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Rapport de Recherche

Inductive Characterizations of Finite Interval Orders and Semiorders

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Inductive Characterizations of Finite Interval Orders and Semiorders

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Abstract

We introduce an inductive definition for two classes of orders. By simple proofs, we show that one corresponds to the interval orders class and that the other is exactly the semiorders class. To conclude we consider most of the known equivalence theorems on interval orders and on semiorders, and we give simple and direct inductive proofs of these equivalences with ours characterizations.

Key words: antichain, characterization, decomposition, finite order, inductive definition, interval order, partially ordered sets, semiorder.

1 Introduction

In the following we are only concerned by finite orders. We present two classes of orders defined inductively by decompositions of the order's ground sets into three sets (X_1, X_2, Z) the two firsts being none void. These decompositions rely upon two classical order's notions which are the antichain and the series composition. Particularly, we assume both that X_2 and Z forms an antichain, and that X_1 and X_2 are in series composition. The difference between the two classes is achieved by a further condition on the order relations between elements of X_1 and elements of Z. In section 3 and in section 4, we give simple proofs of the fact that these classes are characterized by forbidden suborders. These characterizations provide the equivalence of one class with the interval orders class, and the equivalence of the other class with the semiorders class. The inductive definition thus obtained for the interval orders and for the semiorders are, up to our knowledge, so far unknown. In section 5 and in section 6 we consider most of the known equivalence theorems on interval orders and on semiorders, and we give simple and direct inductive proofs of these equivalences with ours

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characterizations. Note that, in 2003, B. Balof and K.P. Bogart [2] already give inductive proofs for the Fishburn and Mirkin characterization theorem of interval orders and for the Scott-Suppes characterization theorem of semiorders.

2 Preliminaries

For general order terminology and known results we refer to B.S.W. Schröder [11] and to W.T. Trotter [13]. We only recall some of them and fix some notations. An order P is a couple $(V(P), <_P)$ such that $<_P$ is a transitive and antireflexive binary relation with ground set $V(P)$. Given an order P, $Max(P)$ (resp. $Min(P)$) denotes the set of its maximal (resp. minimal) elements. For every $x \in V(P)$, $\downarrow_P^{[} x$ (resp. $\uparrow_P^{[} x$) denotes the predecessor (resp. successor) set of x. For every $A, B \subseteq V(P)$, $P[A]$ denotes the sub-order of P induced by A, that is $P[A] = (A, (A \times A) \cap <_P)$, and $P[A] <_P P[B]$ means that $\forall a \in A$, $\forall b \in B$ holds $a <_P b$. Given two orders $P_1 = (V(P_1), <_{P_1})$ and $P_2 = (V(P_2), <_{P_2})$ on disjoint ground sets, the series composition of P_1 and P_2 is the order $P_1 \otimes P_2 = (V(P_1) \cup V(P_2), (V(P_1) \times V(P_2)) \cup <_{P_1} \cup <_{P_2})$. Notice that if $P_1 = \emptyset$ then $P_1 \otimes P_2 = P_2$, and that the behaviour is similar when $P_2 = \emptyset$. An order P is said to be an antichain if $<_P = \emptyset$. Similarly, a subset A of $V(P)$ is an antichain of P if its elements are pairwise incomparable in P, that is, if $(A \times A) \cap <_P = \emptyset$. The set of all the antichains of P is denoted $A(P)$.

3 Recursively Antichain-Series Decomposable Orders and Interval Orders

Definition 1 *An order P is Antichain-Series decomposable if its ground set $V(P)$ is the disjoint union of three sets X_1, X_2 and Z which fulfill (i) X_1 and X_2 are none void, (ii) $P[X_1] <_P P[X_2]$, and (iii) $P[X_2 \cup Z]$ is an antichain. The family (X_1, X_2, Z) is then called an Antichain-Series decomposition of P.*

Remark 1 *Since $P[X_1] <_P P[X_2]$, and since $P[X_2 \cup Z]$ is an antichain, it then immediately follows that $X_2 \cup Z = Max(P)$. So, taking $X_2 \cup Z = Max(P)$ for condition (iii) in the definition would not have induced a loss of generality.*

Definition 2 *An order P is Recursively Antichain-Series decomposable if either it is an antichain, or it has an Antichain-Series decomposition, say (X_1, X_2, Z), such that $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable.*

Theorem 1 *An order P is Recursively Antichain-Series decomposable if and only if it has no sub-order isomorphic to the $2 \oplus 2$ order (see Figure 1 (a)).*

Proof. To show the forward implication we proceed by contradiction. Let P be a Recursively Antichain-Series decomposable order having sub-orders isomorphic to the $2 \oplus 2$ and assume that P is of minimal cardinality over all such orders. Let $\{a, b, c, d\}$ be a subset of $V(P)$ inducing a $2 \oplus 2$ sub-order with $a <_P b$ and

$c <_P d$, and let (X_1, X_2, Z) be an Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable. Then, as P is minimal in cardinality, we have that $\{a, b, c, d\} \cap X_2 \neq \emptyset$. Now, without loss of generality, assume that $b \in X_2$. Consequently, we obtain that $\{c, d\} \cap X_1 = \emptyset$, and thus that $\{c, d\} \subseteq (X_2 \cup Z)$: which contradicts that $P[X_2 \cup Z]$ is an antichain.

To show the backward implication we proceed by induction on $|V(P)|$. The base case being obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$ and, to avoid trivial cases, that P is not an antichain. Then, as P has no sub-order isomorphic to the $2 \oplus 2$, it follows that for $x \in V(P)$ with maximal ideal size we have that $\downarrow_P^{[} x = V(P) \backslash Max(P)$. Then, with $X_1 = V(P) \backslash Max(P)$, $X_2 = \{x\}$ and $Z = Max(P) \setminus \{x\}$, we obtain an Antichain-Series decomposition of P: note that $X_1 \neq \emptyset$ since P is not an antichain. We now conclude by induction hypothesis on $P[X_1 \cup Z]$. $\square$

This characterization by forbidden sub-order provides the equivalence between Recursively Antichain-Series decomposable orders and interval orders. Recall that an order $P = (V(P), <_P)$ is said to be an *interval order* if it can be represented by assigning a real interval $I_x = [l(x), r(x)]$ to each element $x \in V(P)$, such that $x <_P y$ if and only if $r(x) <_{\mathbb{R}} l(y)$ for all $x, y \in V(P)$. The family $(I_x)_{x \in V(P)}$ is then said to be an *interval representation* of P. These orders, introduced in 1914 by N. Wiener [14], have since extensively been studied and several characterizations have been obtained: see the books of B.S.W. Schröder [11], of P.C. Fishburn [4] and of W.T. Trotter [13]. However, up to our knowledge, none of them is inductive. We obtain such a characterization by the following one due to P.C. Fishburn [3] and B.G. Mirkin [6].

Theorem 2 [P.C. Fishburn [3], B.G. Mirkin [6]] *An order P is an interval order if and only if it has no sub-order isomorphic to the $2 \oplus 2$ order.*

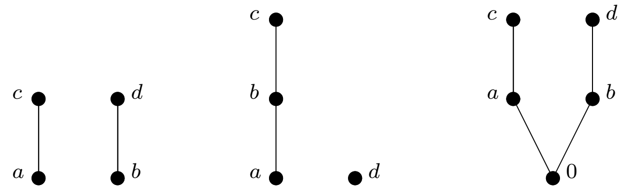
Now directly from both Theorem 1 and Theorem 2 we obtain:

Corollary 1 *An order is an interval order if and only if it is Recursively Antichain-Series decomposable.*

Remark 2 *If we replace condition (iii), in the definition of Antichain-Series decomposable orders, by "$P[X_2]$ and $P[Z]$ are both antichains", then the order of Figure 1 (c) would be Recursively Antichain-Series decomposable by taking, for example, $X_2 = \{c, d\}$ and $Z = \{a, b\}$.*

4 Recursively Full-Antichain-Series Decomposable Orders and Semiorders

Definition 3 *An order P is Full-Antichain-Series decomposable if its ground set $V(P)$ is the disjoint union of three sets X_1, X_2 and Z which fulfill (i) X_1 and X_2 are none void, (ii) $P[X_1] <_P P[X_2]$, (iii) $P[X_2 \cup Z]$ is an antichain, and (iv) $\forall z \in Z$, we have $(X_1 \setminus Max(P[X_1])) \subseteq \downarrow_P^{[} z$. The family (X_1, X_2, Z) is then called a Full-Antichain-Series decomposition of P.*



(a) **(b)** **(c)**

Figure 1: (a): the $2 \oplus 2$ order; (b): the $3 \oplus 1$ order; (c): a none interval order.

Definition 4 *An order P is Recursively Full-Antichain-Series decomposable if either it is an antichain, or it has a Full-Antichain-Series decomposition, say (X_1, X_2, Z), such that $P[X_1 \cup Z]$ is still Recursively Full-Antichain-Series decomposable.*

Theorem 3 *An order P is Recursively Full-Antichain-Series decomposable if and only if it has no sub-order isomorphic neither to the $2 \oplus 2$ order nor to the $3 \oplus 1$ order (see Figure 1 (a) and (b)).*

Proof. To show the forward implication, as in the proof of Theorem 1, we proceed by contradiction. Thus, let P a Recursively Full-Antichain-Series decomposable order the $3 \oplus 1$ order, and assume that P is of minimal cardinality over all such orders. As, by Theorem 1, P cannot contain a $2 \oplus 2$ sub-order, let $\{a, b, c, d\}$ be a subset of $V(P)$ inducing a $3 \oplus 1$ sub-order with $a <_P b <_P c$. Let (X_1, X_2, Z) be a Full-Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is Recursively Full-Antichain-Series decomposable. Then, as P is minimal in cardinality, we have that $\{a, b, c, d\} \cap X_2 \neq \emptyset$. Now, either $d \in X_2$ and thus $\{a, b, c\} \subseteq Z$ which contradicts that Z is an antichain. Or, $c \in X_2$ and thus we have both $\{a, b\} \subseteq X_1$ and $d \in Z$, consequently $a \in (X_1 \setminus Max(P[X_1]))$ but now $a \parallel_P d$: a contradiction.

To show the backward implication, again, we follow the same steps than in the proof of Theorem 1. Simply notice that with $X_2 = \{x\}$ where $x \in V(P)$ and x has a maximal ideal size, with $X_1 = V(P) \setminus Max(P)$, and with $Z = Max(P) \setminus \{x\}$, we obtain a Full-Antichain-Series decomposition of P. Indeed, if there exists $z \in Z$ and $y_1 \in (X_1 \setminus Max(P[X_1])) \setminus \downarrow^{[}_{P} z$ then $y_1 \parallel_P z$ and thus, due to the transitivity, for every $y_2 \in Max(P[X_1])$ such that $y_1 <_P y_2$ we have that $y_2 \parallel_P z$ since $z \in Max(P)$ (see Remark 1). Consequently with $\{y_1, y_2, x, z\}$ we obtain a sub-order of P isomorphic to a $3 \oplus 1$: a contradiction. □

An order $P = (V(P), <_P)$ is said to be a *semiorder* if it can be represented by assigning a real interval $I_x = [l(x), r(x)]$ of unit length, that is $r(x) - l(x) = 1$, to each element $x \in V(P)$, such that $x <_P y$ if and only if $r(x) <_{\mathbb{R}} l(y)$ for all $x, y \in V(P)$. These orders have been introduced in 1956 by R.D. Luce [5], but, up to our knowledge, none of the characterizations since established is inductive. To obtain such a characterization we use the next one due to D. Scott and P. Suppes [12].

Theorem 4 [D. Scott and P. Suppes [12]] *An order P is a semiorder if and only if it has no sub-order isomorphic neither to the $2 \oplus 2$ order nor to the $3 \oplus 1$ order.*

Now directly from both Theorem 3 and Theorem 4 we obtain:

Corollary 2 *An order is a semiorder if and only if it is Recursively Full-Antichain-Series decomposable.*

5 Interval Orders's Equivalence Theorems

In this section we recall most of the known characterization theorems on interval orders. For each of these characterizations we give a simple and direct equivalence proof with the class of the Recursively Antichain-Series decomposable orders.

5.1 Direct Equivalences

We begin by showing that Recursively Antichain-Series decomposable orders are exactly those orders having an interval representation. Combining this result with our previous Theorem 1 this gives the classical Fishburn and Mirkin characterization theorem of interval orders. Notice that the inductive proof of our forward implication has, on certain points, some similarities with the inductive proof, of the Fisburn and Mirkin theorem, given by B. Balof and K.P. Bogart [2].

Theorem 5 *An order P is Recursively Antichain-Series decomposable if and only if it is an interval order.*

Proof. For the two implications we proceed by induction on $|V(P)|$. As the base cases are obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid trivial cases we also assume that P is not an antichain.

For the forward implication let (X_1, X_2, Z) be an Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable. Let $(I_x)_{x \in V(P[X_1 \cup Z])}$ be an interval representation of $P[X_1 \cup Z]$ obtained by induction hypothesis. Then let $gr = max\{r(x) : x \in X_1\}$ and let $r_Z = max\{gr + \frac{3}{2}, max\{r(x) : x \in Z\}\}$. Let $I'(x) = I(x)$ for $x \in X_1$, let $I'(x) = [l(x), r_Z]$ for $x \in Z$ and let $I'(x) = [gr + \frac{1}{2}, r_Z]$ for $x \in X_2$. It is then immediate that for every $x_2 \in X_2$ we have both that for every $x_1 \in X_1$ holds $r(x_1) <_{\mathbb{R}} l(x_2)$, and that for every $x_z \in Z$ holds $[l(x_2), r(x_2)] \cap [l(x_z), r(x_z)] \neq \emptyset$. Also, as we only increase interval's right end points for the elements of Z and as $Z \subseteq Max(P[X_1 \cup Z])$, we have that $(I'_x)_{x \in V(P[X_1 \cup Z])}$ is still an interval representation of $P[X_1 \cup Z]$. Thus, $(I'_x)_{x \in V(P)}$ is a suitable interval representation for P.

For the backward implication let $(I_x)_{x \in V(P)}$ be an interval representation of P. Let $gl = max\{l(x) : x \in V(P)\}$, let $X_2 = \{x : x \in V(P) : l(x) = gl\}$, let $Z = \{x : x \in V(P) : (l(x) \neq gl) \ and \ (r(x) \geq gl)\}$ and let $X_1 = V(P) \setminus (X_2 \cup Z)$.

Since obviously $X_1 <_P X_2$, $X_2 \cap Z = \emptyset$ and $(X_2 \cup Z) \in A(P)$, the result follows immediatly by induction hypothesis. Notice that the assumption "P is not an antichain" implies that $X_1 \neq \emptyset$. $\square$

Recall that an antichain C, of an order P, is said to be maximal if every element of $V(P)$ is either in C or comparable with some element of C. The set of all the maximal antichains forms a lattice by ordering its corresponding ideals by inclusion. That is, given A and B two maximal antichains, we have A is strictly less than B if and only if $(\downarrow_P^{[} A \cup A) \subsetneq (\downarrow_P^{[} B \cup B)$. We denote this order by $AM(P)$, and we refer to it as the lattice of maximal antichains of P. Notice that its greatest and its least element are respectively $Max(P)$ and $Min(P)$, and notice that the following well known property comes directly from the definition of the order relation.

Property 1 *Let P be an order and let $A, B \in V(AM(P))$. If $B <_{AM(P)} A$, then for every $x \in A \setminus B$ and for every $C \in V(AM(P))$ such that $C \leq_{AM(P)} B$ we have that $x \notin C$.*

In 1991, K. Reuter [9] shows that an order is an interval order if and only if its lattice of maximal antichains is totally ordered. In the following, we give a version of that theorem for Recursively Antichain-Series decomposable orders.

Theorem 6 *An order P is Recursively Antichain-Series decomposable if and only if its lattice of maximal antichains is totally ordered.*

Proof. For the two implications we proceed by induction on $|V(P)|$. As the base cases are obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid trivial cases we also assume that P is not an antichain

For the forward implication, let (X_1, X_2, Z) be an Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable. Then, the result follows quite immediately by induction hypothesis on $P[X_1 \cup Z]$. Indeed, due to the series compositions $X_1 <_P X_2$, if $Max(P[X_1 \cup Z]) = Z$ then $V(AM(P)) = (V(AM(P[X_1 \cup Z])) \setminus \{Z\}) \cup \{X_2 \cup Z\}$, and otherwise $V(AM(P)) = V(AM(P[X_1 \cup Z])) \cup \{X_2 \cup Z\}$.

For the backward implication, assume that $AM(P)$ is totaly ordered, let $\top$ be its greatest element, and let $pred_\top$ be its (unique) immediate predecessor in $AM(P)$. Let $X_2 = \top \setminus pred_\top$, let $Z = \top \setminus X_2$ and let $X_1 = V(P) \setminus \top$. Since $\top = Max(P)$, in order to conclude by induction hypothesis, it remains to show that both $AM(P[X_1 \cup Z])$ is totally ordered and that $X_1 <_P X_2$. For the former case, it suffices to show that $V(AM(P)) \setminus \{\top\} = V(AM(P[X_1 \cup Z]))$ since for every $x \in X_1 \cup Z$ we have that $\downarrow_{P[X_1 \cup Z]}^{[} x = \downarrow_P^{[} x$. First, for every $A \in V(AM(P)) \setminus \{\top\}$, it follows, by Property 1, that $A \subseteq V(P[X_1 \cup Z])$, and consequently that $A \in V(AM(P[X_1 \cup Z]))$. Second, by contradiction, let $A \in V(AM(P[X_1 \cup Z])) \setminus V(AM(P))$. Then, there exists $Y \subseteq X_2$ such that $(A \cup Y) \in V(AM(P))$. Consequently, we have that $Y = X_2$ and then follows that $A = Z$. Which contradicts that $pred_\top \in V(AM(P[X_1 \cup Z]))$ and that $Z \subsetneq pred_\top$ ($Z = \top \cap pred_\top$ and $Z \neq pred_\top$ since the element of $AM(P)$ are maximal for inclusion). For the latter case, assume that there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1 \parallel_P x_2$. Thus, there exists $A \in (V(AM(P)) \setminus \{\top\})$ such that $\{x_1, x_2\} \subseteq A$: which leads to a contradiction with Property 1 and $x_2 \in \top \setminus pred_\top$. $\square$

For any order P and for any disjoint subsets A and B of $V(P)$, $[A|B]_P$ denotes, if there exists one, a linear extension of $P[A \cup B]$ such that $a <_{[A|B]_P} b$ whenever both $a \in A$, $b \in B$ and $a \parallel_{P[A \cup B]} b$. By convention if $A = B = \emptyset$ we say that $[A|B]_P$ exists. In 1978, I. Rabinovitch [8] shows that an order P is an interval order if and only if $[A|B]_P$ exists for every two disjoint subsets A and B of $V(P)$. We give here a version of that theorem for Recursively Antichain-Series decomposable orders.

Theorem 7 *An order P is Recursively Antichain-Series decomposable if and only if $[A|B]_P$ exists for every two disjoint subsets A and B of $V(P)$.*

Proof. For the forward implication we proceed by induction on $|V(P)|$. As the base case is obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid trivial cases we also assume that P is not an antichain. By aim of simplicity, for every $x \in V(P)$, we denote $P[V(P) \setminus \{x\}]$ by $P - x$. Let (X_1, X_2, Z) be an Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable. Thus, for any $x \in X_2$ and for $Z' = Z \cup (X_2 \setminus \{x\})$, we have that $(X_1, \{x\}, Z')$ is still an Antichain-Series decomposition of P such that $P[X_1 \cup Z']$ is Recursively Antichain-Series decomposable: it is sufficient to use the $(X_1, X_2 - \{x\}, Z)$ Antichain-Series decomposition of $P - x$ when $X_2 \neq \{x\}$. Let A and B be two any disjoint subsets of $V(P)$. Firstly, if $x \notin A$, then, quite immediately from the induction hypothesis on $P - x$, we have that $[A|B]_P$ exists. Indeed, when $x \in B$, then, for example, we can set $[A|B]_P = [A|(B \setminus \{x\})]_{P-x} \otimes \{x\}$. Secondly, assume that $x \in A$: by induction hypothesis we consider $[(A \setminus \{x\})|B]_{P-x}$. Let $B^< = \{b : b \in B : b <_P x\}$ and let $b^<_{max}$ be the greatest element of $B^<$ in $[(A \setminus \{x\})|B]_{P-x}$. Let $B^\parallel = \{b : b \in B : b \parallel_P x\}$ and let $b^\parallel_{min}$ be the least element of $B^\parallel$ in $[(A \setminus \{x\})|B]_{P-x}$. Then we have that $A \cap \{y : y \in A \cup B : b^\parallel_{min} \leq_{[(A\setminus\{x\})|B]_{P-x}} y \leq_{[(A\setminus\{x\})|B]_{P-x}} b^<_{max}\} = \emptyset$. Indeed, by contradiction, let a be an element belonging to this intersection: notice that $a \neq x$. Then, by definition of $[(A \setminus \{x\})|B]_{P-x}$, it follows that $b^\parallel_{min} <_{P-x} a$ and thus that $b^\parallel_{min} <_P a$. Consequently, due to transitivity and by definition of $b^\parallel_{min}$, we have that $a \parallel_P x$. Thus $\{b^\parallel_{min}, a\} \subseteq Z'$ which contradicts that Z' is an antichain. Now let $D^\parallel = \{b : b \in B^\parallel : b^\parallel_{min} \leq_{[(A\setminus\{x\})|B]_{P-x}} b <_{[(A\setminus\{x\})|B]_{P-x}} b^<_{max}\}$. As, for every $y \in D^\parallel$ and for every $t \in B^<$, we don't have $y <_P t$, and as for every $a \in (A \cap \downarrow^{[}_{P} x)$ we have $a <_{[(A\setminus\{x\})|B]_{P-x}} b^\parallel_{min}$, it then follows that there exists $[A|B]_P = [(A \setminus \{x\})|B]_{P-x}[\{y : y \in (A \cup B) \setminus D^\parallel : y \leq_{[(A\setminus\{x\})|B]_{P-x}} b^<_{max}\}] \otimes (\{x\}, \emptyset) \otimes [(A \setminus \{x\})|B]_{P-x}[D^\parallel] \otimes [(A \setminus \{x\})|B]_{P-x}[\{y : y \in A \cup B : b^<_{max} <_{[(A\setminus\{x\})|B]_{P-x}} y\}]$.

For the backward implication we proceed by contrapositive: by Theorem 1 assume that $\{a, b, c, d\}$ is a subset of $V(P)$ inducing a $2 \oplus 2$ sub-order such that $a <_P b$ and $c <_P d$. But now, $[A|B]_P$ does not exist for $A = \{b, d\}$ and $B = \{a, c\}$. $\square$

5.2 Two More Equivalences

In this subsection we are interested by the characterizations of interval orders in terms of their succesors sets and predecessors sets. First, in order to simplify the proofs, we need to precise the correlations between P and $P[X_1 \cup Z]$ when we consider Antichain-Series decompositions maximal for inclusion on X_2.

Lemma 1 *Let P be a Recursively Antichain-Series decomposable order, and let (X_1, X_2, Z) be its (unique) Antichain-Series decomposition maximal for inclusion on X_2. Then, $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable and $Max(P[X_1 \cup Z]) \cap X_1 \neq \emptyset$.*

Proof. Let (X_1, X_2, Z) be an Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable and such that $A = \{x :\ x \in Z :\ \downarrow^{[}_{P} x = X_1\}$ is of minimal cardinality over all such decomposition.

If $A = \emptyset$, then clearly (X_1, X_2, Z) is maximal for inclusion on X_2 and $Max(P[X_1 \cup Z]) \cap X_1 \neq \emptyset$. Indeed, if $Max(P[X_1 \cup Z]) \cap X_1 = \emptyset$ then in every (X'_1, X'_2, Z') Antichain-Series decomposition of $P[X_1 \cup Z]$ holds $X_1 = X'_1$ thus the fact that $X'_2 \neq \emptyset$ contradicts $A = \emptyset$.

If $A \neq \emptyset$, we have that $Max(P[X_1 \cup Z]) = Z$ and then whenever (X'_1, X'_2, Z') is an Antichain-Series decomposition of $P[X_1 \cup Z]$ holds $X_1 = X'_1$. Thus, we obtain that $X'_2 \subseteq A$. Consequently $(X_1, X_2 \cup X'_2, Z \setminus X'_2)$ is an Antichain-Series decomposition of P such that $P[X_1 \cup (Z \setminus X'_2)]$ is still Recursively Antichain-Series decomposable: which contradicts the minimality of A. $\square$

For any order P, we define $U(P) = \{\uparrow^{[}_{P} x :\ x \in V(P)\}$ and $D(P) = \{\downarrow^{[}_{P} x :\ x \in V(P)\}$. It is well known, see C.H. Papadimitriou and M. Yannakakis [7], that interval orders are characterized by the fact that their predecessors sets can be totaly ordered by inclusion, that is: an order P is an interval order if and only if $D(P)$ is linearly ordered by inclusion. We present next a rewriting of that theorem for Recursively Antichain-Series decomposable orders.

Theorem 8 *An order P is Recursively Antichain-Series decomposable if and only if $D(P)$ is linearly ordered by inclusion.*

Proof. For the two implications we proceed by induction on $|V(P)|$. As the base cases are obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid trivial cases we also assume that P is not an antichain.

For the forward implication, let (X_1, X_2, Z) be an Antichain-Series decomposition of P maximal for inclusion on X_2. Notice that this implies that for all $x, y \in X_2$ we have that $\downarrow^{[}_{P} x = \downarrow^{[}_{P} y$, and that for every $x \in X_2$ and $z \in Z$ we have that $\downarrow^{[}_{P} z \subsetneq \downarrow^{[}_{P} x$ and $\downarrow^{[}_{P} x = X_1$. Thus, as for every $x \in X_1 \cup Z$ we have that $\downarrow_P^{[} x = \downarrow_{P[X_1 \cup Z]}^{[} x$, and as by Lemma 1 $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable, we immediately conclude by induction hypothesis on $P[X_1 \cup Z]$ since $D(P) = D(P[X_1 \cup Z]) \cup \{X_1\}$.

For the backward implication, let X_1 be the greatest element of $D(P)$, let $X_2 = \{x : x \in V(P) : \downarrow_P^{[} x = X_1\}$ and let $Z = V(P) \setminus (X_1 \cup X_2)$. Notice that, as X_1 is the greatest element of $D(P)$, we have that both $X_2 \neq \emptyset$ and $X_2 \subseteq Max(P)$. To conclude that (X_1, X_2, Z) is an Antichain-Series decomposition of P it suffices to notice that $Z \subseteq Max(P)$ since, otherwise, the predecessor set containing $z \in Z \setminus Max(P)$ cannot be included in X_1. Since by construction $D(P[X_1 \cup Z]) = D(P) \setminus \{X_1\}$, by induction hypothesis we conclude that $P[X_1 \cup Z]$ is Recursively Antichain-Series decomposable. $\square$

Remark 3 *Due to its stability under duality, it follows that an order P is Recursively Antichain-Series decomposable if and only if $U(P)$ is linearly ordered by inclusion.*

We are now interested in the following characterization, due to P.C. Fishburn, which gives an interval representation of the order by mean of the cardinals of the equivalence classes on the predecessors and on the successors sets. First, for any order P and for every $x \in V(P)$, we define $bl_P(x) = |\{d \in D(P) :\ d \subsetneq \downarrow_P^{[} x\}|$, and $br_P(x) = |\{u \in U(P) :\ \uparrow_P^{[} x \subsetneq u\}|$.

Theorem 9 [P.C. Fishburn [4]] *An order P is an interval order if and only if $(x <_P y \Longleftrightarrow br_P(x) <_{\mathbb{N}} bl_P(y))$.*

In order to establish the same equivalences with the Recursively Antichain-Series decomposable orders's class, we need more insights into the structure of the Antichain-Series decompositions.

Remark 4 *Let (X_1, X_2, Z) be an Antichain-Series decomposition of P, then we obtain the following relations between the successors and the predecessors in P and in $P[X_1 \cup Z]$.*

For the predecessors sets:

1. *if $x \in X_1 \cup Z$, we have $\downarrow_{P[X_1 \cup Z]}^{[} x = \downarrow_P^{[} x \subseteq X_1$,*
2. *if $x \in X_2$, we have $\downarrow_P^{[} x = X_1$.*

For the successors sets:

3. *if $x \in X_2$, we have $\uparrow_P^{[} x = \emptyset$,*
4. *if $x \in Z$, we have $\uparrow_P^{[} x = \uparrow_{P[X_1 \cup Z]}^{[} x = \emptyset$,*
5. *if $x \in X_1$, we have $\uparrow_P^{[} x = \uparrow_{P[X_1 \cup Z]}^{[} x \cup X_2$.*

Consequently we obtain:

6. $D(P) = D(P[X_1 \cup Z]) \cup \{X_1\}$, *and*

7. *either* $Max(P[X_1 \cup Z]) \cap X_1 = \emptyset$, *and then* $U(P) = \{\emptyset\} \cup \{U \cup X_2 : U \in U(P[X_1 \cup Z]) : U \neq \emptyset\}$,

8. *or* $Max(P[X_1 \cup Z]) \cap X_1 \neq \emptyset$, *and then* $U(P) = \{\emptyset\} \cup \{U \cup X_2 : U \in U(P[X_1 \cup Z])\}$.

Proposition 1 *Let P be a Recursively Antichain-Series decomposable order, and let (X_1, X_2, Z) be its (unique) Antichain-Series decomposition maximal for inclusion on X_2, then:*

1. *$D(P)$ is the disjoint union of $D(P[X_1 \cup Z])$ and $\{X_1\}$, and thus we have that $|D(P)| = |D(P[X_1 \cup Z])| + 1$,*

2. *$U(P) = \{\emptyset\} \cup \{U \cup X_2 : U \in U(P[X_1 \cup Z])\}$, and thus we have that $|U(P)| = |U(P[X_1 \cup Z])| + 1$.*

Proof. Item 1: from the condition of maximality upon X_2 we directly obtain that for every $z \in Z$ we have $\downarrow^{[}_{P} z \subsetneq X_1$. Consequently, the item 1 of Remark 4 can be now written as: if $x \in X_1 \cup Z$, we have $\downarrow^{[}_{P[X_1 \cup Z]} x = \downarrow^{[}_{P} x \subsetneq X_1$. Thus $D(P)$ is the disjoint union of $D(P[X_1 \cup Z])$ and $\{X_1\}$.

Item 2 is an immediate consequence of Lemma 1 and Remark 4 item 8. $\square$

Lemma 2 *Let P be a Recursively Antichain-Series decomposable order, then $|U(P)| = |D(P)|$.*

Proof. We proceed by induction on $|V(P)|$. As the base case is obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid a trivial case we also assume that P is not an antichain. Let (X_1, X_2, Z) be the Antichain-Series decomposition of P maximal for inclusion on X_2, as by Lemma 1 we have that $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable it remains to show, for example, that both $|D(P)| = |D(P[X_1 \cup Z])| + 1$ and $|U(P)| = |U(P[X_1 \cup Z])| + 1$: which corresponds to Proposition 1. $\square$

We can now establish the equivalence.

Theorem 10 *An order P is Recursively Antichain-Series decomposable if and only if ($x <_P y \iff br_P(x) <_{\mathbb{N}} bl_P(y)$).*

Proof. For the two implications we proceed by induction on $|V(P)|$. As the base cases are obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid trivial cases we also assume that P is not an antichain.

For the forward implication, let (X_1, X_2, Z) be the Antichain-Series decomposition of P maximal for inclusion on X_2. By Proposition 1 (1), for $x \in X_1 \cup Z$, we have $bl_P(x) = bl_{P[X_1 \cup Z]}(x)$. By Proposition 1 (2), for $x_1 \in X_1$, we have $br_P(x_1) = br_{P[X_1 \cup Z]}(x_1)$ and for $z \in Z$ we have $br_P(z) = |U(P)| - 1 = |U(P[X_1 \cup Z])|$. As by Lemma 1, $P[X_1 \cup Z]$ is still Recursively Antichain-Series decomposable, the induction hypothesis insures that $x <_{P[X_1 \cup Z]} y \iff$

$br_{P[X_1 \cup Z]}(x) <_{\mathbb{N}} bl_{P[X_1 \cup Z]}(y)$. Consequently, as moreover $Z \subsetneq Max(P[X_1 \cup Z])$ and thus for every $x_1 \in X_1$ and for every $z \in Z$ holds $bl_{P[X_1 \cup Z]}(x_1) \leq_{\mathbb{N}} br_{P[X_1 \cup Z]}(z)$, then, for $x, y \in (X_1 \cup Z)$ we have $x <_P y \Longleftrightarrow br_P(x) <_{\mathbb{N}} bl_P(y)$. Now, for $x_2 \in X_2$, by Proposition 1 (1), we have $bl_P(x_2) = |D(P)| - 1 = |D(P[X_1 \cup Z])|$, and by Proposition 1 (2), we have $br_P(x_2) = |U(P)| - 1 = |U(P[X_1 \cup Z])|$. Consequently, due to Lemma 2, first for every $z \in Z$ and every $x_2 \in X_2$ holds $br_P(z) = bl_P(x_2)$, and second, for every $x_1 \in X_1$ and every $x_2 \in X_2$ holds $br_P(x_1) <_{\mathbb{N}} bl_P(x_2)$ since for every order Q and every $q \in V(Q)$ we have that $br_Q(q) \leq_{\mathbb{N}} |U(Q)| - 1$. So, as both $Z \cup X_2 = Max(P)$ and $X_1 <_P X_2$, we obtain the result we were looking for.

For the backward implication, define $Mbl(P) = max\{bl_P(x) : x \in V(P)\}$ and then let $X_2 = \{x : x \in V(P) : bl_P(x) = Mbl(P)\}$, let $X_1 = \{x : x \in V(P) : br_P(x) < Mbl(P)\}$, and let $Z = V(P) \setminus (X_1 \cup X_2)$. To begin notice that the fact that (X_1, X_2, Z) is an Antichain-Series decomposition of P directly follows from that first $X_1 \neq \emptyset$ since P is not an antichain, that second $X_1 <_P X_2$ since $\forall x_1 \in X_1$ and $\forall x_2 \in X_2$ holds $br_P(x_1) <_{\mathbb{N}} bl_P(x_2)$, and that third $X_2 \cup Z \in A(P)$ since $\forall z \in Z$ and $\forall x_2 \in X_2$ holds $br_P(z) \geq_{\mathbb{N}} bl_P(x_2)$. Now, in order to conclude using the induction hypothesis, it remains to show that $P[X_1 \cup Z]$ fulfills $x <_{P[X_1 \cup Z]} y \Longleftrightarrow br_{P[X_1 \cup Z]}(x) <_{\mathbb{N}} bl_{P[X_1 \cup Z]}(y)$. As (X_1, X_2, Z) is an Antichain-Series decomposition of P, we immediatly deduce from Remark 4 item (1) and item (6), that for every $x \in (X_1 \cup Z)$ holds $bl_{P[X_1 \cup Z]}(x) = bl_P(x)$. Moreover, with $U^\star(P[X_1 \cup Z]) = U(P[X_1 \cup Z]) \setminus \{\emptyset\}$, we obtain that the mapping ϕ from $U^\star(P[X_1 \cup Z])$ to $U(P)$, defined by $\phi(U) = U \cup X_2$, is injective. Then, from Remark 4 item (7) and item (8), it follows that $U(P) \subseteq \{\phi(U) : U \in U^\star(P[X_1 \cup Z])\} \cup \{\emptyset\} \cup \{X_2\}$. Consequently, as for every $x_1 \in X_1$ and for every $U \in U(P[X_1 \cup Z])$, $\uparrow^{[}_{P[X_1 \cup Z]} x_1 \subsetneq U$ implies that both $U \in U^\star(P[X_1 \cup Z])$ and $\uparrow^{[}_{P} x_1 \subsetneq \phi(U)$, then $br_{P[X_1 \cup Z]}(x_1) \leq_{\mathbb{N}} br_P(x_1)$ and thus $br_{P[X_1 \cup Z]}(x_1) = br_P(x_1)$. Indeed, $br_{P[X_1 \cup Z]}(x_1) <_{\mathbb{N}} br_P(x_1)$ implies that there exists $t \in \{\emptyset, X_2\}$ such that $\uparrow^{[}_{P} x_1 \subsetneq t$: which is impossible. Consequently, since $Z \subseteq Max(P[X_1 \cup Z])$, it follows that, for every $x, y \in (X_1 \cup Z)$ we have $x <_{P[X_1 \cup Z]} y \Longleftrightarrow (br_{P[X_1 \cup Z]}(x) <_{\mathbb{N}} bl_{P[X_1 \cup Z]}(y))$. Indeed, on the one hand if $x <_{P[X_1 \cup Z]} y$ then $x \notin Z$ and thus $br_{P[X_1 \cup Z]}(x) = br_P(x)$. On the other hand, for every $z \in Z$ we have that $br_{P[X_1 \cup Z]}(z) = |U(P[X_1 \cup Z])| - 1$ and thus, as by Lemma 2 $U(P[X_1 \cup Z]) = D(P[X_1 \cup Z])$, there doesn't exist $y \in V(X_1 \cup Z)$ such that $br_{P[X_1 \cup Z]}(z) <_{\mathbb{N}} bl_{P[X_1 \cup Z]}(y)$: indeed for every order Q and every $q \in V(Q)$ we have that $bl_Q(q) \leq_{\mathbb{N}} |D(Q)| - 1$. Consequently whenever $br_{P[X_1 \cup Z]}(x) <_{\mathbb{N}} bl_{P[X_1 \cup Z]}(y)$ we have that $x \in X_1$ and thus that $br_{P[X_1 \cup Z]}(x) = br_P(x)$. $\square$

6 Semiorders's Equivalence Theorems

In this section, we consider most of the known equivalence theorems on semiorders, and we give simple and direct inductive proofs of these equivalences with ours characterizations. Notice that in 2003 B. Balof and K.P. Bogart [2] already give inductive proofs of the Fishburn and Mirkin characterization theorem of interval orders and of the Scott-Suppes characterization theorem of semiorders.

We begin by showing that Recursively Full-Antichain-Series decomposable orders are exactly those orders having an interval representation with unit length intervals. Again, combining this result with our previous Theorem 3 this gives the Scott-Suppes charaterization theorem of semiorders. Notice that the inductive proof of our forward implication has, on certain points, some similarities with the inductive proof, of the Scott-Suppes theorem, given by B. Balof and K.P. Bogart [2].

Theorem 11 [D. Scott and P. Suppes [12]] *An order is Recursively Full-Antichain-Series decomposable if and only if it is a semiorder.*

Proof. For the two implications we proceed by induction on $|V(P)|$. As the base cases are obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid trivial cases we also assume that P is not an antichain.

For the forward implication, in order to simplify the proof, we also assume that the interval representation is such that whenever two intervals intersect then this intersection contains more than one point. Let (X_1, X_2, Z) be a Full-Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is still Recursively Full-Antichain-Series decomposable. Let $(I_x = [l(x), r(x)])_{x\in V(P[X_1\cup Z])}$, be an interval representation of $P[X_1 \cup Z]$ fulfilling the induction hypothesis. To begin notice that, if $Z = \emptyset$, we can immediately conclude with the family $(I''_x = [l''_x, r''_x])_{x\in V(P)}$ such that $I''_x = I_x$ for $x \in (X_1 \cup Z)$ and by setting for every $x_2 \in X_2$, $l''(x_2) = 1 + max\{r(x_1) : x_1 \in X_1\}$ and $r''(x_2) = 1 + l''(x_2)$. Now, assume that $Z \neq \emptyset$, let $a = max\{l(x) : x \in Max(P[X_1])\}$, let $b = min\{r(x) : x \in Max(P[X_1])\}$, and notice that $a <_{\mathbb{R}} b$. Then, for every $z \in Z$ let $l'(z) = max\{l(z), \frac{a+b}{2}\}$ and $r'(z) = l'(z) + 1$. Consider the intervals family $(I'_x = [l'(x), r'(x)])_{x\in V(P[X_1\cup Z])}$ defined by $I'_x = I_x$ for every $x \in X_1$ and by $I'_x = [l'(x), r'(x)]$ for every $x \in Z$. Now, we show that $(I'_x)_{x\in V(P[X_1\cup Z])}$ is a same unit length interval representation of $P[X_1 \cup Z]$ such that whenever two intervals intersect then this intersection contains more than one point. Let $a'_Z = max\{l'(z) : z \in Z\}$, let $b'_Z = min\{r'(z) : z \in Z\}$, let $Z' = \{z : z \in Z : l(z) \neq l'(z)\}$ and notice that if $Z' = \emptyset$ then there is nothing to do. Thus assume that $Z' \neq \emptyset$.

First, we show that for every $z' \in Z'$, and for every $x \in (X_1 \cup Z)$, if $I'_x \cap I'_{z'} \neq \emptyset$ then $|I'_x \cap I'_{z'}| \neq 1$. Clearly by contrustuction this is true for $x \in (Max(P[X_1]) \cup Z')$. Moreover, as for every $x \in (X_1 \setminus Max(P[X_1]))$ we have $r'(x) = r(x) <_{\mathbb{R}} a$, we then remain with $x \in Z \setminus Z'$: but as for $x \in Z \setminus Z'$ we have $\frac{a+b}{2} \leq l(x) = l'(x)$, thus the only possibility is that $r'(z') = l(x)$. But, as $r(z') <_{\mathbb{R}} r'(z')$ we now obtain a contradiction with the fact that $(I_x)_{x\in V(P[X_1\cup Z])}$ is an interval representation of $P[X_1 \cup Z]$.

Second, we show that for every $z' \in Z'$ and for every $z \in Z$ holds $I'_{z'} \cap I'_z \neq \emptyset$: which is true if for example $a'_Z <_{\mathbb{R}} b'_Z$. So, assume that $b'_Z <_{\mathbb{R}} a'_Z$. Since $Z' \neq \emptyset$, by construction, we have that $b'_Z = \frac{a+b}{2} + 1$, this means that $\frac{a+b}{2} + 1 <_{\mathbb{R}} a'_Z$ and thus that there exists $z \in Z \setminus Z'$ such that $\frac{a+b}{2} + 1 <_{\mathbb{R}} l(z)$. Then, as $r(z') <_{\mathbb{R}}$

$r'(z') = \frac{a+b}{2} + 1$ we have a contradiction with the fact that $(I_x)_{x \in V(P[X_1 \cup Z])}$ is an interval representation of $P[X_1 \cup Z]$. So, we have that $a'_Z \leq_{\mathbb{R}} b'_Z$ and, since an intersection cannot be reduced to one point, thus we have $a'_Z <_{\mathbb{R}} b'_Z$. Since for every $z' \in Z'$, on the one hand for every $x_1 \in Max(P[X_1])$ holds $I'_{z'} \cap I'_{x_1} \neq \emptyset$, and on the other hand for every $x_1 \in (X_1 \setminus Max(P[X_1]))$ holds $r'(x_1) = r(x_1) <_{\mathbb{R}} a < \frac{a+b}{2} = l'(z')$, then $(I'_x)_{x \in V(P[X_1 \cup Z])}$ is an unit length interval representation of $P[X_1 \cup Z]$.

Now, as for every $x_1 \in X_1$ we have that $r'(x) = r(x) \leq_{\mathbb{R}} a + 1$ and as $\frac{3a+b+4}{4} <_{\mathbb{R}} \frac{a+b}{2} + 1 \leq_{\mathbb{R}} b'_Z$ we can immediately conclude with the family $(I''_x = [l''_x, r''_x])_{x \in V(P)}$ such that $I''_x = I'_x$ for $x \in (X_1 \cup Z)$ and by setting for every $x_2 \in X_2$, $l''(x_2) = max\{a'_Z, \frac{3a+b+4}{4}\}$ and $r''(x_2) = 1 + l''(x_2)$.

For the backward implication, let $(I_x = [l(x), r(x)])_{x \in V(P)}$, be an interval representation of P such that all intervals have the same unit length. Let $l_m = max\{l(x) : x \in V(P)\}$, let $r = max\{r(x) : x \in V(P) : r(x) <_{\mathbb{R}} l_m\}$. First, notice that, since P is not an antichain, r is well defined and then the set $X_1 = \{x : x \in V(P) : r(x) \leq_{\mathbb{R}} r\}$ is none void. Second, let $X_2 = \{x : x \in V(P) : r <_{\mathbb{R}} l(x)\}$, then $X_2 \neq \emptyset$, due to l_m, and moreover we have $X_1 <_P X_2$. Third, let $Z = V(P) \setminus (X_1 \cup X_2)$. Then, immediatly from the definitions of l_m, X_2 and Z, we obtain that $Max(P)$ is the disjoint union of Z and X_2. So, since by induction hypothesis $P[X_1 \cup Z]$ is clearly Recursively Full-Antichain-Series decomposable, in order to conclude it remains to show the condition (iv) of Definition 3. On the contrary, assume that both there exists $x \in (X_1 \setminus Max(P[X_1]))$ and there exists $z \in Z$ such that $x \parallel_P z$ (notice that obviously we don't have $z \leq_P x$). Thus, we have that $l(z) \leq_{\mathbb{R}} r(x)$ and $r <_{\mathbb{R}} r(z)$. But now, for any $y \in Max(P[X_1])$ such that $x <_P y$ we have that $l(z) \leq_{\mathbb{R}} r(x) <_{\mathbb{R}} l(y) \leq_{\mathbb{R}} r(y) \leq r <_{\mathbb{R}} r(z)$. Then we get a contradiction with the fact that, in $(I_x)_{x \in V(P)}$, all the intervals have the same length. $\square$

The equivalence between unit interval graphs and proper interval graphs, due to F.S. Roberts [10], has this immediate and well known interpretation for orders: an order P is a semiorder if and only if it can be represented by assigning a positive real interval to each of its elements such that no interval is strictly included in an other. In the following we present a version of that theorem for Recursively Full-Antichain-Series decomposable orders.

Theorem 12 *An order P is Recursively Full-Antichain-Series decomposable if and only if it can be represented by assigning a positive real interval to each of its elements such that no interval is strictly included in an other.*

Proof. For the two implications we proceed by induction on $|V(P)|$. As the base cases are obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid trivial cases we also assume that P is not an antichain.

For the forward implication, let (X_1, X_2, Z) be a Full-Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is still Recursively Full-Antichain-Series decomposable. Let $(I_x = [l(x), r(x)])_{x \in V(P[X_1 \cup Z])}$, be an interval representation of $P[X_1 \cup Z]$ such that no interval is strictly included in an other: which exists by induction hypothesis. First notice that, if $Z = \emptyset$, we can immediately conclude by setting for every $x_2 \in X_2$, $l(x_2) = r(x_2) = 1 + max\{r(x_1) : x_1 \in X_1\}$. Now, assume that $Z \neq \emptyset$, let $l_Z = min\{l(z) : z \in Z\}$ and let $r = -1$ if $(X_1 \setminus Max(P[X_1])) = \emptyset$, and $r = max\{r(x) : x \in (X_1 \setminus Max(P[X_1]))\}$ otherwise. Let $r_m = min\{r(x) : x \in Max(P[X_1])\}$, notice that $r <_{\mathbb{R}} r_m$ since, by definition of r, there exists $x \in Max(P[X_1])$ such that $r <_{\mathbb{R}} l(x)$ and thus $\forall y \in X_1$ with $r(y) <_{\mathbb{R}} l(x)$ we have $y \notin Max(P[X_1])$. As, by condition (iv) of Definition 3, we have that $r <_{\mathbb{R}} l_Z$, with $l = min\{l_Z, r_m\}$ we obtain $r <_{\mathbb{R}} l$. Thus there exists a mapping α from $\{x : x \in Max(P[X_1]) : r <_{\mathbb{R}} l(x)\}$ into $]r, l[$ such that for all $x, y \in \{x : x \in Max(P[X_1]) : r <_{\mathbb{R}} l(x)\}$ we have $l(x) <_{\mathbb{R}} l(y)$ if and only if $\alpha(x) <_{\mathbb{R}} \alpha(y)$ and $l(x) = l(y)$ if and only if $\alpha(x) = \alpha(y)$. Now, let $r_M = max\{r(x) : x \in (X_1 \cup Z)\}$. Then, there exists a mapping β from Z into $[r_M + 1, r_M + 2]$ such that for all $x, y \in Z$ we have $r(x) <_{\mathbb{R}} r(y)$ if and only if $\beta(x) <_{\mathbb{R}} \beta(y)$ and $r(x) = r(y)$ if and only if $\beta(x) = \beta(y)$. Now, consider the family $(I'_x)_{x \in V(P[X_1 \cup Z])}$ defined by:

- $I'_x = I_x$ if $x \in X_1$ and $l(x) \leq_{\mathbb{R}} r$,
- $I'_x = [\alpha(x), r(x)]$ if $x \in Max(P[X_1])$ and $r <_{\mathbb{R}} l(x)$,
- $I'_x = [l(x), \beta(x)]$ if $x \in Z$.

To show that $(I'_x = [l'(x), r'(x)])_{x \in V(P[X_1 \cup Z])}$ is an interval representation of $P[X_1 \cup Z]$ we proceed by contradiction: as for every $x \in (X_1 \cup Z)$ we have $I_x \subseteq I'_x$, this means that there exist $x, y \in (X_1 \cup Z)$ such that $x <_{P[X_1 \cup Z]} y$ and $I'_x \cap I'_y \neq \emptyset$. Consequently, we have that $x \notin Z$ and thus that $r'(x) = r(x)$. This implies that $l'(y) \neq l(y)$ and so that both $y \in Max(P[X_1])$ and $r <_{\mathbb{R}} l(y)$. Thus, we get that $x \notin Max(P[X_1])$ and then that $r'(x) = r(x) \leq_{\mathbb{R}} r$, which implies that $l'(y) \leq_{\mathbb{R}} r$: a contradiction with α into $]r, l[$. To show that, in $(I'_x)_{x \in V(P[X_1 \cup Z])}$, no interval is strictly included in an other, we also proceed by contradiction: let $x, y \in V(P[X_1 \cup Z])$ such that $I'_x \subsetneq I'_y$. Notice that by definition either $l'(y) = l(y)$ or $r'(y) = r(y)$. First, assume that both $l'(y) = l(y)$ and $r'(y) = r(y)$, but then we have that $I_x \subseteq I'_x \subsetneq I'_y = I_y$: a contradiction. Second, assume that both $l'(y) \neq l(y)$ and $r'(y) = r(y)$, then $y \in Max(P[X_1])$ and $r <_{\mathbb{R}} l(y)$. Thus, due to α, either $x \in X_1$ and $l(x) \leq_{\mathbb{R}} r$, or $x \in Z$. In the former case we get a contradiction with $r <_{\mathbb{R}} l'(y) \leq_{\mathbb{R}} r'(x)$ and $r'(x) = r(x) \leq_{\mathbb{R}} r$. In the latter case we get a contradiction with β into $[r_M + 1, r_M + 2]$ and $r'(y) \leq_{\mathbb{R}} r_M$. Third, assume that both $l'(y) = l(y)$ and $r'(y) \neq r(y)$, then $y \in Z$. Thus, due to β, either $x \in X_1$ and $l(x) \leq_{\mathbb{R}} r$, or $x \in Max(P[X_1])$ and $r <_{\mathbb{R}} l(x)$. In the former case we get a contradiction with $r <_{\mathbb{R}} l(y) = l'(y)$ (condition (iv) of Definition 3) and $l'(y) \leq_{\mathbb{R}} l'(x) = l(x) \leq_{\mathbb{R}} r$. In the latter case we get a contradiction with α into $]r, l[$ and $l \leq_{\mathbb{R}} l(y) = l'(y)$. By setting for every $x_2 \in X_2$, $l'(x_2) = r_M + 1$ and $r'(x_2) = 2 + l'(x_2)$, we now obtain an interval representation of P such that no interval is strictly included in an other. To finish with positive real intervals, we only have to translate every interval to the right with a same value being more than one.

Starting with $(I_x = [l(x), r(x)])_{x \in V(P)}$, an interval representation of P such that no interval is strictly included in an other, the backward implication follows exactly the same lines than the backward implication of Theorem 11 by changing the last sentence in: "Then we get a contradiction with the fact that, in $(I_x)_{x \in V(P)}$, no interval is strictly included in an other.". $\square$

In 1976, P. Avery [1] shows that an order P is Recursively Full-Antichain-Series decomposable if and only if it has a linear extension L such that for every $x, y \in V(P)$ if $x \leq_L y$ then both $\downarrow^{[}_{P} x \subseteq \downarrow^{[}_{P} y$ and $\uparrow^{[}_{P} y \subseteq \uparrow^{[}_{P} x$. We give next a version of that theorem for Recursively Full-Antichain-Series decomposable orders.

Theorem 13 *An order P is Recursively Full-Antichain-Series decomposable if and only if it has a linear extension L such that for every $x, y \in V(P)$ if $x \leq_L y$ then both (a) $\downarrow^{[}_{P} x \subseteq \downarrow^{[}_{P} y$ and (b) $\uparrow^{[}_{P} y \subseteq \uparrow^{[}_{P} x$.*

Proof. For the two implications we proceed by induction on $|V(P)|$. As the base cases are obvious with $|V(P)| = 1$, we simply have to consider that $|V(P)| \geq 2$. To avoid trivial cases we also assume that P is not an antichain.

For the forward implication, let (X_1, X_2, Z) be a Full-Antichain-Series decomposition of P such that $P[X_1 \cup Z]$ is still Recursively Full-Antichain-Series decomposable. Let $L_{P[X_1 \cup Z]}$ be a linear extension of $P[X_1 \cup Z]$ fulfilling condition (a) and condition (b): which exists by induction hypothesis. Then $L'_{P[X_1 \cup Z]} = L_{P[X_1 \cup Z]}[X_1] \otimes L_Z$, where L_Z is any total order on Z, is a linear extension of $P[X_1 \cup Z]$ still fulfilling condition (a) and condition (b). Indeed, on the one hand, as $\forall z \in Z$ we have $(X_1 \setminus Max(P[X_1])) \subseteq \downarrow^{[}_{P} z$, then for every $x_1 \in X_1$ and for every $z \in Z$ holds $\downarrow^{[}_{P[X_1 \cup Z]} x_1 \subseteq \downarrow^{[}_{P[X_1 \cup Z]} z$. On the other hand we have $Z \subseteq Max(P[X_1 \cup Z])$. Now, let $L_P = L'_{P[X_1 \cup Z]} \otimes L_{X_2}$, where L_{X_2} is any total order on X_2. Then, as $X_2 \subseteq Max(P)$ and $X_1 <_P X_2$, L_P is clearly a linear extension of P fulfilling condition (a) and condition (b).

For the backward implication, let L be a linear extension of P fulfilling condition (a) and condition (b), and let g_L be the greatest element of L. Let $X_2 = \{x : x \in V(P) : \downarrow^{[}_{P} x = \downarrow^{[}_{P} g_L\}$, let $X_1 = \downarrow^{[}_{P} g_L$ and let $Z = V(P) \setminus (X_1 \cup X_2)$. Then $L[X_1 \cup Z]$ is a linear extension of $P[X_1 \cup Z]$ and, as for every $x \in (X_1 \cup Z)$ we have that $\downarrow^{[}_{P[X_1 \cup Z]} x = \downarrow^{[}_{P} x \setminus X_2$ and $\uparrow^{[}_{P[X_1 \cup Z]} x = \uparrow^{[}_{P} x \setminus X_2$, then clearly $L[X_1 \cup Z]$ still fulfills condition (a) and condition (b). Thus it remains to show that (X_1, X_2, Z) is a Full Antichain-Series decomposition of P. First notice that by definition we have $X_1 <_P X_2$. Moreover, since P is not an antichain, condition (a) and the fact that $g_L \in X_2$ implies that $X_1 \neq \emptyset$. Now, since g_L is the greatest element of L, again due to condition (a), we have that for every $x \in V(P)$ either $x \in Max(P)$ or $x \in \downarrow^{[}_{P} g_L$ holds. Consequently we have that $(Z \cup X_2) = Max(P)$, and thus it only remains to show that $\forall z \in Z$, we have $(X_1 \setminus Max(P[X_1])) \subseteq \downarrow^{[}_{P} z$: which is an immediate consequence of the fact that X_1 is an initial section of L: that is that there exists $y \in V(P)$ such that $X_1 = \{x \in V(P) : x \leq_L y\}$. Indeed, since X_1 is an initial section of L, for every $z \in Z$ and for every $x_1 \in X_1$ holds $x_1 <_L z$ and thus, by condition (a), we have that $\downarrow^{[}_{P} x_1 \subseteq \downarrow^{[}_{P} z$. To conclude, notice that condition (b) immediately implies that X_1 is an initial section of L. $\square$

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