedge \\ \bigwedge\limits_{i \in I} \neg^6 (\exists \bar{x} \bar{y} \bar{z}_i\, \delta_i^c \wedge \delta_i^p \wedge \varphi_0) \end{array} \right]$$

This rule cannot be applied on the first level of the working formula. Thus in case of the symbol $\neg^7$ being the first symbol of the working formula, a part of this formula is of the form $\neg^7 \neg^8 \neg^9$.

This procedure is repeated with the other parts, at the end, we obtain a working formula of the form

$$\neg^7 (\exists \varepsilon\, true \wedge \bigwedge_{i \in I} \neg^8 (\exists \bar{x}_i\, \alpha_i^c \wedge \alpha_i^p \wedge \bigwedge_{j \in J_i} \neg^9 (\exists \bar{y}_{ij}\, \beta_{ij}^c \wedge \beta_{ij}^p))),$$

which is a final working formula.

**Property 4.4.2** Every formula is equivalent in $T$ either to *true* or to *false* or to a disjunction of general solved formulas, which has at least one free variable and which is equivalent neither to *true* nor to *false* in $T$.

**Proof L** et $\varphi$ be a formula. According to Property 4.3.1 the formula $\varphi$ is equivalent in $T$ to a final working formula $\phi$ i.e. to a disjunction of general solved formulas. If this disjunction is empty then $\phi$ is the formula *false* thus $T \models \neg\varphi$. If the disjunction is not empty two cases arise:

- the disjunction $\phi$ contains at least a general solved formula $\phi_i$ without free variables then according to Property 4.1.5 $\phi_i$ is the formula *true* then according to the third point of our algorithm section 4.4 the formula $\phi$ is reduced to *true* and thus $T \models \varphi$.

- the disjunction $\phi$ does not contain general solved formulas without free variables, then according to Property 4.1.5 each of these general solved formulas are neither equivalent to *true* nor to *false* thus neither $T \models \varphi$ nor $T \models \neg\varphi$.

Thus, in the particular case where $\varphi$ has no free variables one at least of these properties holds: $T \models \varphi$, $T \models \neg\varphi$ and since $T$ has at least a model [10] we have either $T \models \varphi$ or $T \models \neg\varphi$. Thus $T$ is complete.