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Extension of First-Order Theories into Trees

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Abstract

We present in this paper an automatic way to combine any first-order theory T with the theory of finite or infinite trees. First of all, we present a new class of theories that we call *zero-infinite-decomposable* and show that every decomposable theory T accepts a decision procedure in the form of six rewriting which for every first order proposition give either true or false in T. We present then the axiomatization T^* of the extension of T into trees and show that if T is flexible then its extension into trees T^* is zero-infinite-decomposable and thus complete. The flexible theories are theories having elegant properties which enable us to eliminate quantifiers in particular cases.

1 Introduction

The theory of finite or infinite trees plays a fundamental role in programming. Recall that Alain Colmerauer has described the execution of Prolog II, III and IV programs in terms of solving equations and disequations in this theory [6, 9, 2]. He has first introduced in Prolog II the unification of infinite terms together with a predicate of non-equality [8]. He has then integrated in Prolog III the domain of rational numbers together with the operations of addition and subtraction and a linear dense order relation without endpoints [5, 7]. He also gave a general algorithm to check the satisfiability of a system of equations, inequations and disequations on a combination of trees and rational numbers. Finally, in Prolog IV, the notions of list, interval and boolean have been added [10, 2].

We present in this paper an idea of a general extension of the model of Prolog IV by allowing the user to incorporate universal and existential quantifiers to Prolog closes and to decide the validity or not validity of any first-order proposition (sentence) in a combination of trees and first-order theories. For that:

(1) we give an automatic way to generate the axiomatization of the combination of any first order theory T with the theory of finite or infinite trees,

(2) we present simple conditions on T and only on T so that the combination of T with the theory of finite or infinite trees is complete and accepts a decision algorithm in the form of six rewriting rules which for every proposition give either true or false.

One of major difficulties in this work resides in the fact that the two theories can possibly have non-disjoint signatures. Moreover, the theory of finite or infinite trees does not accept full elimination of quantifiers.

The emergence of general constraint-based paradigms, such as constraint logic programming [19], constrained resolution [3] and what is generally referred to as *theory reasoning* [1], rises the problem of combining decision procedure for solving general first order constraints. Initial combinations results were provided by R. Shostak in [27] and in [28]. Shostak's approach is limited in scope and not very modular. A rather general and completely modular combination method was proposed by G. Nelson and D. Oppen in [21] and then slightly revised in [22]. Given, for $i = 1, ..., n$ a procedure P_i that decides the satisfiability of quantifier-free formulas in the theory $T_1 \cup ... \cup T_n$. A declarative and non-deterministic view of the procedure was suggested by Oppen in [24]. In [30], C. Tinelli and H.Harandi followed up on this suggestion describing a non-deterministic version of the Nelson-Oppen approach combination procedure and providing a simpler correctness proof. A similar approach had also been followed by C. Ringeissen in [26] which describes the procedure as a set of a derivation rules applied non-deterministically.

All the works mentioned above share one major restriction on the constraint languages of the component reasoners: they must have disjoint signatures, i.e. no function and relation symbols in common. (The only exception is the equality symbol which is however regarded as a logical constant). This restriction has proven really hard to lift. A testament of this is that, more than two decades after Nelson and Oppen's original work, their combination results are still state of the art.

Results on non-disjoint signatures do exists, but they are quit limited. To start with, some results on the union of non-disjoint equational theories can be obtained as a byproduct of the research on the combination of term rewriting systems. Modular properties of term rewriting systems have been extensively investigated (see the overviews in [23] and [18]). Using some of these properties it is possible to derive combination results for the word problem in the union of equational theories sharing constructors[1]. Outside the work on modular term rewriting, the first combination result for the word problem in the union of non-disjoint constraint theories were given in [16] as a consequence of some combination techniques based on an adequate notion of (shared) constructors. C. Ringeissen used similar ideas later in [25] to extend the Nelson-Oppen method to theories sharing constructors in a sense closed to that of [16].

Recently, C. Tinelli and C. Ringeissen have provided some sufficient conditions for the Nelson-Oppen combinability by using a concept of stable Σ-freeness [29], a natural extension of Nelson-Oppen's stable-infiniteness requirement for theories with non-disjoint signatures. As for us, we present a natural way to combine the theory of finite or infinite trees with any first order theory T which can possibly have a non-disjoint signature. A such theory is denoted by T^* and does not accept full elimination of quantifiers which makes the decision procedure not evident. To show the completeness of T^* we give simple conditions on T and only on T so that its combination with the theory of finite or infinite trees, i.e. T^*, is complete and accepts a decision procedure which using only six rewriting rules is able to decide the validity or not validity of any first order constraints in T^*.

This paper is organized in five sections followed by a conclusion. This introduction is the first section. In Section 2, we recall the basic definitions of signature, model, theory and vectorial quantifier. In section 3, after having presented a new quantifier called *zero-infinite*, we preset a new class of theories that we call *zero-infinite-decomposable*. The main idea behind this class of the-

[1]The word problem in an equational theory T is the problem of determining whether a given equation $s = t$ is valid in T, or equivalently, whether a disequation $\neg(s = t)$ is (un)satisfiable in T. In a term rewriting system, a constructor is a function symbol that does not appear as the top symbol of a rewrite rule's left-hand-side.

ories consists in decomposing each quantified conjunction of atomic formulas into three embedded sequences of quantifications having very particular properties, which can be expressed with the help of three special quantifiers denoted by $\exists ?$, $\exists !$, $\exists_{o\infty}^{\Psi(u)}$ and called *at-most-one*, *exactly-one*, *zero-infinite*. We end this section by six rewriting rules which for every zero-infinite-decomposable theory T and for every proposition φ give either true or false in T. The correctness of our algorithm shows the completeness of the zero-infinite-decomposable theories. In Section 4, we recall the structure of finite or infinite trees and present the Maher axiomatization [20] of this structure. We give then a general way to generate the axioms of T^* using those of T and present the standard model M^* of T^*. In section 5 we introduce the flexible theories and show that if T is flexible then T^* is zero-infinite-decomposable and thus complete. Finally, in section 6, we give a full proof of the flexibility of the theory of ordered additive rational or real numbers together with the operations of addition, subtraction and a linear dense order relation without endpoints.

The zero-infinite-decomposable theories, the decision procedure in zero-infinite-decomposable theories, the axiomatization of T^* and the flexible theories are our main contribution in this paper.

2 Preliminaries

Let V be an infinite set of *variables*. Let S be a set of symbols, called *a signature* and partitioned into two disjoint sub-sets: the set F of *function symbols* and the set R of *relation symbols*. To each function symbol and relation is linked a non-negative integer n called *its arity*. An n-ary symbol is a symbol of arity n. A 0-ary function symbol is called *a constant*.

An *S-formula* is an expression of the one of the eleven following forms:

$$\begin{gathered} s = t,\ rt_1 \dots t_n,\ \mathit{true},\ \mathit{false}, \\ \neg\varphi,\ (\varphi \wedge \psi),\ (\varphi \vee \psi),\ (\varphi \rightarrow \psi),\ (\varphi \leftrightarrow \psi), \\ (\forall x\, \varphi),\ (\exists x\, \varphi), \end{gathered} \tag{1}$$

with $x \in V$, r an n-ary relation symbol taken from F, φ and ψ shorter S-formulas, s, t and the t_i's S-terms, that are expressions of the one of the two following forms

$$x,\ ft_1 \dots t_n,$$

with x taken from V, f an n-ary function symbol taken from F and the t_i shorter S-terms

The S-formulas of the first line of (1) are called *atomic*, and *flat* if they are of the one of the five following forms:

$$\mathit{true},\ \mathit{false},\ x_0 = fx_1 ... x_n,\ x_0 = x_1,\ rx_1 ... x_n,$$

with the x_i's possibly non-distinct variables taken from V, $f \in F$ and $r \in R$.

If φ is an S-formula then we denote by $var(\varphi)$ the set of the free variables of φ. An *S-proposition* is an S-formula without free variables. The set of the S-terms and the S-formulas represent *a first-order language with equality*.

An S-*structure* is a couple $M = (D, F)$, where D is a non-empty set of *individuals* of M and F a set of functions and relations in D. We call *instantiation* or *valuation* of an S-formula φ by individuals of M, the $(S \cup D)$-formula obtained from φ by replacing each free occurrence of a free variable x in φ by the same individual i of D and by considering each element of D as 0-ary function symbol.

An S-*theory* T is a set of S-propositions. We say that the S-structure M is a model of T if for each element φ of T, $M \models \varphi$. If φ is an S-formula, we write $T \models \varphi$ if for each S-model M of T, $M \models \varphi$. A theory T is called *complete* if for every proposition φ, one and only one of the following properties holds: $T \models \varphi$, $T \models \neg\varphi$.

Let M be a model. Let $\bar{x} = x_1 \dots x_n$ and $\bar{y} = y_1 \dots y_n$ be two words on $\mathbf{v}$ of the same length. Let φ and $\varphi(\bar{x})$ be M-formulas. We write

$$\begin{array}{ll} \exists \bar{x}\, \varphi & \text{for } \exists x_1 ... \exists x_n\, \varphi, \\ \forall \bar{x}\, \varphi & \text{for } \forall x_1 ... \forall x_n\, \varphi, \\ \exists ? \bar{x}\, \varphi(\bar{x}) & \text{for } \forall \bar{x} \forall \bar{y}\, \varphi(\bar{x}) \wedge \varphi(\bar{y}) \rightarrow \bigwedge_{i \in \{1,\dots,n\}} x_i = y_i, \\ \exists ! \bar{x}\, \varphi & \text{for } (\exists \bar{x}\, \varphi) \wedge (\exists ? \bar{x}\, \varphi). \end{array}$$

The word $\bar{x}$, which can be the empty word ε, is called *vector of variables.* Note that the formulas $\exists ? \varepsilon \varphi$ and $\exists ! \varepsilon \varphi$ are respectively equivalent to *true* and to φ in any model M.

Property 2.0.1 *If* $T \models \exists ? \bar{x}\, \varphi$ *then*

$$T \models (\exists \bar{x}\, \varphi \wedge \neg \phi) \leftrightarrow ((\exists \bar{x} \varphi) \wedge \neg (\exists \bar{x}\, \varphi \wedge \phi)). \tag{2}$$

Proof. Let M be a model of T and let $\exists \bar{x}\, \varphi' \wedge \neg \phi'$ be an instantiation of $\exists \bar{x}\, \varphi \wedge \neg \phi$ by individuals of M. Let us denote by φ'_1 the M-formula $(\exists \bar{x}\, \varphi' \wedge \neg \phi')$ and by φ'_2 the M-formula $(\exists \bar{x}\, \varphi') \wedge \neg(\exists \bar{x} \varphi' \wedge \phi')$. To show the equivalence (2), it is enough to show that

$$M \models \varphi'_1 \leftrightarrow \varphi'_2. \tag{3}$$

If $M \models \neg(\exists \bar{x}\, \varphi')$ then $M \models \neg \varphi'_1$ and $M \models \neg \varphi'_2$, thus the equivalence (3) holds. If $M \models \exists \bar{x}\, \varphi'$. Since $T \models \exists ? \bar{x}\, \varphi'$, there exists a unique vector $\bar{i}$ of individuals of M such that $M \models \varphi'_{\bar{x} \leftarrow \bar{i}}$. Two cases arise:

If $M \models \neg(\phi'_{\bar{x} \leftarrow \bar{i}})$, then $M \models (\varphi' \wedge \neg \phi')_{\bar{x} \leftarrow \bar{i}}$, thus $M \models \varphi'_1$. Since $\bar{i}$ is unique and since $M \models \neg(\phi'_{\bar{x} \leftarrow \bar{i}})$, there exists no vector $\bar{u}$ of individuals of M such that $M \models (\varphi' \wedge \phi')_{\bar{x} \leftarrow \bar{u}}$. Consequently, $M \models \neg(\exists \bar{x}\, \varphi' \wedge \phi')$ and thus $M \models \varphi'_2$. We have $M \models \varphi'_1$ and $M \models \varphi'_2$, thus, the equivalence (3) holds.

If $M \models \phi'_{\bar{x} \leftarrow \bar{i}}$, then $M \models (\varphi' \wedge \phi')_{\bar{x} \leftarrow \bar{i}}$ and thus $M \models \neg \varphi'_2$. Since $\bar{i}$ is unique and since $M \models \phi'_{\bar{x} \leftarrow \bar{i}}$, there exists no vector $\bar{u}$ of individuals of M such that $M \models (\varphi' \wedge \neg \phi')_{\bar{x} \leftarrow \bar{u}}$. Consequently, $M \models \neg(\exists \bar{x}\, \varphi' \wedge \neg \phi')$ and thus $M \models \neg \varphi'_1$. We have $M \models \neg \varphi'_1$ and $M \models \neg \varphi'_2$, thus, the equivalence (3) holds.

Corollary 2.0.2 *If* $T \models \exists ? \bar{x}\, \varphi$ *then*

$$T \models (\exists \bar{x}\, \varphi \wedge \bigwedge_{i \in I} \neg \phi_i) \leftrightarrow ((\exists \bar{x} \varphi) \wedge \bigwedge_{i \in I} \neg (\exists \bar{x}\, \varphi \wedge \phi_i)).$$

Proof. Let ψ be the formula $\neg(\bigwedge_{i\in I}\neg\phi_i)$. The formula $\exists\bar{x}\,\varphi\wedge\bigwedge_{i\in I}\neg\phi_i$, is equivalent in T to $\exists\bar{x}\,\varphi\wedge\neg\psi$. Since $T\models\exists?\bar{x}\,\varphi$, then according to Property 2.0.1 the preceding formula is equivalent in T to $(\exists\bar{x}\,\varphi)\wedge\neg(\exists\bar{x}\,\varphi\wedge\psi)$, which is equivalent in T to $(\exists\bar{x}\,\varphi)\wedge\neg(\exists\bar{x}\,\varphi\wedge\neg(\bigwedge_{i\in I}\neg\phi_i))$, thus to $(\exists\bar{x}\,\varphi)\wedge\neg(\exists\bar{x}\,\varphi\wedge(\bigvee_{i\in I}\phi_i))$, which is equivalent in T to $(\exists\bar{x}\,\varphi)\wedge\neg(\exists\bar{x}\,(\bigvee_{i\in I}(\varphi\wedge\phi_i)))$, thus to $(\exists\bar{x}\,\varphi)\wedge\neg(\bigvee_{i\in I}(\exists\bar{x}\,\varphi\wedge\phi_i))$, which is finally equivalent in T to

$$(\exists\bar{x}\,\varphi)\wedge\bigwedge_{i\in I}\neg(\exists\bar{x}\,\varphi\wedge\phi_i).$$

Corollary 2.0.3 *If $T\models\ \psi\rightarrow(\exists!\bar{x}\,\varphi)$ then*

$$T\models(\psi\wedge(\exists\bar{x}\,\varphi\wedge\bigwedge_{i\in I}\neg\phi_i))\leftrightarrow(\psi\wedge\bigwedge_{i\in I}\neg(\exists\bar{x}\,\varphi\wedge\phi_i)).$$

Property 2.0.4 *If $T\models\exists?\bar{y}\phi$ and if all the variables of $\bar{y}$ has no free occurrences in φ then*

$$T\models(\exists\bar{x}\,\varphi\wedge\neg(\exists\bar{y}\,\phi\wedge\psi))\leftrightarrow\begin{bmatrix}(\exists\bar{x}\,\varphi\wedge\neg(\exists\bar{y}\,\phi))\\ \vee\\ (\exists\overline{xy}\,\varphi\wedge\phi\wedge\neg\psi)\end{bmatrix}.$$

Proof. The formula

$$\exists\bar{x}\,\varphi\wedge\neg(\exists\bar{y}\,\phi\wedge\psi),$$

is equivalent in T to

$$\exists\bar{x}\,\varphi\wedge\neg(\exists\bar{y}\,\phi\wedge\neg(\neg\psi)),$$

which according to Property 2.0.1 is equivalent in T to

$$\exists\bar{x}\,\varphi\wedge\neg((\exists\bar{y}\,\phi)\wedge\neg(\exists\bar{y}\phi\wedge\neg\psi)),$$

i.e. to

$$\exists\bar{x}\,\varphi\wedge((\neg(\exists\bar{y}\,\phi))\vee(\exists\bar{y}\phi\wedge\neg\psi)),$$

i.e. to

$$\begin{bmatrix}(\exists\bar{x}\,\varphi\wedge\neg(\exists\bar{y}\,\phi))\\ \vee\\ (\exists\bar{x}\,\varphi\wedge(\exists\bar{y}\phi\wedge\neg\psi))\end{bmatrix}.$$

Since all the variables of $\bar{y}$ has no free occurrences in φ, then the preceding formula is equivalent in T to

$$\begin{bmatrix}(\exists\bar{x}\,\varphi\wedge\neg(\exists\bar{y}\,\phi))\\ \vee\\ (\exists\bar{x}\bar{y}\,\varphi\wedge\phi\wedge\neg\psi)\end{bmatrix}.$$

3 Zero-infnite-decomposable theories

In this section, let us fix a signature $S^*=F^*\cup R^*$. Thus, we can allow ourself to remove the prefix S^* from the following words: formulas, equations, theories and models. We will also use the abbreviation *wnfv* for *"without new free variables"*. We say that an S-formula φ is equivalent to a wnfv S-formula ψ in T if $T\models\varphi\leftrightarrow\psi$ and ψ does not contain other free variables than those of φ.

3.1 Zero-infinite quantifier [15]

Let M be a model and T a theory. Let $\Psi(u)$ be a set of formulas having at most one free variable u. Let φ and φ_j be M-formulas.

Definition 3.1.1 *We write*

$$M \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x), \tag{4}$$

if for each instantiation $\exists x\, \varphi'(x)$ of $\exists x\, \varphi(x)$ by individuals of M one of the following properties holds:

- *the set of the individuals i of M such that $M \models \varphi'(i)$, is infinite,*
- *for every finite sub-set $\{\psi_1(u), .., \psi_n(u)\}$ of elements of $\Psi(u)$, the set of the individuals i of M such that $M \models \varphi'(i) \wedge \bigwedge_{j\in\{1,\dots,n\}} \neg\psi_j(i)$ is infinite.*

We write $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$, if for every model M of T we have $M \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$.

This infinite quantifier holds only for infinite models, i.e. models whose set of elements are infinite. Note that if $\Psi(u) = \{\mathit{false}\}$ then (4) simply means that if $M \models \exists x\, \varphi(x)$ then M contains an infinity of individuals i such that $M \models \varphi(i)$. The intuitions behind this definition come from an aim to eliminate a conjunction of the form $\bigwedge_{i\in I} \neg\psi_i(x)$ in complex formulas of the form $\exists \bar{x}\, \varphi(x) \wedge \bigwedge_{i\in I} \neg\psi_i(x)$ where I is a finite (possibly empty) set and the $\psi_i(x)$ are formulas which do not accept full elimination of quantifiers.

Property 3.1.2 *Let J be a finite possibly empty set. If $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$ and if for each φ_j, one at least of the following properties holds:*

- *$T \models \exists ?x\, \varphi_j$,*
- *there exists $\psi_j(u) \in \Psi(u)$ such that $T \models \forall x\, \varphi_j \rightarrow \psi_j(x)$,*

then

$$T \models (\exists x\, \varphi(x) \wedge \textstyle\bigwedge_{j\in J} \neg\varphi_j) \leftrightarrow (\exists x\, \varphi(x)).$$

Proof. Let $\exists x\, \varphi'(x)$ be an instantiation of $\exists x\, \varphi(x)$ by individuals of M. Let us show that if the conditions of this property hold, then

$$M \models (\exists x\, \varphi'(x) \wedge \textstyle\bigwedge_{j\in J} \neg\varphi_j(x)) \leftrightarrow (\exists x\, \varphi'(x)). \tag{5}$$

Let J' be the set of the $j \in J$ such that $M \models \exists ?x\, \varphi_j(x)$ and let m be its cardinality. Since for all $j \in J'$, $M \models \exists ?x\, \varphi'_j(x)$, then it is enough that M contains at least $m+1$ individuals, to warrant the existence of an individual $i \in M$ such that

$$M \models \bigwedge_{j\in J'} \neg\varphi'_j(i). \tag{6}$$

On the other hand, since $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$ and according to Definition 3.1.1 of the zero-infinite quantifier, two cases arise:

(1) Either, $M \models \neg(\exists x\, \varphi'(x))$, thus $M \models \neg(\exists \bar{x}\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi_j(x))$ and thus the equivalence (5) holds in M.

(2) Or, for every finite sub-set $\{\psi_1(u), ..., \psi_n(u)\}$ of $\Psi(u)$, the set of the individuals i of M such that $M \models \varphi'(i) \wedge \bigwedge_{j=1}^{n} \neg\psi_j(i)$ is infinite. Thus, since for all $j \in J - J'$ we have $M \models \forall x\, \varphi_j(x) \rightarrow \psi_j(x)$, then there exists an infinite set ξ of individuals i of M such that $M \models \varphi'(i) \wedge \bigwedge_{j\in J-J'} \neg\varphi_j(i)$. Since ξ is infinite, then it contains at least $m+1$ individuals and thus according to (6), there exists at least an individual $i \in \xi$ such that $M \models \varphi'(i) \wedge (\bigwedge_{j\in J-J'} \neg\varphi'_j(i)) \wedge (\bigwedge_{k\in J'} \neg\varphi'_k(i))$ and thus such that

$$M \models \exists x\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi'_j(x).$$

Since $M \models \exists x\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi_j(x)$, then $M \models \exists x\, \varphi'(x)$ and thus the equivalence (5) holds in M.

3.2 Zero-infinite-decomposable theory [15]

Definition 3.2.1 *A theory T is called* zero-infinite-decomposable *if there exists a set $\Psi(u)$ of formulas, having at least one free variable u, a set A of formulas closed under conjunction, a set A' of formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$, and a sub-set A'' of A such that:*

1. *every formula of the form $\exists\bar{x}\, \alpha \wedge \psi$, with $\alpha \in A$ and ψ a formula, is equivalent in T to a wnfv formula of the form:*

$$\exists\bar{x}'\, \alpha' \wedge (\exists\bar{x}''\, \alpha'' \wedge (\exists\bar{x}'''\, \alpha''' \wedge \psi)),$$

   with $\exists\bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$, $\alpha''' \in A$ and $T \models \forall\bar{x}''\alpha'' \rightarrow \exists!\bar{x}'''\alpha'''$,

2. *if $\exists\bar{x}'\alpha' \in A'$ then $T \models \exists?\bar{x}'\, \alpha'$ and for every free variable y in $\exists\bar{x}'\alpha'$, one at least of the following properties holds:*
   - *$T \models \exists?y\bar{x}'\, \alpha'$,*
   - *there exists $\psi(u) \in \Psi(u)$ such that $T \models \forall y\, (\exists\bar{x}'\, \alpha') \rightarrow \psi(y)$,*

3. *if $\alpha'' \in A''$ then*
   - *the formula $\neg\alpha''$ is equivalent in T to a wnfv formula of the form $\bigvee_{i\in I} \alpha_i$ with $\alpha_i \in A$,*
   - *for every x'', the formula $\exists x''\alpha''$ is equivalent in T to a wnfv formula which belongs to A'',*
   - *for every variable x'', $T \models \exists_{o\,\infty}^{\Psi(u)} x''\, \alpha''$,*

4. *every conjunction of flat formulas is equivalent in T to a wnfv disjunction of elements of A,*

5. *if the formula $\exists\bar{x}'\alpha' \wedge \alpha''$ with $\exists\bar{x}'\alpha' \in A'$ and $\alpha'' \in A''$ has no free variables then $\bar{x}$ is the empty vector, α' is the formula true and α'' is either the formula true or false.*

3.3 A decision procedure for zero-infinite-decompoable theories [14]

Let T be a zero-infinite-decomposable theory. The sets $\Psi(u)$, A, A' and A'' are known and fixed.

Definition 3.3.1 *A normalized formula φ of depth $d \geq 1$ is a formula of the form $\neg(\exists \bar{x}\, \alpha \wedge \bigwedge_{i\in I} \varphi_i)$, where I is a finite possibly empty set, $\alpha \in A$, the φ_i are normalized formulas of depth d_i with $d = 1 + \max\{0, d_1, ..., d_n\}$, and all the quantified variables have distinct names and different form those of the free variables.*

Property 3.3.2 *Every formula is equivalent in T to a normalized formula.*

Definition 3.3.3 *A* final formula *is a normalized formula of the form*

$$\neg(\exists \bar{x}'\, \alpha' \wedge \alpha'' \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}'_i\, \beta'_i)), \tag{7}$$

with I a finite possibly empty set, $\exists \bar{x}'\alpha' \in A'$, $\alpha'' \in A''$, $\exists \bar{y}'_i \beta'_i \in A'$, α'' is different from the formula false*, all the β'_i's are different from the formulas* true *and* false*.*

Property 3.3.4 *Let φ be a conjunction of final formulas without free variables. The conjunction φ is either the formula* true *or the formula* $\neg$true.

Property 3.3.5 *Every normalized formula is equivalent in T to a conjunction of final formulas.*

Proof. We give bellow six rewriting rules which transform a normalized formula of any depth d into a conjunction of final formulas equivalent in T. To apply the rule $p_1 \Longrightarrow p_2$ on a normalized formula p means to replace in p, the sub-formula p_1 by the formula p_2, by considering the connector $\wedge$ associative and commutative.

$$(1)\quad \neg\begin{bmatrix} \exists\bar{x}\,\alpha\wedge\varphi\wedge \\ \neg(\exists\bar{y}\,\mathit{true}) \end{bmatrix} \quad\Longrightarrow\quad \mathit{true}$$

$$(2)\quad \neg\begin{bmatrix} \exists\bar{x}\,\alpha\wedge \mathit{false}\wedge\varphi \end{bmatrix} \quad\Longrightarrow\quad \mathit{true}$$

$$(3)\quad \neg\begin{bmatrix} \exists\bar{x}\,\alpha\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i) \end{bmatrix} \quad\Longrightarrow\quad \neg\begin{bmatrix} \exists\bar{x}'\bar{x}''\,\alpha'\wedge\alpha''\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^* \end{bmatrix}$$

$$(4)\quad \neg\begin{bmatrix} \exists\bar{x}\,\alpha\wedge\varphi\wedge \\ \neg(\exists\bar{y}'\,\beta'\wedge\beta'') \end{bmatrix} \quad\Longrightarrow\quad \begin{bmatrix} \neg(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'))\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)^* \end{bmatrix}$$

$$(5)\quad \neg\begin{bmatrix} \exists\bar{x}\,\alpha\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{y}'_i\,\beta'_i) \end{bmatrix} \quad\Longrightarrow\quad \neg\begin{bmatrix} \exists\bar{x}'\,\alpha'\wedge\alpha''_* \\ \bigwedge_{i\in I'}\neg(\exists\bar{y}'_i\,\beta'_i) \end{bmatrix}$$

$$(6)\quad \neg\begin{bmatrix} \exists\bar{x}\,\alpha\wedge\varphi\wedge \\ \neg\begin{bmatrix} \exists\bar{y}'\,\beta'\wedge\beta''\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{z}'_i\,\delta'_i) \end{bmatrix} \end{bmatrix} \quad\Longrightarrow\quad \begin{bmatrix} \neg(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'\wedge\beta''))\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{x}\bar{y}'\bar{z}_i\,\alpha\wedge\beta'\wedge\beta''\wedge\delta'_i\wedge\varphi)^* \end{bmatrix}$$

with α an element of A, φ a conjunction of working formulas and I a finite possibly empty set. In the rule (3), the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\bar{x}'''\,\alpha'''))$ with $\exists\bar{x}'\,\alpha'\in A'$, $\alpha''\in A''$, $\alpha'''\in A$, $T\models\forall\bar{x}''\alpha''\rightarrow\exists!\bar{x}'''\alpha'''$ and $\exists\bar{x}'''\,\alpha'''$ is different from $\exists\varepsilon\,\mathit{true}$. All the β_i belong to A. The formula $(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^*$ is the formula $(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)$ in which we have renamed the variables which occur in $\bar{x}'''$ by distinct names and different from those of the free variables. In the rule (4), the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\varepsilon\,\mathit{true}))$ with $\exists\bar{x}'\,\alpha'\in A'$ and $\alpha''\in A''$. The formula $\exists\bar{y}'\,\beta'$ belongs to A'. The formula β'' belongs to A'' and is different from the formula *true*. Moreover, $T\models(\neg\beta'')\leftrightarrow\bigvee_{i\in I}\beta''_i$ with $\beta''_i\in A$. The formula $(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)^*$ is the formula $(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables. In the rule (5), the formula $\exists\bar{x}\,\alpha$ is not of the form $\exists\bar{x}\,\alpha_1\wedge\alpha_2$ with $\exists\bar{x}\,\alpha_1\in A'$ and $\alpha_2\in A''$, and is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\varepsilon\,\mathit{true}))$ with $\exists\bar{x}'\,\alpha'\in A'$ and $\alpha''\in A''$. Each formula $\exists\bar{y}'_i\,\beta'_i$ belongs to A'. The set I' is the set of the $i\in I$ such that $\exists\bar{y}'_i\beta'_i$ has no occurrences of any variable of $\bar{x}''$. Moreover, $T\models(\exists\bar{x}''\alpha'')\leftrightarrow\alpha''_*$ with $\alpha''_*\in A''$. In the rule (6), $I\neq\emptyset$, $\exists\bar{y}'\,\beta'\in A'$, $\exists\bar{z}'_i\,\delta'_i\in A'$ and $\beta''\in A''$. The formula $(\exists\bar{x}\bar{y}'\bar{z}_i\,\alpha\wedge\beta'\wedge\beta''\wedge\delta'_i\wedge\varphi)^*$ is the formula $(\exists\bar{x}\bar{y}'\bar{z}_i\,\alpha\wedge\beta'\wedge\beta''\wedge\delta'_i\wedge\varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables.

Correctness of the rules: Let us show that for each rule of the form $p\Longrightarrow p'$ we have $T\models p\leftrightarrow p'$ and the formula p' remains a conjunction of working formulas. It is clear that the rules 1 and 2 are correct in T.

Correctness of the rule (3):

$$\neg\begin{bmatrix} \exists\bar{x}\,\alpha\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i) \end{bmatrix} \Longrightarrow \neg\begin{bmatrix} \exists\bar{x}'\bar{x}''\,\alpha'\wedge\alpha''\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^* \end{bmatrix}$$

where the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\bar{x}'''\,\alpha'''))$ with $\exists\bar{x}'\,\alpha'\in A'$, $\alpha''\in A''$, $\alpha'''\in A$, $T\models\forall\bar{x}''\alpha''\rightarrow\exists!\bar{x}'''\alpha'''$ and $\exists\bar{x}'''\,\alpha'''$ different from $\exists\varepsilon\,\mathit{true}$. The formula $(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^*$ is the formula $(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)$ in which we have renamed the variables which occur in $\bar{x}'''$ by distinct names and different from those of the free variables.

Let us show the correctness of this rule. According to the conditions of this rule, the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\bar{x}'''\,\alpha'''))$ with $\exists\bar{x}'\,\alpha'\in A'$, $\alpha''\in A''$, $\alpha'''\in A$, $T\models\forall\bar{x}''\alpha''\rightarrow\exists!\bar{x}'''\alpha'''$ and $\exists\bar{x}'''\,\alpha'''$ different from $\exists\varepsilon\,\mathit{true}$. Thus the left hand side of this rule is equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\alpha''\wedge(\exists\bar{x}'''\alpha'''\wedge\bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i)))).$$

According to Corollary 2.0.3, the preceding formula is equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\alpha''\wedge\bigwedge_{i\in I}\neg(\exists\bar{x}'''\alpha'''\wedge(\exists\bar{y}_i\,\beta_i)))).$$

According to the definition of working formula, the quantified variables have distinct names and different from those of the free variables. We can then lift the quantifications $\exists\bar{y}_i$. The preceding formula is thus equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\alpha''\wedge\bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i))),$$

which, by renaming the variables which occur in $\bar{x}'''$ by distinct names and different from those of the free variables, is equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\alpha''\wedge\bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^*)),$$

thus, the rule (3) is correct in T.

Correctness of the rule (4):

$$\neg\begin{bmatrix} \exists\bar{x}\,\alpha\wedge\varphi\wedge \\ \neg(\exists\bar{y}'\,\beta'\wedge\beta'') \end{bmatrix} \Longrightarrow \begin{bmatrix} \neg(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'))\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)^* \end{bmatrix}$$

where the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\varepsilon\,\mathit{true}))$ with $\exists\bar{x}'\,\alpha'\in A'$ and $\alpha''\in A''$. The formula $\exists\bar{y}'\,\beta'$ belongs to A'. The formula β'' belongs to A'' and is not of the form true. Moreover, $T\models(\neg\beta'')\leftrightarrow\bigvee_{i\in I}\beta''_i$ with $\beta''_i\in A$. The formula $(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)^*$ is the formula $(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables.

Since $\exists \bar{y}' \beta' \in A'$, then according to the second point of Definition 3.2.1, we have $T \models \exists ? \bar{y}' \beta'$, thus according to Corollary 2.0.4, the left hand side of our rule is equivalent in T to

$$\neg \begin{bmatrix} (\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \vee \\ (\exists \bar{x}\bar{y}'\, \alpha \wedge \varphi \wedge \beta' \wedge \neg \beta'') \end{bmatrix}.$$

Since $T \models (\neg \beta'') \leftrightarrow (\bigvee_{i \in I} \beta''_i)$ (always possible according to the condition 3 of Definition 3.2.1), then the preceding formula is equivalent in T to

$$\neg \begin{bmatrix} (\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \vee \\ (\exists \bar{x}\bar{y}'\, \alpha \wedge \varphi \wedge \beta' \wedge (\bigvee_{i \in I} \beta''_i)) \end{bmatrix},$$

i.e. to

$$\neg \begin{bmatrix} (\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \vee \\ (\exists \bar{x}\bar{y}'\, \bigvee_{i \in I} (\alpha \wedge \varphi \wedge \beta' \wedge \beta''_i)) \end{bmatrix},$$

i.e. to

$$\neg \begin{bmatrix} (\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \vee \\ \bigvee_{i \in I} (\exists \bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi) \end{bmatrix},$$

and thus to

$$\begin{bmatrix} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi) \end{bmatrix},$$

which, by denoting by $(\exists \bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi)^*$ the formula $(\exists \bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables, is equivalent in T to

$$\begin{bmatrix} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi)^* \end{bmatrix}.$$

Thus, the rule (4) is correct in T.

Correctness of the rule (5):

$$\neg \begin{bmatrix} \exists \bar{x}\, \alpha \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{y}'_i\, \beta'_i) \end{bmatrix} \Longrightarrow \neg \begin{bmatrix} \exists \bar{x}'\, \alpha' \wedge \alpha''_* \\ \bigwedge_{i \in I'} \neg(\exists \bar{y}'_i\, \beta'_i) \end{bmatrix}$$

where the formula $\exists \bar{x}\, \alpha$ is not of the form $\exists \bar{x}\, \alpha_1 \wedge \alpha_2$ with $\exists \bar{x}\, \alpha_1 \in A'$ and $\alpha_2 \in A''$ and is equivalent in T to a decomposed formula of the form $\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \varepsilon\, \mathit{true}))$ with $\exists \bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$. Each formula $\exists \bar{y}'_i\, \beta'_i$ belongs to A'. I' is the set of the $i \in I$ such that $\exists \bar{y}'_i \beta'_i$ has no occurrences of the variables of $\bar{x}''$. Moreover, $T \models (\exists \bar{x}'' \alpha'') \leftrightarrow \alpha''_*$ with $\alpha''_* \in A''$.

Let us show the correctness of this rule. According to the conditions of this rule, its left hand side is equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha' \wedge (\exists\bar{x}''\alpha'' \wedge \bigwedge_{i\in I} \neg(\exists\bar{y}'_i\,\beta'_i))),$$

with $\exists\bar{x}'\,\alpha' \in A'$, $\alpha'' \in A''$ and all the $\exists\bar{y}'_i\,\beta'_i$ belong to A'. Let us denote by I_1, the set of the $i \in I$ such that x''_n has no occurrences in $\exists\bar{y}'_i\beta'_i$. The preceding formula is equivalent in T to

$$\neg(\exists\bar{x}'\alpha' \wedge (\exists x''_1...\exists x''_{n-1} \begin{bmatrix} (\bigwedge_{i\in I_1} \neg(\exists\bar{y}'_i\beta'_i))\wedge \\ (\exists x''_n\,\alpha'' \wedge \bigwedge_{i\in I-I_1} \neg(\exists\bar{y}'_i\beta'_i)) \end{bmatrix})). \tag{8}$$

Since $\alpha'' \in A''$ and $\exists\bar{y}'_i\beta'_i \in A'$, then according to Property 3.1.2 and the conditions 2 and 3 of Definition 3.2.1, the formula (8) is equivalent in T to

$$\neg(\exists\bar{x}'\alpha' \wedge (\exists x''_1...\exists x''_{n-1} \begin{bmatrix} (\bigwedge_{i\in I_1} \neg(\exists\bar{y}'_i\beta'_i))\wedge \\ (\exists x''_n\,\alpha'') \end{bmatrix})).$$

Since $T \models (\exists x''_n\,\alpha'') \leftrightarrow \alpha''_n$ with $\alpha''_n \in A''$ (always possible according to the condition 3 of Definition 3.2.1), then the preceding formula is equivalent in T to

$$\neg(\exists\bar{x}'\alpha' \wedge (\exists x''_1...\exists x''_{n-1}((\bigwedge_{i\in I_1} \neg(\exists\bar{y}'_i\beta'_i)) \wedge \alpha''_n))), \tag{9}$$

thus to

$$\neg(\exists\bar{x}'\alpha' \wedge (\exists x''_1...\exists x''_{n-1}\,\alpha''_n \wedge \bigwedge_{i\in I_1} \neg(\exists\bar{y}'_i\beta'_i))). \tag{10}$$

By repeating the four last steps $(n-1)$ times and by denoting by I_k the set of the $i \in I_{k-1}$ such that $x''_{(n-k+1)}$ has no occurrences in $\exists\bar{y}'_i\beta'_i$, the preceding formula is equivalent in T to

$$\neg(\exists\bar{x}'\alpha' \wedge \alpha''_1 \wedge \textstyle\bigwedge_{i\in I_n} \neg(\exists\bar{y}'_i\beta'_i)).$$

Thus, the rule (5) is correct in T.

Correctness of the rule (6):

$$\neg\begin{bmatrix} \exists\bar{x}\,\alpha \wedge \varphi\wedge \\ \neg\begin{bmatrix} \exists\bar{y}'\,\beta' \wedge \beta'' \\ \bigwedge_{i\in I} \neg(\exists\bar{z}'_i\,\delta'_i) \end{bmatrix} \end{bmatrix} \Longrightarrow \begin{bmatrix} \neg(\exists\bar{x}\,\alpha \wedge \varphi \wedge \neg(\exists\bar{y}'\,\beta' \wedge \beta''))\wedge \\ \bigwedge_{i\in I} \neg(\exists\bar{x}\bar{y}'\bar{z}'_i\,\alpha \wedge \beta' \wedge \beta'' \wedge \delta'_i \wedge \varphi)^* \end{bmatrix}$$

where $I \neq \emptyset$, $\exists\bar{y}'\,\beta' \in A'$, $\beta'' \in A''$ and $\exists\bar{z}'_i\,\delta'_i \in A'$. The formula $(\exists\bar{x}\bar{y}'\bar{z}_i\,\alpha \wedge \beta' \wedge \beta'' \wedge \delta'_i \wedge \varphi)^*$ is the formula $(\exists\bar{x}\bar{y}'\bar{z}_i\,\alpha \wedge \beta' \wedge \beta'' \wedge \delta'_i \wedge \varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables.

Let us show the correctness of this rule. Since $\exists\bar{y}'\beta' \in A'$, then according to the second point of Definition 3.2.1, we have $T \models \exists?\bar{y}'\beta'$, thus $T \models \exists?\bar{y}'\beta' \wedge \beta''$.

Thus, according to Corollary 2.0.2, the left hand side of this rule is equivalent in T to

$$\neg\left[\begin{array}{l} \exists\bar{x}\,\alpha\wedge\varphi\wedge \\ \neg\left[\ (\exists\bar{y}'\,\beta'\wedge\beta'')\wedge\bigwedge_{i\in I}\neg(\exists\bar{y}'\,\beta'\wedge\beta''\wedge(\exists\bar{z}'_i\,\delta'_i))\ \right] \end{array}\right],$$

i.e. to

$$\neg\left[\begin{array}{l} \exists\bar{x}\,\alpha\wedge\varphi\wedge \\ \neg\left[\ (\exists\bar{y}'\,\beta'\wedge\beta'')\wedge\bigwedge_{i\in I}\neg(\exists\bar{y}'\bar{z}'_i\,\beta'\wedge\beta''\wedge\delta'_i)\ \right] \end{array}\right],$$

thus to

$$\neg\left[\begin{array}{l} \exists\bar{x}'\,\alpha\wedge\varphi\wedge \\ \left[\ (\neg(\exists\bar{y}'\,\beta'\wedge\beta''))\vee\bigvee_{i\in I}(\exists\bar{y}'\bar{z}'_i\,\beta'\wedge\beta''\wedge\delta'_i)\ \right] \end{array}\right].$$

After having distributed the $\wedge$ on the $\vee$ and lifted the quantifications $\exists\bar{y}'\bar{z}'_i$ we get

$$\neg\left[\begin{array}{l} (\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'\wedge\beta''))\vee \\ \bigvee_{i\in I}(\exists\bar{x}\bar{y}'\bar{z}'_i\,\alpha\wedge\varphi\wedge\beta'\wedge\beta''\wedge\delta'_i) \end{array}\right],$$

which is equivalent in T to

$$\left[\begin{array}{l} \neg(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'\wedge\beta''))\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{x}\bar{y}'\bar{z}'_i\,\alpha\wedge\varphi\wedge\beta'\wedge\beta''\wedge\delta'_i) \end{array}\right],$$

which, by denoting by $(\exists\bar{x}\bar{y}'\bar{z}'_i\,\alpha\wedge\varphi\wedge\beta'\wedge\beta''\wedge\delta'_i)$ the formula $(\exists\bar{x}\bar{y}'\bar{z}'_i\,\alpha\wedge\varphi\wedge\beta'\wedge\beta''\wedge\delta'_i)$ in wich we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables, is equivalent in T to

$$\left[\begin{array}{l} \neg(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'\wedge\beta''))\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{x}\bar{y}'\bar{z}'_i\,\alpha\wedge\varphi\wedge\beta'\wedge\beta''\wedge\delta'_i)^* \end{array}\right].$$

Thus, the rule (6) is correct in T.

3.3.6 The decision procedure

Let ψ be a formula without free variables, the *decision* of ψ proceeds as follows:

1. Transform the formula ψ into a normalized formula φ which is equivalent to ψ in T.

2. While it is possible, apply the rewriting rules on φ. At the end, we obtain a conjunction ϕ of final formulas.

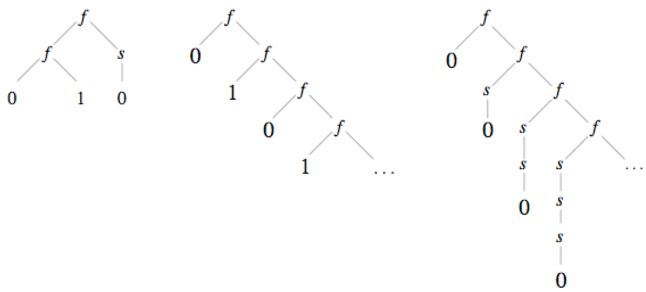
According to Property 3.3.5, the application of the rules on a formula ψ without free variables produces a wnfv conjunction ϕ of final formulas, i.e. a conjunction ϕ of final formulas without free variables. According to Property 3.3.4, ϕ is either the formula *true*, or the formula $\neg$*true*, i.e. the formula *false*.

Corollary 3.3.7 *If T is zero-infinite-decomposable then T is complete and accepts a decision procedure in the form of six rewriting rules which for every proposition give either* true *or* false *in T.*

4 Extension of first-order theories into trees

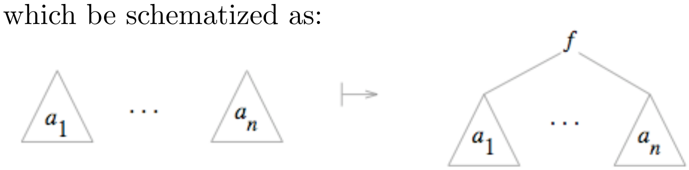
4.1 The structure of finite or infinite trees

Trees are well known objects in the computer science world. Here are some of them:



Their nodes are labeled by the symbols 0,1,s,f, of respective arities 0,0,1,2, taken from a set F of functional symbols which we assume to be infinite. While the first tree is a *finite tree* (it has a finite set of nodes), the two others are *infinite trees* and have an infinite set of nodes. We denote by A the set of all trees[2] constructed on F.

We introduce in A a set of construction operations[3], one for each element $f \in F$, which is the mapping $(a_1, ..., a_n) \rightarrow b$, where n is the arity of f and b the tree whose initial node is labeled by f and the sequence of suns is $(a_1, ..., a_n)$, and which be schematized as:



We thus obtain *the structure of finite or infinite trees* constructed on F, which we denote by (A, F).

4.2 Theory of finite or infinite trees

Let S be a signature containing only an infinite set of function symbols F. Michael Maher has introduced the S-theory of finite or infinite trees [20]. The axiomatization of this S-theory is the set of the S-propositions of the one of

[2]More precisely, we define first a node to be a word constructed on the set of strictly positive integers. A *tree* a on F, is then a mapping of type $a : E \rightarrow F$, where E is a non-empty set of nodes, each one $i_1 \dots i_k$ (with $k \geq 0$) satisfies two conditions: (1) if $k > 0$ then $i_1 \dots i_{k-1} \in E$ and (2) if the arity of $a(i_1 \dots i_k)$ is n, then the set of the nodes E of the form $i_1 \dots i_k i_{k+1}$ is obtained by giving to i_{k+1} the values $1, ..., n$.

[3]In fact, the construction operation linked to the n-ary symbol f of F is the mapping $(a_1, ..., a_n) \rightarrow b$, where the a_i's are any trees and b is the tree defined as follows from the a_i's and their set of nodes E_i's: the set E of nodes of a is $\{\varepsilon\} \cup \{ix | x \in E_i \text{ and } i \in \{1, ..., n\}$ and, for each $x \in E$, if $x = \varepsilon$, then $a(x) = f$ and if x is of the form iy, with i being an integer, $a(x) = a_i(y)$.

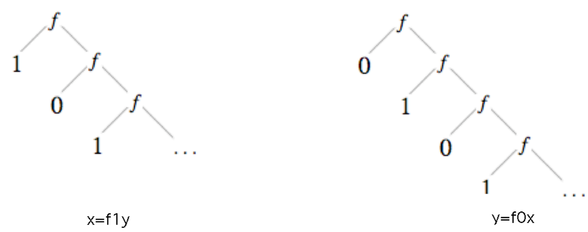
the following forms:

$$\begin{array}{ll} 1 & \forall \bar{x} \forall \bar{y}\, f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i, \\ 2 & \forall \bar{x} \forall \bar{y}\, \neg f\bar{x} = g\bar{y}, \\ 3 & \forall \bar{x} \exists! \bar{z}\, \bigwedge_i z_i = f_i(\bar{z}, \bar{x}), \end{array}$$

with f and g two distinct function symbols taken from F, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $f_i(\bar{x}, \bar{z})$ an S-term which begins with an element of F followed by variables taken from $\bar{x}\bar{z}$.

The first axiom is called *axiom of explosion*, the second one is called *axiom of conflict of symbols* and the last one is called *axiom of unique solution*.

We show that this theory has as model the structure of finite or infinite trees [12]. For example, using axiom 3, we have $T \models \exists! xy\, x = f1y \wedge y = f0x$. The individuals x and y represents the two following trees in the structure of finite or infinite trees:



4.3 Axiomatization of the theory $T + Tree$ or T^*

Let us fix now a signature S containing a set F of function symbols and a set R of relation symbols, as well as a signature S^* containing:

- an infinite set $F^* = F \cup F_A$ where F_A is an infinite set of function symbols disjoint from F.
- a set $R^* = R \cup \{p\}$ of relation symbols, containing R, and an 1-ary relation symbol p.

Let T be an S-theory. The extension of the S-theory T into trees is the S^*-theory denoted by T^* and whose set of axioms is the infinite set of the following S^*-propositions, with $\bar{x}$ a vector of variables x_i and $\bar{y}$ a vector of variables y_i:

1. Explosion: $\forall \bar{x} \forall \bar{y}\, \neg pf\bar{x} \wedge \neg pf\bar{y} \wedge f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$, for all $f \in F^*$.
2. Conflict of symbols: $\forall \bar{x} \forall \bar{y}\, f\bar{x} = g\bar{y} \rightarrow pf\bar{x} \wedge pg\bar{y}$, with f and g two distinct function symbols taken from F^*.
3. Unique solution

$$\forall \bar{x} \forall \bar{y}\ (\bigwedge_i px_i) \wedge (\bigwedge_j \neg py_j) \rightarrow \exists! \bar{z} \bigwedge_k (\neg pz_i \wedge z_k = f_k(\bar{x}, \bar{y}, \bar{z})),$$

   where $\bar{z}$ is a vector of distinct variables z_i, $f_k(\bar{x}, \bar{y}, \bar{z})$ an S^*-term which begins with a function symbol $f_k \in F^*$ followed by variables taken from $\bar{x}\bar{y}\bar{z}$, moreover, if $f_k \in F$, then the S^*-term $f_k(\bar{x}, \bar{y}, \bar{z})$ contains at least one variable from $\bar{y}\bar{z}$.

4. Relations of R: $\forall \bar{x}\ r\bar{x} \to \bigwedge_i px_i$, for all $r \in R$.

5. Operations of F: $\forall \bar{x}\ pf\bar{x} \leftrightarrow \bigwedge_i px_i$, for all $f \in F$.

6. Elements not in T: $\forall \bar{x}\ \neg pf\bar{x}$, for all $f \in F^* - F$.

7. Existence of an element satisfying p: $\exists x\, px$, (only if F does not contain 0-ary function symbols)

8. Extension of the axioms of T into trees: all axioms obtained by the following transformations of each axiom φ of T : While it is possible replace every sub-formula of φ which is of the form $\exists \bar{x}\, \psi$, but not of the form $\exists \bar{x}\, (\bigwedge px_i) \wedge \psi'$, by $\exists \bar{x}\, (\bigwedge px_i) \wedge \psi$ and every sub-formula of φ which is of the form $\forall \bar{x}\, \psi$, but not of the form $\forall \bar{x}\, (\bigwedge px_i) \to \psi'$, by $\forall \bar{x}\, (\bigwedge px_i) \to \psi$.

4.4 The standard model M^* of T^*

Let $M = (\mathcal{M}, \mathcal{F}, \mathcal{R})$ be an S-model of an S-theory T with $\mathcal{F}$ a set of functions in $\mathcal{M}$ subscripted by the elements of F and $\mathcal{R}$ a set of relations in $\mathcal{M}$ subscripted by the elements of R.

Let $M^* = (\mathcal{M}^*, \mathcal{F}^*, \mathcal{R}^*)$ be an S^*-model with $\mathcal{F}^*$ an infinite set of functions subscripted by the elements of F^* and containing the set $\mathcal{F}$, and $\mathcal{R}^* = \mathcal{R} \cup \{p\}$ a set of relations subscripted by the elements of R^* and containing the set $\mathcal{R}$ as well as an 1-ary relation p.

The extension into trees T^* of the S-theory T has as standard model the extension into trees of the S-model M, i.e. the S^*-model $M^* = (\mathcal{M}^*, \mathcal{F}^*, \mathcal{R}^*)$ defined as follows[4] :

Domain of M^* : The domain $\mathcal{M}^*$ is the set of the finite or infinite trees labeled by $F^* \cup \mathcal{M}$ by considering each n-ary symbol in F^* as a label of arity n and each individual of $\mathcal{M}$ as a label of arity 0 and such that each sub-tree labeled by $F \cup \mathcal{M}$ is evaluated in $\mathcal{M}$ and reduced to a leaf labeled by an element of $\mathcal{M}$. Since F^* does not contain function symbols of arity 0 then all the leaves of any tree a taken from $\mathcal{M}^*$ belong to $\mathcal{M}$. We understand now the semantic meaning of an extension into trees of any theory T which is finally nothing else a construction of trees on the individuals of each model M_i of T without creating new leaves that does not belong to $\mathcal{M}_\rangle$.

Operations of M^* : To each n-ary function symbol f in F^* is associated the application $f^{M^*} : \mathcal{M}^{*n} \to \mathcal{M}^*$ such that $f(a_1, .., a_n)$ is the result of f on $(a_1, .., a_n)$ in $\mathcal{M}$, if $f \in F$ and $a_i \in \mathcal{M}$ for all $i \in \{1, ..., n\}$, and is the tree whose root is labeled f and whose suns are $(a_1, .., a_n)$ else.

Relations of M^* : To each n-ary relation symbol r of $R^* - \{p\}$ is associated the set $r^{M^*} = r^M$. To the relation symbol p is associated the set $p^{M^*} = \mathcal{M}$.

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[4]By denoting by $(f^{M^*})_{f \in F^*}$ and $(r^{M^*})_{r \in R^*}$ for $\mathcal{F}^*$ respectively $\mathcal{R}^*$.

4.5 Examples

4.5.1 Extension into trees of the empty theory

Let $S = \emptyset$ be an empty signature and T be the S-empty theory. This empty theory has as model every non-empty set without any other restrictions. Let $S^* = F^* \cup \{p\}$ be a signature such that F^* is an infinite set of function symbols, each one having a non-nul arity, and p a relation symbol of arity 1. The extension into trees of T is the S^*-theory T^* whose set of axioms is the set of the following propositions:

1. Explosion: for all $f \in F^*$:

$$\forall \bar{x} \forall \bar{y}\ \neg pf\bar{x} \wedge \neg pf\bar{y} \wedge f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$$

2. Conflict of symbols: Let f and g be two distinct function symbols taken from F^*:

$$\forall \bar{x} \forall \bar{y}\ f\bar{x} = g\bar{y} \rightarrow pf\bar{x} \wedge pg\bar{y}$$

3. Unique solution

$$\forall \bar{x} \forall \bar{y}\ (\bigwedge_i px_i) \wedge (\bigwedge_j \neg py_j) \rightarrow \exists ! \bar{z} \bigwedge_k (\neg pz_i \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z}))$$

   where $\bar{z}$ is a vector of distinct variables z_i, $t_k(\bar{x}, \bar{y}, \bar{z})$ an S^*-term which begins by a function symbol $f_k \in F^*$ followed by variables taken from $\bar{x}, \bar{y}, \bar{z}$,

4. Elements not in T: for all $f \in F^*$,

$$\forall \bar{x}\ \neg pf\bar{x}$$

5. Existence of an element satisfying p :

$$\exists x\, px.$$

We can simplify this axiomatization using Axiom 4. We will also replace the relation symbol p by $leaf$ in order to clarify the intuitions of the our axiomatization. Thus, we get the following axiomatization:

1. Explosion: for all $f \in F^*$:

$$\forall \bar{x} \forall \bar{y}\ f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$$

2. Conflict of symbols: Let f and g be two distinct function symbols taken from F^* :

$$\forall \bar{x} \forall \bar{y}\ f\bar{x} = g\bar{y}$$

3. Unique solution
$$\forall \bar{x} \exists! \bar{z} \bigwedge_k z_k = t_k(\bar{x}, \bar{z})$$
where $\bar{z}$ is a vector of distinct variables z_i, $t_k(\bar{x}, \bar{z})$ an S^*-term which begins by a function symbol $f_k \in F^*$ followed by variables taken from $\bar{x}$ or $\bar{z}$,

4. Elements not in T: for all $f \in F^*$,
$$\forall \bar{x}\ \neg leaf\ f\bar{x}$$

5. Existence of an element satisfying $leaf$:
$$\exists x\, leaf\, x.$$

This axiomatization is the axiomatization of the theory of finite or infinite trees of M. Maher [20], built on the set F^* and increased by the relation symbol $leaf$ of arity 1 which distinguishes the leaves from the other trees. Nevertheless, this axiomatization forces each models of T^* to contain at least a tree reduced to a leaf. This small restriction is due to the fact that according to the definition of model, each model M of the empty theory contains at least one individual. Thus, the extension M^* of the model M contains at least an individual which will be reduced to a leaf.

4.5.2 Extension into trees ${T_{ord}}^*$ of the linear dense order relation without endpoints T_{ord}

Let F be an empty set of function symbols and let R be a set of relation symbols containing only the relation symbol $<$ of arity 2. If t_1 and t_2 are terms, then we write $t_1 < t_2$ for $<(t_1, t_2)$. Let T_{ord} the theory of the linear dense order relation without endpoints, whose signature is $S = F \cup R$ and whose axioms are the following propositions:

1 $\forall x \neg x < x$,
2 $\forall x \forall y \forall z\, (x < y \wedge y < z) \rightarrow x < z$,
3 $\forall x \forall y\, x < y \vee x = y \vee y < x$,
4 $\forall x \forall y\, x < y \rightarrow (\exists z\, x < z \wedge z < y)$,
5 $\forall x\, \exists y\, x < y$,
6 $\forall x\, \exists y\, y < x$.

Let now F^* be an infinite set of function symbols each one of a non-nul arity and $R^* = \{<, p\}$ a set of relation symbol containing the symbol $<$ as well as the relation symbol p. Let S^* be the signature $F^* \cup R^*$. According to the transformations of axioms in Section 4.3, the axiomatization of the extension into trees of the theory T_{ord} is the S^*-theory ${T_{ord}}^*$ whose axioms are the following propositions:

1 $\forall\bar{x}\forall\bar{y}\ \neg pf\bar{x} \wedge \neg pf\bar{y} \wedge f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$
2 $\forall\bar{x}\forall\bar{y}\ f\bar{x} = g\bar{y} \rightarrow pf\bar{x} \wedge pg\bar{y}$
3 $\forall\bar{x}\forall\bar{y}\ (\bigwedge_i px_i) \wedge (\bigwedge_j \neg py_j) \rightarrow \exists!\bar{z} \bigwedge_k (\neg pz_i \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z}))$
4 $\forall x\forall y\, x < y \rightarrow (px \wedge py)$,
5 $\forall\bar{x}\, \neg pf\bar{x}$,
6 $\exists x\, px$,
7 $\forall x\, px \rightarrow \neg x < x$,
8 $\forall x\forall y\forall z\, px \wedge py \wedge pz \rightarrow ((x < y \wedge y < z) \rightarrow x < z)$,
9 $\forall x\forall y\, (px \wedge py) \rightarrow (x < y \vee x = y \vee y < x)$,
10 $\forall x\forall y\, (px \wedge py) \rightarrow (x < y \rightarrow (\exists z\, pz \wedge x < z \wedge z < y))$,
11 $\forall x\, px\ \rightarrow (\exists y\, py \wedge x < y)$,
12 $\forall x\, px\ \rightarrow (\exists y\, py \wedge y < x)$,

where f and g are distinct function symbols taken from F^*, x, y, z variables, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $t_i(\bar{x}, \bar{y}, \bar{z})$ a term which begins by an element de F^* followed by variables taken from $\bar{x}$, $\bar{y}$ or $\bar{z}$.

According to axiom 5, and by replacing the relation symbol p by the relation symbol p, this axiomatization is simplified to

1 $\forall\bar{x}\forall\bar{y}\ f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$
2 $\forall\bar{x}\forall\bar{y}\ \neg(f\bar{x} = g\bar{y})$
3 $\forall\bar{x}\ \exists!\bar{z} \bigwedge_k z_k = t_k(\bar{x}, \bar{z})$
4 $\forall x\forall y\, x < y \rightarrow (p\, x \wedge p\, y)$,
5 $\forall\bar{x}\, \neg p\, f\bar{x}$,
6 $\exists x\, p\, x$,
7 $\forall x\, p\, x \rightarrow \neg x < x$,
8 $\forall x\forall y\forall z\, p\, x \wedge p\, y \wedge p\, z \rightarrow ((x < y \wedge y < z) \rightarrow x < z)$,
9 $\forall x\forall y\, (p\, x \wedge p\, y) \rightarrow (x < y \vee x = y \vee y < x)$,
10 $\forall x\forall y\, (p\, x \wedge p\, y) \rightarrow (x < y \rightarrow (\exists z\, p\, z \wedge x < z \wedge z < y))$,
11 $\forall x\, p\, x\ \rightarrow (\exists y\, p\, y \wedge x < y)$,
12 $\forall x\, p\, x\ \rightarrow (\exists y\, p\, y \wedge y < x)$,

where f and g are distinct function symbols taken from F^*, x, y, z variables, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $t_i(\bar{z}, \bar{x})$ a term which begins by an element of F^* followed by variables taken from $\bar{x}$ or $\bar{z}$.

5 Completeness of T^*

We have given a general axiomatization of T^* using the axioms of T, what about the completeness of T^*? Are all the extensions into trees complete theories? While in [15] we have shown the completeness of a combination of trees and rational numbers, in this paper the challenge is to use general properties that hold not only for rational numbers but for a large set of different theories T_i and that make T_i^* zero-infinite-decomposable and thus complete.

Let $S = F \cup R$ be a signature and T an S-theory. Let $S^* = F^* \cup R^*$ be another signature with F^* an infinite set of function symbols containing F and $R^* = R \cup \{p\}$. Let T^* be the S^*-theory of the extension of T into trees.

Suppose that the variables of V are ordered by a linear dense order relation without endpoints denoted $\succ$.

5.1 Flexible theory

Definition 5.1.1 *We call* leader *of an S-equation α the greatest variable x in α, according to the order $\succ$, such that $T \models \exists! x\alpha$.*

Definition 5.1.2 *A conjunction of S-atomic formulas α is called* formated *in T if*

- *α does not contain sub-formulas of the form $f_1 = f_2$ or $r f_1 ... f_n$ or $y = x$, where all the f_i's are 0-ary function symbols taken from F, $r \in R$ and $x \succ y$,*
- *each S-equation of α has a distinct leader which has no occurrences in other S-equations or S-relations of α,*
- *if α' is the conjunction of all the S-equations of α then for all $x \in var(\alpha')$ we have $T \models \exists ? x\, \alpha'$.*

Definition 5.1.3 *The theory T is called* flexible *if for each conjunction α of S-equations and for each conjunction β of S-relations:*

1. *$\alpha \wedge \beta$ is equivalent in T to a formated conjunction of atomic formulas wnfv,*
2. *the S-formula $\neg\beta$ is equivalent in T to a disjunction wnfv of S-equations and S-relations,*
3. *for all $x \in V$*
   - *the S-formula $\exists x\, \beta$ is equivalent in T to* false, *or to a wnfv conjunction of S-relations,*
   - *for all $x \in V$, we have $T \models \exists_{o\,\infty}^{\{faux\}} x\, \beta$.*

Let us present now the main result of this chapter:

Theorem 5.1.4 *If T is flexible then T^* is complete.*

To show this theorem we will first introduce structured formulas much more complex than the conjunctions of atomic formulas and that we call *blocks*. We will then show using these blocks that if T is flexible then T^* is zero-infinite-decomposable and thus complete.

5.2 Blocks and solved blocks in T^*

Definition 5.2.1 *A* block *is a conjunction α of S^*-formulas of the form*

- *true, false, px, $\neg px$,*
- *$x = y$, $x = f x_1 \ldots x_n$, with $f \in F^*$,*

- $t_1 = t_2 \wedge \bigwedge_{i=1}^{n} px_i$, *where* t_1 *and* t_2 *are* S*-terms and* $var(t_1 = t_2) = \{x_1, \ldots, x_n\}$,
- $rt_1 \ldots t_n$, *with* $r \in R$ *and the* t_i S*-terms,*

and such that α *contains* px *or* $\neg px$ *for all variables* $x \in \mathrm{var}(\alpha)$. *A* relational block *is a block which does not contain* S^**-equations. An* equational block *is a block which does not contain* S*-relations and where each variable has at least an occurrence in an* S^**-equation.*

Example 5.2.2 *Let us consider the* S^**-theory* ${T_{ord}}^*$. *The following* S^**-formula is a block*

$$x = fxy \wedge z = gxy \wedge \neg px \wedge py \wedge pz.$$

While the S^**-formula*

$$fxy = gyx \wedge px \wedge py,$$

is not a block because fxy *and* gyx *are not* S*-terms but* S^**-terms. The* S^**-formula*

$$fxy < gyx \wedge px \wedge py,$$

is not a block because fxy *and* gyx *are* S^**-terms and not* S*-terms. The block*

$$x = fxy \wedge y = z \wedge \neg px \wedge py \wedge pz,$$

is an equational block, while the block

$$x = fxy \wedge y = z \wedge \neg px \wedge py \wedge pz \wedge pw,$$

is not equational because the variable w *does not occur in any equation of this block.*

Definition 5.2.3 *Let* α *be a block and* $\bar{x}$ *be a vector of variables. A variable* u *is called* reachable *in* $\exists \bar{x} \alpha$ *if* u *is a free variable in* $\exists \bar{x} \alpha$, *or* α *has a sub-formula of the form* $y = t(u) \wedge \neg p\, y$ *with* $t(u)$ *an* S^**-term containing* u *and* y *a reachable variable. In the last case, the equation* $y = t(u)$ *is called reachable in* $\exists \bar{x} \alpha$.

Example 5.2.4 *Let us consider the* S^**-theory* ${T_{ord}}^*$. *In the formula*

$$\exists yz\, x = fxy \wedge y = z \wedge \neg px \wedge py \wedge pz,$$

the variables x *and* y *as well as the equation* $x = fxy$ *are reachable. The variable* z *as well as the equation* $y = x$ *are not reachable because the preceding formula does not contain sub-formulas of the form* $\neg py$.

From the general axiomatization of T^*, given in Section 4.3, and more exactly from axioms 1 and 2, we have the following property

Property 5.2.5 *Let* α *be a block. If all the variables of* $\bar{x}$ *are reachable in* $\exists \bar{x} \alpha$, *then* $T^* \models \exists ? \bar{x} \alpha$.

Definition 5.2.6 A block α is called *well-typed* if α does not contain sub-formulas of the form:

- $p\,x \wedge \neg p\,x$,
- $x = h\bar{y} \wedge p\,x$, with $h \in F^* - F$,
- $x = f_0 \wedge \neg p\,x$, with f_0 a constante of F,
- $x_0 = f x_1 ... x_n \wedge \neg p\,x_0 \wedge \bigwedge_{i=1}^{n} p\,x_i$, with $f \in F$,
- $x_0 = f x_1 ... x_n \wedge p\,x_0 \wedge \neg p\,x_i$, with $f \in F^*$
- $x_0 = x_1 \wedge p\,x_0 \wedge \neg p\,x_1$,
- $x_0 = x_1 \wedge \neg p\,x_0 \wedge p\,x_1$,
- $rt_1 ... t_n \wedge \neg p\,x_i$ with $r \in R$ and $x_i \in var(rt_1 ... t_n)$.

Definition 5.2.7 *Let α be a well-typed block. An S^*-equation of α of the form $t_1 = t_2$ is called* tree-equation *in α if for all $x \in var(t_1 = t_2)$, px is a sub-formula of α. It is called* non-tree-equation *in α if there exists $x \in var(t_1 = t_2)$ such that $\neg px$ is a sub-formula of α.*

In a block well-typed α every equation is either a tree-equation or a non-tree-equation. This property holds since in a well-typed block there exists no sub-formulas of the form $\neg px \wedge px$. Note also that all the non-tree-equations of a well-typed block α are of the form $x = y$ or $x = f\bar{y}$ with $f \in F^*$.

Definition 5.2.8 *Let α be a well-typed block. Let $x = t$, with t a term, be an S^*-tree-equation of α. The variable x is called α-leader of the S^*-equation $x = t$. Let $t_1 = t_2$, with t_1 and t_2 two S-terms, be an S^*-non-tree-equation of α. We call α-leader of the S^*-equation $t_1 = t_2$ the greatest variable x_k in $var(t_1 = t_2)$ according to the order $\succ$ such that $T \models \exists! x_k\, t_1 = t_2$.*

Example 5.2.9 *Let us consider the theories T_{ord} and $T_{ord}{}^*$. Let x, y, z be variables with $x \succ y \succ z$. Let α be the block*

$$x = fxy \wedge z = y \wedge \neg px \wedge py \wedge pz.$$

The variable x is α-leader of the S^*-equation $x = fxy$. The variable y is α-leader of the S^*-equation $z = y$ because $T_{ord} \models \exists! y\, z = y$ and $y \succ z$.

Definition 5.2.10 *A block α is called* $(\succ)$-solved *in T^* if*

1. *α is a well-typed block which does not contain sub-formulas of the form* false $\wedge\, \beta$ *with β a formula different from the formula true,*
2. *each S^*-equation of α has a distinct α-leader which does not occur in the S-relations of α,*
3. *every conjunction of S-equations and S-relations is formated in T.*

Note that from the last point of this definition and according to the definition of the formated formulas in T, we deduce that if $x = y$ is a sub-formula of the $(\succ)$-solved block α then $x \succ y$. Note also that every S^*-equation of the form $x = y$ is also an S-equation.

Example 5.2.11 *Let us consider the theory $T_{ord}{}^*$. Let x, y, z be variables with $x \succ y \succ w \succ z$. The block*

$$x = fxy \wedge y = w \wedge w = z \neg px \wedge py \wedge pz \wedge pw,$$

is not $(\succ)$-solved because w is the leader of the S-equation $w = z$ and occurs also in the S-equation $y = w$. The blocks false, true, and

$$x = fxy \wedge y = z \wedge w = z \wedge \neg px \wedge py \wedge pz \wedge pw,$$

are $(\succ)$-solved.

Property 5.2.12 *If T is flexible then every block is equivalent in T^* to a wnfv $(\succ)$-solved block.*

Proof. Let us introduce the following rewriting rules which transform a block into a wnfv $(\succ)$-solved block in T^* for every flexible theory T. To apply the rule $p_1 \Longrightarrow p_2$ to the block p means to replace in p, a sub-formula p_1 by the formula p_2, by considering the connector $\wedge$ associative and commutative.

$$
\begin{array}{lll}
(1) & p\,x \wedge \neg p\,x & \Longrightarrow \; \mathit{false} \\
(2) & x = h\bar{y} \wedge p\,x & \Longrightarrow \; \mathit{false} \\
(3) & x = f_0 \wedge \neg p\,x & \Longrightarrow \; \mathit{false} \\
(4) & x_0 = f x_1 ... x_n \wedge \neg p\,x_0 \wedge \bigwedge_{i=1}^{n} p\,x_i & \Longrightarrow \; \mathit{false} \\
(5) & x_0 = g x_1 ... x_n \wedge p\,x_0 \wedge \neg p\,x_i & \Longrightarrow \; \mathit{false} \\
(6) & x_0 = x_1 \wedge p\,x_0 \wedge \neg p\,x_1 & \Longrightarrow \; \mathit{false} \\
(7) & x_0 = x_1 \wedge \neg p\,x_0 \wedge p\,x_1 & \Longrightarrow \; \mathit{false} \\
(8) & r t_1 ... t_n \wedge \neg p\,z & \Longrightarrow \; \mathit{false} \\
(9) & \mathit{false} \wedge \alpha & \Longrightarrow \; \mathit{false}, \\
(10) & x = f_1 y_1 ... y_m \wedge x = f_2 z_1 ... z_n \wedge \neg px & \Longrightarrow \; \mathit{false}, \\
(11) & x = x & \Longrightarrow \; \mathit{true} \\
(12) & x = g y_1 ... y_n \wedge x = g z_1 ... z_n \wedge \neg px & \Longrightarrow \; x = g y_1 ... y_n \wedge \bigwedge_{i \in 1..n} y_i = z_i \wedge \neg px, \\
(13) & x = y \wedge x = g z_1 ... z_n \wedge \neg px & \Longrightarrow \; x = y \wedge y = g z_1 ... z_n \wedge \neg px, \\
(14) & x = y \wedge x = z \wedge \neg px & \Longrightarrow \; x = y \wedge z = y \wedge \neg px \\
(15) & y = x \wedge \neg px & \Longrightarrow \; x = y \wedge \neg px \\
(16) & \alpha \wedge \bigwedge_{i \in I} px_i & \Longrightarrow \; \alpha' \wedge \bigwedge_{i \in I} px_i
\end{array}
$$

with $h \in F^* - F$, f_0 a constant of F, $f \in F$, $g \in F^*$ and f_1 and f_2 two distinct elements of F^*. In the rule (8), $r \in R$ and $z \in var(rt_1...t_n)$, the rules (13), (14) and (15) are applied only if $x \succ y$. In the rule (16), α is a non-formated conjunction in T of S-atomic formulas, $var(\alpha) = \{x_1..., x_n\}$, $I = \{1, ..., n\}$ is a finite possibly empty set and α' is a formated conjunction (according to the order $\succ$) of S-atomic formulas equivalent to α in T[5]. Let us show now that

[5] The formula α' always exists since T is flexible.

every repeated application of the preceding rules on a block α terminates, keeps the equivalence in T^* and produces a wnfv $(\succ)$-solved block β.

Proof first part: every repeated application of the rules on a block terminates. Since the variables which occur in our formulas are ordered by the linear dense order relation $\succ$, we can number them by positive integers such that $x \succ y \leftrightarrow no(x) > no(y)$, where $no(x)$ is the positive integer associated to the variable x. Let us consider the 5-tuple $(n_1, n_2, n_3, n_4, n_5)$ where the n_i are the following non-negative integers:

- n_1 is the number of sub-formulas of the form $x = fy_1...y_n$, with $f \in F^*$,
- n_2 is a function which gives 1 if the formula contains a non-formated conjunction in T of S-atomic formulas and 0 otherwise,
- n_3 is the number of occurrences of atomic formulas,
- n_4 is the sum of $no(x)$ for every occurrence of a variable x,
- n_5 is the number of sub-formulas of the form $y = x$ with $x \succ y$.

for each rule there exists a row i such that the application of this rule decreases or does not change the value of the n_j with $1 \leq j < i$, and decreases the value of n_i. The row i is equal to: 3 for the rules (1)...(10), 4 for the rule (11), 1 for the rule (12), 4 for the rules (13) and (14), 5 for the rule (15) and 2 for the rule (16). To each sequence of formulas obtained by finite application of the rules, we can associate a series of 5-tuples $(n_1, n_2, n_3, n_4, n_5)$ which is strictly decreasing in the lexicographic order. Since these n_i's are positive integers they can not be negative and thus this series is finite and the application of our rules terminates.

Proof, second part: The rules keep the equivalence in T^*. The rule (1) is evident in T^*. The rule (2) comes from axiom 6 of T^*. The rules (3) and (4) come from axiom 5 of T^*. The rule (5) comes from axioms 5 and 6. The rules (6) and (7) come from the properties of the equality. The rule (8) comes from axiom 4 of T^*. The rule (9) is evident. The rule (10) comes from axiom 2 of T^*. The rule (11) is evident in T^*. The rule (12) comes from axiom 1 of T^*. The rules (13), (14) and (15) are evident in T^* and come from the properties of the equality. The rule (16) holds since T is flexible and using axioms 4,5 and 8 of T^* which enable to move from properties on T to properties on T^* by introducing typing constraints.

Proof third part: every finite application of these rules on a block produces a $(\succ)$-solved block equivalent in T^*. Let us suppose that the obtained formula is not a $(\succ)$-solved block and no-rules can be applied. Thus, at least one of the three conditions of Definition 5.2.10 does not hold. According to which condition 1 or 2 or 3 does not hold one at least of the rules (1),..., (9) or (10),(12),(13),(14), (16) or (11),(15), (16) can be applied which contradicts our supposition.

Property 5.2.13 *Let α be an equational $(\succ)$-solved block, different from the formula false. Let α^* be the conjunction of the sub-formulas of α of the form*

py or $\neg py$ with y a variable of α which is not α-leader in the S^*-equations of α. Let $\bar{x}$ be the set of the α-leaders of the S^*-equations of α. We have $T^* \models \alpha^* \rightarrow \exists! \bar{x}\alpha$.

This property comes from axiom 3 of T^* and using the fact that the block α is $(\succ)$-solved.

Example 5.2.14 *Let us consider the theory $\mathcal{T}_{ord}$. We have*

$$T^*_{ad} \models pw \wedge pz \rightarrow (\exists! xy\, x = fxw \wedge y = z \wedge \neg px \wedge py \wedge pz \wedge pw).$$

But we have not

$$T^*_{ad} \models \exists! xy\, x = fxw \wedge y = z \wedge \neg px \wedge py \wedge pz \wedge pw$$

because if we instantiate z by a tree-value for example $f1$ (f a 1-ary function symbol taken from $F^* - \{+, -, 0, 1\}$) then y will be a tree which contradicts the fact that we have py.

5.3 T^* is zero-infinite-decomposable

To show Theorem 5.1.4, it is enough to show the following theorem:

Theorem 5.3.1 *If T is flexible then T^* is zero-infinite-decomposable.*

Proof. Let T be a flexible theory. Let us show that T^* satisfies the fifth conditions of Definition 3.2.1. Let us denote by F_0 the set of the constants of F. The sets $\Psi(u)$, A, A' and A'' are chosen as follows:

Choice of the sets $\Psi(u)$, A, A' and A''

- $\Psi(u)$ is the set of the S^*-formulas of the form $\exists \bar{y}\, u = f\bar{y} \wedge \neg p\, u$, with f a function symbol taken from $F^* - F_0$.
- A is the set of the blocks.
- A' is the set of the S^*-formulas of the form $\exists \bar{x}' \alpha'$, where
  - α' is an equational $(\succ)$-solved block, different from the formula *false*, and such that the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}' \alpha'$,
  - all the variables of $\bar{x}'$ and all the S^*-tree-equations of α' are reachable in $\exists \bar{x}' \alpha'$,
  - all the variables of the S^*-non-tree-equations of α' are reachable in $\exists \bar{x}' \alpha'$,
- A'' is the set of the $(\succ)$-solved relational blocks.

Note that the set A is closed for the conjunction and A'' is a sub-set of A.

T^* satisfies the first condition of Definition 3.2.1

Let us show that every formula of the form $\exists\overline{x}\,\alpha \wedge \psi$, with $\alpha \in A$ and ψ any formula is equivalent in T^* to a wnfv S^*-formula of the form

$$\exists\overline{x}'\,\alpha' \wedge (\exists\overline{x}''\,\alpha'' \wedge (\exists\overline{x}'''\,\alpha''' \wedge \psi))), \tag{11}$$

with $\exists\overline{x}'\alpha' \in A'$, $\alpha'' \in A''$, $\alpha''' \in A$ and $T^* \models \forall\bar{x}''\,\alpha'' \rightarrow \exists!\bar{x}'''\alpha'''$.

Let us choose the order $\succ$ such that all the variables of $\bar{x}$ are greater than the free variables of $\exists\bar{x}\alpha$. Let β be the $(\succ)$-solved block of α, (β exists according to Property 5.2.12). Let X be the set of the variables of the vector $\bar{x}$. Let Y_{rea} be the set of the reachable variables in $\exists\bar{x}\beta$ and let Y_{nrea} be the set of the non reachable variables in $\exists\bar{x}\beta$. Let us rename the variables of $Y_{nrea} \cap X$ which have at least one occurrence in a non-tree-equation of β by variables greater than all the other variables of β. Note that these variables are quantified and thus we can rename them and keep the equivalence. Let β^* be the $(\succ)$-solved block of β. Let *Lead* be the set of the β^*-leaders of the S^*-equations of β^*. If $faux$ is a sub-formula of β^* then $\bar{x}' = \bar{x}'' = \bar{x}''' = \varepsilon$, $\alpha' = \mathit{true}$, $\alpha'' = \mathit{false}$ and $\alpha''' = \mathit{true}$. Else

– $\bar{x}'$ contains the variables of $X \cap Y_{rea}$,
– $\bar{x}''$ contains the variables of $(X - Y_{rea}) - \mathit{Lead}$.
– $\bar{x}'''$ contains the variables of $(X - Y_{rea}) \cap \mathit{Lead}$.
– α' is of the form $\alpha'_1 \wedge \alpha'_2$ where α'_1 is the conjunction of (1) all the tree-equations of β^* which are reachable in $\exists\bar{x}\beta^*$, (2) all the non-tree-equations of β^* whose β^*-leader is not element of $Y_{nacc} \cap X$. The formula α'_2 is the conjunction of all the sub-formulas of β^* of the form $p\,x$ or $\neg p\,x$ with x having at least an occurrence in α'_1.
– α'' is of the form $\alpha''_1 \wedge \alpha''_2$ where α''_1 is the conjunction of all the sub-formulas of β^* of the form $p\,x$ or $\neg p\,x$ with $x \notin \bar{x}'''$. The formula α''_2 is the conjunction of all the sub-formulas of β^* of the form $rt_1...t_n$ with $r \in R$ and t_i S-terms.
– α''' is of the form $\alpha'''_1 \wedge \alpha'''_2$ where α'''_1 is the conjunction of (1) all the S^*-tree-equations of β^* which are not reachable in $\exists\bar{x}\beta^*$, (2) all the S^*-non-tree-equations of β^* whose β^*-leaders belong to $Y_{nrea} \cap X$. The formula α'''_2 is the conjunction of all the sub-formulas of β^* of the form $p\,x$ or $\neg p\,x$ with x having at least an occurrence in α'''_1.

According to our construction, it is clear that $\exists\bar{x}'\alpha' \in A'$, $\alpha'' \in A''$ and $\alpha''' \in A$. Moreover, according to axiom 3 of T^* and Property 5.2.13 we have $\mathcal{T}_{ord} \models \forall\bar{x}''\alpha'' \rightarrow \exists!\bar{x}'''\alpha'''$. Let us show now that (11) and $\exists\bar{x}\alpha \wedge \psi$ are equivalents in T^*. Let X', X'' and X''' be the sets of the variables of the vectors[6] of $\bar{x}'$, $\bar{x}''$ and $\bar{x}'''$. If β^* is the formula *false*, then the equivalence of the decomposition is evident. Else, β^* is a $(\succ)$-solved block which does not contain the sub-formula *false*. Thus, according to our construction we have $X = X' \cup X'' \cup X'''$, $X' \cap X'' = \emptyset$, $X' \cap X''' = \emptyset$, $X'' \cap X''' = \emptyset$, for all $x''_i \in X''$ we have $x''_i \notin var(\alpha')$ and for all $x'''_i \in X'''$ we have $x'''_i \notin var(\alpha' \wedge \alpha'')$. These properties come from the definition of $(\succ)$-solved block and the order $\succ$ which has been chosen such that the quantified non-reachable variables are greater than the quantified

[6]Of course, if $\bar{x} = \varepsilon$ then $X = \emptyset$

reachable variables which are greater than the free variables in $\exists\bar{x}\beta^*$. On the other hand, each S^*-equation and each S^*-relation of β^* occurs in $\alpha' \wedge \alpha'' \wedge \alpha'''$ and each S^*-equation and each S^*-relation of $\alpha' \wedge \alpha'' \wedge \alpha'''$ occurs in β^* and thus $T^* \models \beta^* \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$. We have shown that the quantifications are coherent and the equivalence $T^* \models \beta^* \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$ holds. According to Property 5.2.12 we have $T^* \models \alpha \leftrightarrow \beta^*$ and thus the decomposition keeps the equivalence in T^*.

T^* satisfies the second condition of Definition 3.2.1

Let us show that T^* satisfies the second condition of Definition 3.2.1, i.e. if $\exists\bar{x}'\alpha' \in A'$ then $T^* \models \exists?\bar{x}'\alpha'$. Since $\exists\bar{x}'\alpha' \in A'$ and according to the choice of the set A', the variables of $\bar{x}'$ are reachable in $\exists\bar{x}'\alpha'$. Thus, according to Property 5.2.5 we get $T^* \models \exists?\bar{x}'\alpha'$.

Let us show now that if y is a free variable in $\exists\bar{x}'\alpha'$ then $T^* \models \exists?y\bar{x}'\,\alpha'$, or there exists $\psi(u) \in \Psi(u)$ such that $T^* \models \forall y\,(\exists\bar{x}'\,\alpha') \rightarrow \psi(y)$. Let y be a free variable of $\exists\bar{x}'\alpha'$. Since α' is an equational $(\succ)$-solved block different from *false*, then three cases arise:

Either, y occurs in a sub-formula of α' of the form $y = t(\bar{x}', \bar{z}', y) \wedge \neg py$, where $\bar{z}'$ is the set of the free variables of $\exists\bar{x}'\alpha'$ which are different from y, $t(\bar{x}', \bar{z}', y)$ is a term which begins by an element of $F^* - F_0$, followed by variables taken from $\bar{x}'$ or $\bar{z}'$ or $\{y\}$. In this case, the formula $\exists\bar{x}'\alpha'$ implies in T^* the formula

$$\exists\bar{x}'\,y = t(\bar{x}', \bar{z}', y) \wedge \neg p\,y,$$

which implies in T^* the formula

$$\exists\bar{x}'\bar{z}'w\,y = t(\bar{x}', \bar{z}', w) \wedge \neg p\,y, \tag{12}$$

where $y = t(\bar{x}', \bar{z}', w)$ is the formula $y = t(\bar{x}', \bar{z}', y)$ in which we have replaced every free occurrence of y in the term $t(\bar{x}', \bar{z}', y)$ by the variable w. According to the choice of the set $\Psi(u)$ defined in Section 5.3, the formula (12) belongs to $\Psi(y)$.

Or, y occurs in a sub-formula of α' of the form $y = z \wedge \neg py$. In this case, since y is α'-leader of the equation $y = z$, then we have $y \succ z$ (because α' is $(\succ)$-solved), and thus, z is a free variable in $\exists\bar{x}'\alpha'$ because the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists\bar{x}'\alpha'$ (thus greater than y). On the other hand, since α' is a $(\succ)$-solved block, y is not α'-leader in another equation of α' (because all the α'-leaders are distinct), thus the variable y can not occur in another left hand sides of an S^*-equation of α' (because $\neg py$ is a sub-formula of the well-typed block α'). Thus, since the variables of $\bar{x}$ are reachable in $\exists\bar{x}'\alpha'$ (according to the choice of the set A' in Section 5.3) then all the variables of $\bar{x}'$ remain reachable in $\exists\bar{x}'y\,\alpha'$. Moreover, for each value of the free variable z, there exists at most a value for y. Thus, according to Property 5.2.5 we have $T^* \models \exists?\bar{x}'y\,\alpha'$.

Or, y occurs only in sub-formulas of the form

$$x_0 = t(y) \;\; or \;\; t_1 = t_2, \tag{13}$$

with $x_0 = t(y)$ an S^*-tree-equation of α', $t(y)$ an S^*-term which begins by an element of $f \in F^*$ and contains at least an occurrence of the variable y, and $t_1 = t_2$ an S^*-non-tree-equation of α' containing at least an occurrence of y. Let us recall that according to the choice of the set A' (section 5.3), $\bar{x}'$ contains the quantified reachable variables in $\exists \bar{x}' \, \alpha'$ and all the tree-equations of α' are reachable in $\exists \bar{x}' \alpha'$. Two cases arise: (1) If y occurs in a tree-equation of α', then since y does not occur in another left hand side of a tree-equation of α', then the variables of $\bar{x}'y$ remain reachable in $\exists \bar{x}'y \, \alpha'$ and thus according to Property 5.2.5 we get $T^* \models \exists ? \bar{x}'y \, \alpha'$. (2) If y occurs only in non-tree-equations of α', then according to the choice of the set A' the variables of $\bar{x}'$ are reachable in $\exists \bar{x}' \, \alpha'$. Since y does not occur in a tree-equation of α', then the variables of $\bar{x}'$ remain reachable in $\exists \bar{x}'y \, \alpha'$. Moreover, since α' is $(\succ)$-solved then its S-equations are formated and thus $T^* \models \exists ? y \, \alpha$ and thus according to Property 5.2.5 we get $T^* \models \exists ? \bar{x}'y \, \alpha'$.

In a ll the cases, T^* satisfies the second condition of Definition 3.2.1.

T^* satisfies the third condition of Definition 3.2.1

T^* satisfies the first point of the third condition of Definition 3.2.1

Let us show that if $\alpha'' \in A''$ then the formula $\neg \alpha''$ is equivalent in T^* to a disjunction of elements of A, i.e. to a disjunction of blocks. Let α'' be an S^*-formula which belongs to A''.

According to the choice of the set A'' given in Section 5.3, either α'' is the formula *false* and thus $\neg \alpha''$ is the formula *true* which belongs to A'', or α'' is a $(\succ)$-solved relational block of the form

$$\beta \wedge (\bigwedge_{x \in X} px) \wedge (\bigwedge_{y \in Y} \neg py),$$

with β a conjunction of S-relations of the form $rt_1 ... t_n$ with $r \in R$ and $var(\beta) \subseteq X$. According to the second point of the definition of flexible theory, we have $T \models \neg \beta \leftrightarrow \beta'$ where β' is a disjunction of S-relations and S-equations. Thus, according to the axiomatization of T^* (more exactly Axioms 4,5 and 8), the S^*-formula $\neg \alpha''$ is equivalent in T^* to an S^*-formula wnfv of the form

$$(\bigvee_{k \in K} (\beta'_k \wedge p_k)) \vee (\bigvee_{x \in X} \neg px) \vee (\bigvee_{y \in Y} py),$$

where each β'_k is either an S-equation or an S-relation, p_k is a conjunction of S^*-formulas of the form px for every variable $x \in var(\beta'_k)$. It is obvious that this formula is a disjunction of blocks. Thus T^* satisfies the first point of the third condition of Definition 3.2.1.

T^* satisfies the second point of the third condition of Definition 3.2.1

Let us show that if $\alpha'' \in A''$ then for every variable x'', the S^*-formula $\exists x'' \alpha''$ is equivalent in T^* to an element of A''. Let α'' be an S^*-formula of A'', three cases arise:

(1) If x'' has no occurrences in α'', then the S^*-formula $\exists x''\, \alpha''$ is equivalent in T^* to α'' which belongs to A''.

(2) If the S^*-formula $\exists x''\, \alpha''$ is of the form $\exists x''\, \alpha''_1 \wedge \neg p\, x''$ with $\alpha''_1 \in A''$ and x'' has no occurrences in α''_1, then the S^*-formula $\exists x''\, \alpha''$ is equivalent in T^* to $\alpha''_1 \wedge (\exists x''\, \neg p\, x'')$, which according to axiom 3 of T^* is equivalent in T^* to α''_1, which belongs to A''.

(3) If the S^*-formula $\exists x''\, \alpha''$ is of the form

$$\exists x''\, \alpha''_1 \wedge \varphi,$$

with α''_1 a conjunction of S^*-relations with $x'' \notin var(\alpha''_1)$ and φ a relational block containing only S^*-relations of the form px'' or $rt_1...t_n$ with $r \in R$ and $x'' \in var(rt_1...t_n)$, then the formula $\exists x''\, \alpha''$ is equivalent in T^* to

$$\alpha''_1 \wedge \phi \wedge (\exists x''\, \varphi), \tag{14}$$

with ϕ the conjunction of the typing constraints of φ of the form px or $\neg px$ with $x \in var(\alpha''_1)$. Thus, the formula $\alpha''_1 \wedge \phi$ is a relational $(\succ)$-solved block. If φ is reduced to the formula px, then according to axiom 7 of T^*, the formula (14) is equivalent in T^* to $\alpha''_1 \wedge \phi$, which belongs to A''. Else, φ is of the form $\varphi_1 \wedge \varphi_2 \wedge px''$ where φ_1 is the conjunction of the typing constraints of φ which have no occurrences of x'', and φ_2 is the conjunction of the relations of φ of the form $rt_1...t_n$ with $r \in R$ and the t_i's S-terms. According to the last point of the definition of flexible theory, the formula $\exists x''\, \varphi_2$ is equivalent in T to *false* or to a conjunction φ'_2 wnfv of S-relations. Thus, according to axioms 8 and 4 of T^*, the formula $\exists x''\, \varphi_1 \wedge \varphi_2 \wedge px''$ is equivalent in T^* either to *false* or to the relational $(\succ)$-solved block wnfv $\varphi_1 \wedge \varphi'_2$. Thus, the formula (14) is equivalent in T^* to *false* or to

$$\alpha''_1 \wedge \phi \wedge \varphi_1 \wedge \varphi'_2,$$

which is a relational $(\succ)$-solved block. Thus, T^* satisfies the second point of the third condition of Definition 3.2.1.

T^* satisfies the third point of the third condition of Definition 3.2.1

Let us first introduce two properties which hold in each S^*-model M^* of T^*. The first one comes from the axiomatization of T^* and introduces the notion of *zero-infinite* in M^*. The second one comes from the last point of the definition of the flexible theories using also axioms 4 and 8 of T^*.

Property 5.3.2 *Let M^* be an S^*-model of T^* and $f \in F^* - F_0$. The set of the individuals i of M^*, such that $M^* \models \exists \overline{x}\, i = f\overline{x} \wedge \neg pi$, is infinite.*

Property 5.3.3 *Let M^* be an S^*-model of T^*. Let $\bigwedge_{j\in J} r_j(x)$ be a conjunction of S-relations, i.e. a conjunction of S-formulas of the form $rt_1...t_n$ with $r \in R$ and the t_i's S-terms. Let $\exists x\, \bigwedge_{j\in J} r'_j(x)$ be an instantiation of $\exists x\, \bigwedge_{j\in J} r_j(x)$ by individuals of M^*. Let $\varphi(x)$ be the formula*

$$p\, x \wedge \bigwedge_{j\in J} r'_j(x). \tag{15}$$

The set of the individuals i of M^* such that $M^* \models \varphi(i)$ is empty or infinite.

Let M^* be an S^*-model of T^*. Recall that $\Psi(u)$ is the set of the formulas of the form $\exists \bar{y}\, u = f\bar{y} \wedge \neg p\, u$, with $f \in F^* - F_0$. Let $\varphi(x)$ be a formula which belongs to A''. Let us show that for every variable x we have $T^* \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$. Let $\exists x\, \varphi'(x)$ be an instantiation of $\exists \bar{x}\, \varphi(x)$ by individuals of M^* such that $M^* \models \exists x\, \varphi'(x)$. Let us show that there exists an infinity of individuals i of M^* which satisfy

$$M^* \models \varphi'(i) \wedge \neg\psi_1(i) \wedge \cdots \wedge \neg\psi_n(i),$$

with $\psi_j(u) \in \Psi(u)$. This condition can be replaced by the following stronger one

$$M^* \models \begin{pmatrix} p\, i \vee \\ \psi_{n+1}(i) \end{pmatrix} \wedge \varphi'(i) \wedge \neg\psi_1(i) \wedge \cdot \cdot \wedge \neg\psi_n(i),$$

where $\psi_{n+1}(u)$ belongs to $\Psi(u)$ and has been chosen different from all the $\psi_1(u), \ldots, \psi_n(u)$, (always possible because $F^* - F$ is infinite according to the definition of F^*). Since for every k between 1 and n, we have

- $T^* \models p\, x \rightarrow \neg\psi_k(x)$
- $T^* \models \psi_{n+1}(x) \rightarrow \neg\psi_k(x)$ (axiom 2 of T^* conflict of symbols).

The preceding condition is simplified to

$$M^* \models (p\, i \wedge \varphi'(i)) \vee (\psi_{n+1}(i) \wedge \varphi'(i)).$$

Thus, knowing $M^* \models \exists x\, \varphi'(x)$, it is enough to show that there exists an infinity of individuals i of M^* such that

$$M^* \models p\, i \wedge \varphi'(i) \;\text{ or }\; M^* \models \psi_{n+1}(i) \wedge \varphi'(i). \tag{16}$$

two cases arise:

Either, the formula $p\, x$ occurs in $\varphi'(x)$. Since $\varphi'(x)$ is an instantiation of an equational $(\succ)$-solved block and $M^* \models \exists x\, \varphi'(x)$, then according to axiom 4 of T^*, we deduce that the S^*-formula $p\, x \wedge \varphi'(x)$ is equivalent in M^* to an S^*-formula of the form (15). According to Property 5.3.3 and since $M^* \models \exists x\, p\, x \wedge \varphi'(x)$, there exists an infinity of individuals i of M^* such that $M^* \models p\, i \wedge \varphi'(i)$ and thus such that (16).

Or, the S^*-formula $p\, x$ does not occur in $\varphi'(x)$. Since $\varphi'(x)$ is an instantiation of a relational $(\succ)$-solved block and $M^* \models \exists x\, \varphi'(x)$, then the S^*-formula $\psi_{n+1}(x) \wedge \varphi'(x)$ is equivalent in M^* to $\psi_{n+1}(x)$. According to Property 5.3.2, there exists an infinity of individuals i of M^* such that $M^* \models \psi_{n+1}(i)$, thus such that $M^* \models \psi_{n+1}(i) \wedge \varphi'(i)$ and thus such that (16).

In all the cases T^* satisfies the third condition of Definition 3.2.1.

T^* satisfies the fourth condition of Definition 3.2.1

Let us show that every conjunction of flat formulas is equivalent in T^* to a disjunction of elements of A. For that, it is enough to show that every flat formula is equivalent in T^* to a disjunction of blocks. Let φ be a flat formula.

If it is of the form *true*, *false* or $p\,x$ then φ is a block. Else the following equivalences after distribution of the $\wedge$ on the $\vee$ give the needed combinations

$$T^* \models rx_0...x_n \leftrightarrow \begin{bmatrix} rx_0...x_n \wedge \\ \bigwedge_{i=0}^{n}(p\,x_i \vee \neg p\,x_i) \end{bmatrix},$$

$$T^* \models x_0 = x_1 \leftrightarrow \begin{bmatrix} x_0 = x_1 \wedge \\ \bigwedge_{i=0}^{1}(p\,x_i \vee \neg p\,x_i) \end{bmatrix},$$

$$T^* \models x_0 = fx_1...x_n \leftrightarrow \begin{bmatrix} x_0 = fx_1...x_n \wedge \\ \bigwedge_{i=0}^{n}(p\,x_i \vee \neg p\,x_i) \end{bmatrix},$$

with $r \in R$ and $f \in F^*$. Thus T^* satisfies the fourth condition of Definition 3.2.1.

T^* satisfies the fifth condition

Let us show that for every S^*-proposition φ of the form $\exists \bar{x}'\alpha' \wedge \alpha''$ with $\exists \bar{x}'\alpha' \in A'$ and $\alpha'' \in A''$, we have $\bar{x} = \varepsilon$, $\alpha' \in \{true, false\}$ and $\alpha'' \in \{true, false\}$. Since φ does not contain free variables, then there exists no reachable variables and no reachable equations in $\exists \bar{x}'\alpha'$. Thus, according to Section 5.3, we have $\bar{x}' = \varepsilon$. According to the choice of the set A', the S^*-formula α' is a $(\succ)$-solved block different from the formula *false*, thus since $\exists \varepsilon\, \alpha'$ does not contain free variables, then α' is the formula *true*[7]. Thus, the S^*-proposition φ is of the form $\exists \varepsilon\, true \wedge \alpha''$. According to the choice of the set A'' given in Section 5.3, α'' is a relational $(\succ)$-solved block. Since it does not contain free variables then it is either the formula *true*, or the formula *false*[8]. Thus, the theory T^* satisfies the fifth condition of Definition 3.2.1.

The theory T^* satisfies all the conditions of Definition 3.2.1. Thus it is zero-infinite-decomposable and thus complete. The theorem 5.1.4 is then proved.

6 Extension into trees T^*_{ad} of ordered additive rational numbers

6.1 Axiomatization

Let $F = \{+, -, 0, 1\}$ be a set of function symbols of respective arities $2, 1, 0, 0$. Let $R = \{<\}$ a set of relation symbols containing only the binary relation symbol $<$. Let S be the signature $F \cup R$.

Note 6.1.1 *Let a be a positive integer and the $t_1, ..., t_n$'s S-terms. Let us denote by*

- *Z the set of the integers,*
- *$t_1 < t_2$, the S-term $< t_1 t_2$,*

[7] The formula α' does not contain sub-formulas of the form $f_1 = f_2$ with f_1 and f_2 constants of F because α' is $(\succ)$-solved and thus all the S-equations are formated.

[8] The formula α'' does not contain sub-formulas of the form $rf_1...f_n$ with $r \in R$ and f_i constants of F because α' is $(\succ)$-solved and thus all the S-relations are formated.

- $t_1 + t_2$, *the* S-*term* $+t_1t_2$,
- $t_1 + t_2 + t_3$, *the* S-*term* $+t_1(+t_2t_3)$,
- $0.t_1$, *the* S-*term* 0,
- $-a.t_1$, *the* S-*term* $\underbrace{(-t_1) + \cdots + (-t_1)}_{a}$,
- $a.t_1$, *the* S-*term* $\underbrace{t_1 + \cdots + t_1}_{a}$.

Let T_{ad} be the S-theory of ordered additive rational numbers. The axiomatization of T_{ad} consists in the set of the following S-propositions

1 $\forall x \forall y\, x + y = y + x$,
2 $\forall x \forall y \forall z\, x + (y + z) = (x + y) + z$,
3 $\forall x\, x + 0 = x$,
4 $\forall x\, x + (-x) = 0$,
5_n $\forall x\, n.x = 0 \rightarrow x = 0$, $\quad (n \neq 0)$
6_n $\forall x\, \exists! y\, n.y = x$, $\quad (n \neq 0)$
7 $\forall x\, \neg x < x$,
8 $\forall x \forall y \forall z\, (x < y \wedge y < z) \rightarrow x < z$,
9 $\forall x \forall y\, (x < y \vee x = y \vee y < x)$,
10 $\forall x \forall y\, x < y \rightarrow (\exists z\, x < z \wedge z < y)$,
11 $\forall x\, \exists y\, x < y$,
12 $\forall x\, \exists y\, y < x$,
13 $\forall x\, \forall y\, \forall z\, x < y \rightarrow (x + z < y + z)$,
14 $0 < 1$.
with n a non-nul integer.

Property 6.1.2

$$T_{ad} \models \sum_{i=1}^{n} a_i.x_i = a_0.1 \leftrightarrow a_k.x_k = \sum_{i=1, i\neq k}^{n} (-a_i).x_i + a_0.1,$$

for every $k \in \{1, ..., n\}$.

Let F^* be an infinite set of function symbols containing the set $\{+, -, 0, 1\}$. Let $R^* = \{<, p\}$ be a set of relation symbols containing the symbol $<$ as well as the relation symbol p. Let S^* be the signature $F^* \cup R^*$. According to the transformations of axioms given in Section 4.3, the axiomatization of T^*_{ad} is the infinite set of the following S^*-propositions:

1 $\forall \bar{x}\forall \bar{y}\,((\neg p\, f\bar{x}) \wedge (\neg p\, f\bar{y}) \wedge f\bar{x} = f\bar{y}) \rightarrow \bigwedge_i x_i = y_i,$
2 $\forall \bar{x}\forall \bar{y}\, f\bar{x} = g\bar{y} \rightarrow p\, f\bar{x} \wedge p\, g\bar{y},$
3 $\forall \bar{x}\forall \bar{y}\,(\bigwedge_{i\in I} p\, x_i) \wedge (\bigwedge_{j\in J} \neg p\, y_j) \rightarrow (\exists! \bar{z}\, \bigwedge_{k\in K}(\neg p\, z_k \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z}))),$
4 $\forall x\forall y\, x < y \rightarrow (p\, x \wedge p\, y),$
5 $\forall x\forall y\, p\, x + y \leftrightarrow p\, x \wedge p\, y,$
6 $\forall x\, p\, - x \leftrightarrow p\, x,$
7 $\forall \bar{x}\, \neg p\, h\bar{x},$
8 $\forall x\forall y\,(p\, x \wedge p\, y) \rightarrow x + y = y + x,$
9 $\forall x\forall y\forall z\,(p\, x \wedge p\, y \wedge p\, z) \rightarrow x + (y + z) = (x + y) + z,$
10 $\forall x\, p\, x \rightarrow x + 0 = x,$
11 $\forall x\, p\, x \rightarrow x + (-x) = 0,$
12_n $\forall x\, p\, x \rightarrow (nx = 0 \rightarrow x = 0), \quad (n \neq 0)$
13_n $\forall x\, p\, x \rightarrow \exists! y\, p\, y \wedge ny = x, \quad (n \neq 0)$
14 $\forall x\, p\, x \rightarrow \neg x < x,$
15 $\forall x\forall y\forall z\, p\, x \wedge p\, y \wedge p\, z \rightarrow ((x < y \wedge y < z) \rightarrow x < z),$
16 $\forall x\forall y\,(p\, x \wedge p\, y) \rightarrow (x < y \vee x = y \vee y < x),$
17 $\forall x\forall y\,(p\, x \wedge p\, y) \rightarrow (x < y \rightarrow (\exists z\, p\, z \wedge x < z \wedge z < y)),$
18 $\forall x\, p\, x \rightarrow (\exists y\, p\, y \wedge x < y),$
19 $\forall x\, p\, x \rightarrow (\exists y\, p\, y \wedge y < x),$
20 $\forall x\, \forall y\, \forall z\,(p\, x \wedge p\, y \wedge\, p\, z) \rightarrow (x < y \rightarrow (x + z < y + z)),$
21 $0 < 1,$

with f and g two distinct function symbols taken from F^*, $h \in F^* - F$, x, y, z variables, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $t_k(\bar{x}, \bar{y}, \bar{z})$ an S^*-term which begins by a function symbol f_k element of F^* followed by variables taken from $\bar{x}$ or $\bar{y}$ or $\bar{z}$. Moreover, if $f_k \in F$ then $t_k(\bar{x}, \bar{y}, \bar{z})$ contains at least a variable of $\bar{y}$ or $\bar{z}$. A similar theory has been introduced by A. Colmerauer to model the execution of Prolog III and Prolog IV [9].

Note that the theory of trees and the theory of additive ordered rational numbers have non-disjoint signatures. In fact, the symbols $+$ and $-$ are tree constructors in the theory of trees and operations of addition and subtraction in the theory of additive ordered rational numbers. Note also that T^*_{ad} does not accept full elimination of quantifiers. For example, the S^*-formula $\exists x\, y = fx$ with $f \in F - \{+, -, 0, 1\}$ can not be simplified anymore in T^*_{ad}.

6.2 Completeness

Theorem 6.2.1 *The extension into trees T^*_{ad} of ordered additive rational numbers T_{ad} is a complete theory.*

According to Theorem 5.1.4, it is enough to show that T_{ad} is flexible to get the completeness of T^*_{ad}. Thus, let us show the following property

Property 6.2.2 *The theory T_{ad} of ordered additive rational numbers is a flexible theory.*

Proof. Let us show that T_{ad} satisfies the three conditions of Definition 5.1.3. In order to simplify this proof, we will remove the prefix S from the words:

equations, relations, terms, formulas, since we will handle only the theory T_{ad} of signature S.

Let us denote by $\sum_{i=1}^{n} t_i$, the term $\overline{t_1 + t_2 + ... + t_n}$+0, where $\overline{t_1 + t_2 + ... + t_n}$ is the term $t_1 + t_2 + ... + t_n$ in which we have removed all the terms t_i which are equal to 0. For $n = 0$ the term $\sum_{i=1}^{n} t_i$ is reduced to the term 0. Formulas of the form $\sum_{i=1}^{n} a_i.x_i = a_0.1$ and $\sum_{i=1}^{n} a_i.x_i < a_0.1$ with $a_i \in Z$ are called *blocks* in T_{ad}. According to Definition 5.1.1, and since for all $x_j \in var(\sum_{i=1}^{n} a_i.x_i = a_0.1)$ with $a_j \neq 0$ we have $T_{ad} \models \exists! x_j \sum_{i=1}^{n} a_i.x_i = a_0.1$, (Property 6.1.2 and axiom 6_n of T_{ad}), then the leader of an equation of the form $\sum_{i=1}^{n} a_i.x_i = a_0.1$ is quit simply the greatest variable x_k with $k \in \{1, ..., n\}$ such that $a_k \neq 0$.

6.2.3 T_{ad} satisfies the first condition of Definition 5.1.3

Let us show that every conjunction of equations and relations is equivalent in T to a formated conjunction of atomic formulas wnfv, i.e. to a conjunction α wnfv of atomic formulas such that

1. α does not contain sub-formulas of the form $f_1 = f_2$ or $r f_1 ... f_n$ or $y = x$, where all the f_i belong to $\{0, 1\}$, $r \in \{<\}$ and $x \succ y$,

2. each equation of α has a distinct leader which have no occurrences in other equations or relations of α,

3. if α' is the conjunction of the equations of α then for all $x \in var(\alpha')$ we have $T_{ad} \models \exists ? x\, \alpha'$.

Let us introduce now the following rules that transform every conjunction of flat formulas either to *false*, or to a wnfv formated conjunction of blocks equivalent in T_{ad}.

$$
\begin{array}{lll}
(1) & \mathit{false} \wedge \alpha & \Longrightarrow \; \mathit{false} \\
(2) & 0 = 0.1 & \Longrightarrow \; \mathit{true} \\
(3) & 0 = a_0.1 & \Longrightarrow \; \mathit{false} \\
(4) & x = y & \Longrightarrow \; x + (-1).y = 0.1 \\
(5) & x = -y & \Longrightarrow \; x + y = 0.1 \\
(6) & x = y + z & \Longrightarrow \; x + (-1).y + (-1).z = 0.1 \\
(7) & \left[\begin{array}{l} \sum_{i=1}^{n} a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^{n} b_i.x_i = b_0.1 \end{array}\right] & \Longrightarrow \; \left[\begin{array}{l} \sum_{i=1}^{n} a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^{n} (b_k a_i - a_k b_i).x_i = (b_k a_0 - a_k b_0).1 \end{array}\right] \\
(8) & \left[\begin{array}{l} \sum_{i=1}^{n} a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^{n} b_i.x_i < b_0.1 \end{array}\right] & \Longrightarrow \; \left[\begin{array}{l} \sum_{i=1}^{n} a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^{n} \lambda (b_k a_i - a_k b_i).x_i < (b_k a_0 - a_k b_0).1 \end{array}\right]
\end{array}
$$

In the rule (3), $a_0 \neq 0$. In the rules (7) and (8), the variable x_k is the leader of the equation $\sum_i a_i.x_i = a_0.1$ and $b_k \neq 0$. In the rule (8), $\lambda = 1$ if $a_k > 0$ and $\lambda = -1$ otherwise. Of course, every repeated application of these rules terminates and produces either *false* or a formated conjunction wnfv of blocks equivalent in T_{ad}.

Let α be a conjunction of atomic formulas. By introducing quantified variables to transform the formulas into flat formulas, α is equivalent in T_{ad} to a formula of the form $\exists \bar{x}\, \beta$ with β a conjunction of flat formulas. Let us choose the order $\succ$ such that the variables of $\bar{x}$ are greater than free variables of $\exists \bar{x}\, \beta$. Let δ be the formula obtained from β after application of the preceding rules. Two cases arise:

Either, δ is the formula *false*, thus the formula $\exists \bar{x}\, \delta$ is equivalent to *false* in T_{ad}, thus the conjunction α is equivalent to *false* in T_{ad} which is a formated atomic formula.

Or, δ is a formated conjunction of blocks such that each variable of $\bar{x}$ has an occurrence as leader in an equation of δ. This restriction comes from the order $\succ$ which has been chosen such that the variables of $\bar{x}$ are greater than the free variables of $\exists \bar{x}\, \beta$. Thus, the formula δ is of the form

$$(\bigwedge_{i \in I} \delta_{x_i}) \wedge \delta^*,$$

where each δ_{x_i} is an equation of δ whose leader x_i is a variable of $\bar{x}$ and where δ^* is a conjunction of blocks which does not contain occurrences of the variables of $\bar{x}$. The formula $\exists \bar{x} \beta$ is then equivalent in T_{ad} to

$$\delta^* \wedge (\exists \bar{x} \bigwedge_{i \in I} \delta_{x_i}),$$

which since each leader x_i does not occur in another equation, is equivalent in T_{ad} to

$$\delta^* \wedge \bigwedge_{i \in I} (\exists x_i\, \delta_{x_i}),$$

which since for each leader x_i we have $T_{ad} \models \exists ! x_i \delta_{x_i}$, is equivalent in T_{ad} to δ^*. Thus, the formula α is equivalent to the formula δ^* which is a formated conjunction of blocks and thus a conjunction of atomic formulas. Then, the theory T_{ad} satisfies the first condition of Definition 5.1.3.

6.2.4 T_{ad} satisfies the second condition of Definition 5.1.3

Let us show that every formula of the form $\neg \alpha$ where α is a conjunction of relations is equivalent in T_{ad} to a disjunction of equations and relations. According to the preceding point, the formula α is equivalent in T_{ad} to a conjunction of blocks of the form $\sum_{j=1}^{n} b_j.x_j < b_0.1$. Since the order is linear then we have

$$T_{ad} \models \neg(\sum_{j=1}^{n} b_j.x_j < b_0.1) \leftrightarrow ((\sum_{j=1}^{n} (-b_j).x_j < (-b_0).1) \vee (\sum_{j=1}^{n} b_j.x_j = b_0.1))$$

Thus, the formula $\neg \alpha$ is equivalent in T_{ad} to a disjunction of blocks, and thus to a disjunction of equations and relations. The theory T_{ad} satisfies the second condition of Definition 5.1.3.

6.2.5 T_{ad} satisfies the third condition of Definition 5.1.3

Let us show that for every conjunction of relations β and every variable x we have:

- the formula $\exists x\,\beta$ is equivalent in T_{ad} either to *false*, or to a wnfv conjunction of relations,
- $T_{ad} \models \exists_{o\,\infty}^{\{faux\}} x\,\beta$.

The first point is evident and comes from the Fourier elimination of quantifiers. The second point holds since the order is dense and without endpoints. Let M be a model of T_{ad}. For every instantiation $\exists x\,\beta'(i)$ of $\exists x\,\beta(i)$ by individuals of M, if $M \models \beta'(i)$ then there exists an infinity of individuals i of M such that $M \models \beta'(i)$, thus $T_{ad} \models \exists_{o\,\infty}^{\{faux\}} x\,\beta$.

The theory T_{ad} is flexible, and thus the extension into trees T^*_{ad} is zero-infinite-decomposable. Consequently it is complete according to Theorem 5.1.4.

7 Conclusion

We have defined in this paper a general idea for the extension of the models of Prolog by giving an automatic way to combine any first order theory T with the theory of finite or infinite trees. To show the completeness of T^* we have introduced the flexible theories and have shown that if T is flexible then T^* zero-infinite-decomposable. The zero-infinite-decomposable theories are first order theories having elegant properties which enable us to decide the validity of any proposition using only six rewriting rules. The main idea behind this rules consists in a local decomposition of quantified conjunctions of hybrid atomic formulas, a partial elimination of quantifiers using the properties of the vectorial quantifiers, and a special distribution to decrease the depth of the formulas.

There exists many practical applications of the extensions into trees of first order theories. First-order constraints on trees can be expressed in a simpler way when they are in the extension into trees of another structure. For example, the constraints representing the moves in two players games introduced by Alain Colmerauer and Thi-Bich-Hanh Dao [11, 12] can be represented by a simpler constraint in the extension into trees of the integers together with the operations of addition and subtraction and a linear dense order relation.

On the other hand, our decision algorithm can decide the validity or not validity of big and complex propositions and can also be applied on formulas having free variables and produces in this case a boolean combination of basic formulas which does not accept full elimination of quantifiers. Unfortunately, this algorithm is not able to detect formulas having free variables and being always equivalent to *false* or *true* in T^*. It does not warrant that a final formula having at least one free variable is neither true nor false in T^* and can not present the solutions of the free variables in a clear and explicit way. This is why our algorithm is called *decision procedure* and not a general algorithm for solving first order constraints. It would be interesting to transform our decision procedure into a general algorithm for solving any first order constraint in T^* and which presents the solutions of the free variables in a clear and explicit way, as it has been done in [11, 12] for the theory of finite trees and finite or infinite trees. This kind of algorithm needs another work completely different from this one, by introducing syntactic and semantic definitions much more complex than the definition of flexible theories given in this paper. The implementation of a such algorithm will enable us to extend the Prolog language by allowing the user to solve any complex first order constraint, with or without free variables, in many combinations of theories around trees.

Currently, we are trying to proof that every extension of a complete theory into trees is complete and may be zero-infinite-decomposable. For that, we expect to add new vectorial quantifiers in the decomposition such as $\exists^n$ which means *there exists* n and $\exists_{n,\infty}^{\Psi(u)}$ which means *there exists* n *or infinite*, in order to increase the size of the set of the zero-infinite-decomposable theories and may be get a much more simple definition than the one defined in this paper. We plan also with Thom Fruehwirth [17] to add to CHR a general mechanism to treat our normalized formulas. This will enable us to implement quickly and easily our algorithms and get a general idea on the expressiveness of first order constraints in combinations of trees and first order theories.

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