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Rapport de Recherche

On Fan Raspaud conjecture

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Abstract

A conjecture of Fan and Raspaud [3] asserts that every bridgeless cubic graph contains three perfect matchings with empty intersection. Kaiser and Raspaud [6] suggested a possible approach to this problem based on the concept of a balanced join in an embedded graph. We give here some new results concerning this conjecture and prove that a minimum counterexample must have at least 32 vertices.

Key words: Cubic graph; Edge-partition;

1 Introduction

Fan and Raspaud [3] conjectured that any bridgeless cubic graph can be provided with three perfect matchings with empty intersection (we shall say also *non intersecting perfect matchings*).

Conjecture 1 *[3] Every bridgeless cubic graph contains perfect matching M_1, M_2, M_3 such that*

$$M_1 \cap M_2 \cap M_3 = \emptyset$$

This conjecture seems to be originated independently by Jackson. Goddyn [5] indeed mentioned this problem proposed by Jackson for $r-$graphs ($r-$regular graphs with an even number of vertices such that all odd cuts have size at least r, as defined by Seymour [8]) in the proceedings of a joint summer research conference on graphs minors which dates back 1991.

Conjecture 2 *[5] There exists $k \geq 2$ such that any r-graph contains $k+1$ perfect matchings with empty intersection.*

Seymour [8] conjectured that:

Conjecture 3 *[8] If $r \geq 4$ then any r-graph has a perfect matching whose deletion yields an (r-1)-graph.*

Hence Seymour's conjecture leads to a specialized form of Jackson's conjecture when dealing with cubic bridgeless graphs and the Fan Raspaud conjecture appears as a refinement of Jackson's conjecture.

A *join* in a graph G is a set $J \subseteq E(G)$ such that the degree of every vertex in G has the same parity as its degree in the graph $(V(G), J)$. A perfect matching being a particular join in a cubic graph Kaiser and Raspaud conjectured in [6]

Conjecture 4 *[6] Every bridgeless cubic graph admits two perfect matching M_1, M_2 and a join J such that*

$$M_1 \cap M_2 \cap J = \emptyset$$

The *oddness* of a cubic graph G is the minimum number of odd circuits in a 2-factor of G. Conjecture 1 being obviously true for cubic graphs with chromatic index 3, we shall be concerned here by bridgeless cubic graphs with chromatic index 4. Hence any 2-factor of such a graph has at least two odd cycles. The class of bridgeless cubic graphs with oddness two is, in some sense, the "easiest" class to manage with in order to tackle some well known conjecture. In [6] Kaiser and Raspaud proved that Conjecture 4 holds true for bridgeless cubic graph of oddness two. Their proof is based on the notion of *balanced join* in the multigraph obtained in contracting the cycles of a two factor. Using an equivalent formulation of this notion in the next section, we shall see that we can get some new results on Conjecture 1 with the help of this technique.

For basic graph-theoretic terms, we refer the reader to Bondy and Murty [1].

2 Preliminary results

Let M be a perfect matching of a cubic graph and let $\mathcal{C} = \{C_1, C_2 \dots C_k\}$ be the 2-factor $G - M$. $A \subseteq M$ is a *balanced* $M-$matching whenever there is a perfect matching M' such that $M \cap M' = A$. That means that each odd cycle of $\mathcal{C}$ is incident to at least one edge in A and the subpaths determined by the ends of M' on the cycles of $\mathcal{C}$ incident to A have odd lengths.

In the following example, M is the perfect matching (thick edges) of the Petersen graph. Taking any edge (ab by example) of this perfect matching we are led to a balanced $M-$matching since the two cycles of length 5 give rise to two paths of length 5 (we have "opened" these paths closed to a and b).

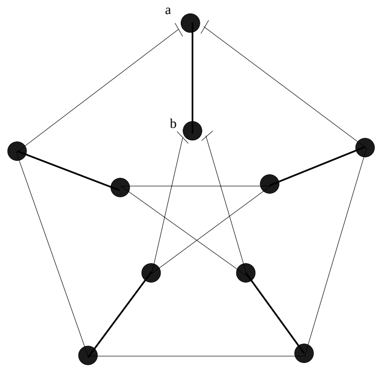

Fig. 1. A balanced $M-$matching

Remark that given a perfect matching M of a bridgeless cubic graph, M is obviously a balanced $M-$matching.

Kaiser and Raspaud [6] introduced this notion via the notion of *balanced join* in the context of a combinatorial representation of graphs embedded on surfaces. They remarked that a natural approach to the Fan Raspaud conjecture would require finding two disjoint balanced joins and hence two balanced $M-$matchings for some perfect matching M. In fact Conjecture 1 and balanced matching are related by the following lemma

Lemma 5 *A bridgeless cubic graph contains* 3 *non intersecting perfect matching if and only if there is a perfect matching M and two balanced disjoint balanced $M-$matchings.*

Proof Assume that M_1, M_2, M_3 are three perfect matchings of G such that $M_1 \cap M_2 \cap M_3 = \emptyset$. Let $M = M_1$, $A = M_1 \cap M_2$ and $B = M_1 \cap M_3$. Since $A \cap B = M_1 \cap M_2 \cap M_3$, A and B are two balanced $M-$matchings with empty intersection.

Conversely, assume that M is a perfect matching and that A and B are two balanced $M-$matchings with empty intersection. Let $M_1 = M$, M_2 be a perfect matching such that $M_2 \cap M_1 = A$ and M_3 be a perfect matching such that $M_3 \cap M_1 = B$. We have $M_1 \cap M_2 \cap M_3 = A \cap B$ and the three perfect matchings M_1, M_2 and M_3 have an empty intersection. □

The following theorem is a corollary of Edmond's Matching Polyhedron Theorem [2]. A simple proof is given by Seymour in [8].

Theorem 6 *Let G be an r-graph. Then there is an integer p and a family $\mathcal{M}$ of perfect matchings such that each edge of G is contained in precisely p members of $\mathcal{M}$.*

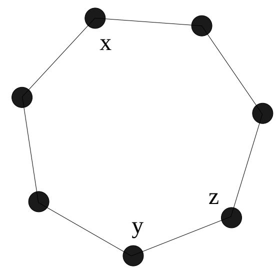

Fig. 2. A balanced triple

Lemma 7 *Let G be a bridgeless cubic graph and let $e = uv$ and $e' = u'v'$ be two edges of G. Then there exists a perfect matching avoiding these two edges.*

Proof Remark that a bridgeless cubic graph is a 3-graph as defined by Seymour. Applying Theorem 6, let $\mathcal{M}$ be a set of perfect matching such that each edge of G is contained in precisely p members of $\mathcal{M}$ (for some fixed integer $p \geq 1$).

Assume first that e and e' have a common end vertex (say u). Then u is incident to a third edge e". Any perfect matching using e" avoids e and e'.

When e and e' have no common end then, let f and g be the two edges incident with u. Assume that any perfect matching using f or g contains also the edge e'. Then e' is contained in $2p$ members of $\mathcal{M}$, impossible. Hence some perfect matchings using f or g must avoid e', as claimed. □

It can be pointed out that Lemma 7 is not extendable, so easily, to a larger set of edges. Indeed, a corollary of Theorem 6 asserts that $\mathcal{M}$ (the family of perfect matching considered) intersects each $3-$edge cut in exactly one edge. Hence for such a $3-$edge cut, there is no perfect matching in $\mathcal{M}$ avoiding this set.

Let C be an odd cycle and let $T = \{x, y, z\}$ a set of three distinct vertices of C. We shall say that C is a *balanced triple* when the three subpaths of C determined by T have odd lengths.

Let $C = x_0x_1 \ldots x_{2k}$ be an odd cyle of length at least 7. Assume that its vertex set is coloured with three colours 1, 2 and 3 such that $2 \leq |A_1| \leq |A_2| \leq |A_3|$, A_i denoting the set of vertices coloured with i, $i = 1, 2, 3$. Then we shall say that C is *good odd cycle.*

Lemma 8 *Any good odd cycle C contains two disjoint balanced triples T and T' intersecting each colour exactly once.*

Proof We shall prove this lemma by induction on $|C|$.

Assume first that C has length 7. Then A_1 and A_2 have exactly two vertices while A_3 must have 3 vertices. We can distinguish, up to isomorphism, 9 subcases

(1) $A_3 = \{x_0, x_1, x_2\}$ $A_1 = \{x_3, x_4\}$ and $A_2 = \{x_5, x_6\}$ then $T = \{x_0, x_3, x_6\}$ and $T' = \{x_1, x_4, x_5\}$ are two disjoint balanced triples.
(2) $A_3 = \{x_0, x_1, x_2\}$ $A_1 = \{x_3, x_5\}$ and $A_2 = \{x_4, x_6\}$ then $T = \{x_2, x_3, x_4\}$ and $T' = \{x_5, x_6, x_0\}$ are two disjoint balanced triples.
(3) $A_3 = \{x_0, x_1, x_2\}$ $A_1 = \{x_3, x_6\}$ and $A_2 = \{x_4, x_5\}$ then $T = \{x_2, x_3, x_4\}$ and $T' = \{x_5, x_6, x_0\}$ are two disjoint balanced triples.
(4) $A_3 = \{x_0, x_1, x_3\}$ $A_1 = \{x_2, x_4\}$ and $A_2 = \{x_5, x_6\}$ then $T = \{x_1, x_4, x_5\}$ and $T' = \{x_2, x_3, x_6\}$ are two disjoint balanced triples.
(5) $A_3 = \{x_0, x_1, x_3\}$ $A_1 = \{x_2, x_5\}$ and $A_2 = \{x_4, x_6\}$ then $T = \{x_1, x_4, x_5\}$ and $T' = \{x_2, x_3, x_6\}$ are two disjoint balanced triples.
(6) $A_3 = \{x_0, x_1, x_3\}$ $A_1 = \{x_2, x_6\}$ and $A_2 = \{x_4, x_5\}$ then $T = \{x_1, x_0, x_6\}$ and $T' = \{x_2, x_3, x_4\}$ are two disjoint balanced triples.
(7) $A_3 = \{x_0, x_1, x_4\}$ $A_1 = \{x_2, x_3\}$ and $A_2 = \{x_5, x_6\}$ then $T = \{x_1, x_2, x_5\}$ and $T' = \{x_0, x_3, x_6\}$ are two disjoint balanced triples.
(8) $A_3 = \{x_0, x_1, x_4\}$ $A_1 = \{x_2, x_5\}$ and $A_2 = \{x_3, x_6\}$ then $T = \{x_1, x_2, x_3\}$ and $T' = \{x_0, x_5, x_6\}$ are two disjoint balanced triples.
(9) $A_3 = \{x_0, x_1, x_4\}$ $A_1 = \{x_2, x_6\}$ and $A_2 = \{x_3, x_5\}$ then $T = \{x_1, x_2, x_3\}$ and $T' = \{x_0, x_5, x_6\}$ are two disjoint balanced triples.

Assume that C is a good odd cycle of length at least 9 and assume that the property holds for any good odd cycle of length $|C| - 2$.

CLAIM 1 *If C has two consecutive vertices x_j and x_{j+1} (j being taken modulo $2k$) in the same set A_i ($i = 1, 2$ or 3) such that $|A_i| \geq 4$, then the property holds.*

Proof Assume that C has two consecutive vertices x_j and x_{j+1} in the same set A_i ($i = 1, 2$ or 3) such that $|A_i| \geq 4$, then delete x_j and x_{j+1} and add the edge $x_{j-1}x_{j+2}$. We get hence a good odd cycle C' of length $|C| - 2$. C' has two disjoint balanced triples T and T' by induction hypothesis and we can check that these two triples are also balanced in C since the edge $x_{j-1}x_{j+2}$ is replaced by the path $x_{j-1}x_jx_{j+1}x_{j+2}$ in C. ∎

CLAIM 2 *If C has two consecutive vertices x_j and x_{j+1} (j being taken modulo $2k$) one of them being in A_i while the other is in $A_{i'}$ ($i \neq i' \in \{1, 2, 3\}$), then the property holds as soon as $|A_i| \geq 3$ and $|A_{i'}| \geq 3$.*

Proof Use the same trick as in the proof of Claim 1 ∎

If $A_3 \geq 4$, we can suppose, by Claim 1 that no two vertices of A_3 are consecutive on C. When $x \in A_3$, x' (its succesor in the natural ordering) is in A_1 or A_2. By Claim 2, the vertices in A_3 have at most two successors in A_1 and at most two successors in A_2. Hence we must have $|A_3| = 4$ and $|A_2| = |A_3| = 2$, impossible. If $|A_3| = 3$ then we must have $|A_2| = |A_3| = 3$ since C has length 9. In that case we certainly have two consecutive vertices with distinct colours and we can apply the above claim 1. □

Let C be an even cycle and let $P = \{x, y\}$ a set of two distinct vertices of C. We shall say that C is a *balanced pair* when the two subpaths of C determined by P have odd lengths.

Let $C = x_0x_1 \dots x_{2k-1}$ be an even cyle of length at least 4. Assume that its vertex set is coloured with three colours 1, 2 and 3. Let A_i be the set of vertices coloured with i, $i = 1, 2, 3$. Assume that $|A_i| = 0$ or 1 for at most one colour, then we shall say that C is *good even cycle.*

Lemma 9 *Any good even cycle C contains two disjoint balanced pairs P_i and P_i' intersecting A_i exactly once each as soon as A_i has at least two vertices ($i = 1, 2, 3$).*

Proof We prove the lemma for $i = 1$. Assume that $|A_1| \geq 2$ and $|A_2| \geq 2$. Assume that x_0 is a vertex in A_2 and let x_i be the first vertex in A_1, x_j be the last vertex in A_1 when running on C in the sens given by x_0x_1. If $i \neq 1$ or $j \neq 2k - 1$ $P = \{x_{i-1}, x_i\}$ and $P' = \{x_j, x_{j+1}\}$ are two distinct balanced pairs intersecting A_1 exactly once each. Assume that $i = 1$ and $j = 2k - 1$. Since A_2 contains another vertex x_l ($1 < l < 2k - 1$). Let x_m be the first vertex in A_1 when running from x_l to x_{2k-1} ($l < m \leq 2k - 1$. Then $P = \{x_0, x_1\}$ and $P' = \{x_{m-1}, x_m\}$ are two disjoint balanced pairs intersecting A_1 exactly once each. □

Lemma 10 *Let C be an even cycle of length $2p \geq 8$ and let x and y be two vertices. Assume that the vertices of $C - \{x, y\}$ are partitioned into A and B with $|A| \geq p - 2$ and $|B| \geq p - 2$. Then there are at least two disjoint balanced pairs intersecting A and B exactly once each.*

Proof Let us colour alternately the vertices of C in red and blue. If A contains at least two red (or blue) vertices u and v and B two blue (or red respectively) vertices u' and v' then $P = \{u, u'\}$ and $P' = \{v, v'\}$ are two disjoint balanced pairs. If A contains a red vertex u and a blue vertex v and, symmetrically, B contains a red vertex u' and a blue vertex v' then $P = \{u, u'\}$ and $P' = \{v, v'\}$ are two disjoint balanced. It is clear that at least one of the above cases must happens and the result follows. □

3 Applications

From now on, we consider that our graphs are cubic, connected and bridgeless (multi-edges are allowed). Moreover we suppose that they are not $3-$edge colourable. Hence these graphs have perfect matchings and any 2-factor have a non null even number of odd cycles. If $X \subset V(G)$ and $Y \subset V(G)$, $d(X,Y)$ is the length of a shortest path between these two sets.

3.1 Graphs with small oddness

Theorem 11 *Let G be a cubic graph of oddness two. Assume that G has a perfect matching M where the $2-$factor $\mathcal{C} = \{C_1, C_2 \ldots C_k\}$ of $G - M$ is such that C_1 and C_2 are the only odd cycles and $d(C_1, C_2) \leq 3$. Then G has three perfect matchings with an empty intersection.*

Proof

If $d(C_1, C_2) = 1$ let uv be an edge joining C_1 and C_2 ($u \in C_1$ and $v \in C_2$). $A = \{uv\}$ is a balanced $M-$matching. Let M_2 be a perfect matching such that $M_2 \cap M = A$. There is certainly a perfect matching M_3 avoiding uv (see Theorem 6). Hence M, M_1 and M_3 are three perfect matchings with an empty intersection.

It can be noticed that $d(C_1, C_2) \neq 2$. Indeed, Let $P = u_1vu_2$ be a shortest path joining $u_1 \in C_1$ to $u_2 \in C_2$, then the cycle of $\mathcal{C}$ containing v cannot be disjoint from C_1 or C_2, impossible.

Assume thus now that $d(C_1, C_2) = 3$ and let $P = u_1u_2u_3u_4$ be a shortest path joining C_1 to C_2 (with $u_1 \in C_1$ and $u_4 \in C_2$). Then $A = \{u_1u_2, u_3u_4\}$ is a balanced $M-$matching. Let M_2 be a perfect matching such that $M_2 \cap M = A$. From Lemma 7 there is a perfect matching M_3 avoiding these two edges of A. Hence M, M_2 and M_3 are three non intersecting perfect matchings □

A graph G is *near-bipartite* whenever there is an edge e of G such that $G - e$ is bipartite.

Theorem 12 *Let G be a cubic graph of oddness two. Assume that G has a perfect matching M where the $2-$factor $\mathcal{C} =$ of $G - M$ has only 3 cycles*

C_1, C_2 (odds) and C_3 (even) such that the subgraph of G induced by C_3 is a near-bipartite graph. Then G has three perfect matchings with an empty intersection.

Proof From Theorem 11, we can suppose that $d(C_1, C_2) \geq 3$. That means that the neighbors of C_1 are contained in C_3 as well as those of C_2. Let us colour the vertices of C_3 with two colours red and blue alternately along C_3. Assume that a and b are two vertices of C_3 with distinct colours such that a is a neighbor of C_1 and b is a neighbor of C_2. Let e and f be the two edges of M so determined by a and b. Then $A = \{e, f\}$ is a balanced $M-$matching. Let M_2 be a perfect matching such that $M \cap M_2 = A$ and M_3 be a perfect matching avoiding A (Lemma 7). Then M, M_1 and M_2 are 3 non intersecting perfect matchings.

It remains thus to assume that the neighbors of C_1 and C_2 have the same colour (say red). G being bridgeless, we have an odd number (at least 3) of edges in M joining C_1 and C_3 (C_2 and C_3 respectively). The remaining vertices of C_3 are matched by edges of M, but we have at least 6 blue vertices more than red vertices in C_3 to be matched and hence at least three pairs of blue vertices must be matched. Let $e \in E(G)$ such that $G - e$ is bipartite, if $e \in C_3$ then C_3 must have odd length, impossible. Hence e is the only chord of C_3 whose ends have the same colour, impossible.

□

Theorem 13 *Assume that G is a cubic graph having a perfect matching M where the $2-$factor $\mathcal{C} = \{C_1, C_2, C_3, C_4 \ldots C_k\}$ of $G - M$ is such that C_1, C_2, C_3 and C_4 are the only odd cycles. Assume moreover that $d(C_1, C_2) = 1$ as well as $d(C_3, C_4) = 1$. Then G has three perfect matchings with an empty intersection.*

Proof Let u_1u_2 be an edge joining C_1 to C_2 and u_3u_4 be an edge joining C_3 to C_4. $A = \{u_1u_2, u_3u_4\}$ is a balanced $M-$matching. Let M_2 be a perfect matching such that $M \cap M_2 = A'$. By Lemma 7, there is a perfect matching M_3 avoiding these two edges. Hence the three perfect matchings M, M_2 and M_3 are non intersecting. □

Theorem 14 *Assume that G has a perfect matching M where the $2-$factor $\mathcal{C}$ has only 4 chordless cycles $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$. Then G has three perfect matchings with an empty intersection.*

Proof By the connectivity of G, every vertex of three cycles of $\mathcal{C}$ (say C_1, C_2 and C_3) are joined to C_4 while no other edge exists. Otherwise the result holds by Theorem 13.

Each cycle of $\mathcal{C}$ has length at least 3 and, hence C_4 has length at least 9. We can colour each vertex $v \in C_4$ with 1, 2 or 3 following the fact the edge of M incident with v has its other end on C_1, C_2 or C_3. From lemma 8, there is two balanced triples T and T' intersecting each colour. These two balanced triples determine two disjoint balanced $M-$matchings. Hence, the result holds from Lemma 5. □

3.2 Good Rings, Good stars

A *good path of index* C_0 is a set P of $k+1$ disjoint cycles $C_0, C_1 \dots C_k$ such that

- C_0 and C_k are the only odd cycles of P
- C_i is joined to C_{i+1} $(0 \leq i \leq k-1)$ by an edge e_i (called *jonction edge of index* C_0)
- the two jonction edges incident to an even cycle determine two odd paths on this cycle

A *good ring* is a set R of disjoint odd cycles $C_0 \dots C_{2p-1}$ and even cycles such that

- C_i is joined to C_{i+1} (i is taken modulo $2p$) by a good path P_i of index C_i whose even cycles are in R
- the good paths involved in R are pairwise disjoint.

A *good star* (*centered in* C_0) is a set S of four disjoint cycles C_0, C_1, C_2, C_3 such that

- C_0 (the center) is chordless and has length at least 7
- C_0 is joined to each other cycle by at least two edges and has no neighbor outside of S
- there is no edge between C_1, C_2 and C_3

Theorem 15 *Assume that G has a perfect matching M where the $2-$factor $\mathcal{C}$ of $G - M$ can be partitioned into good rings, good stars and even cycles. Then G has three perfect matchings with an empty intersection.*

Proof Let $\mathcal{R}$ be the set of good rings of $\mathcal{C}$ and $\mathcal{S}$ be the set of good stars.

Let $R \in \mathcal{R}$, and let $C_0 \dots C_{2p-1}$ be its set of odd cycles. Let us us say that a junction edge of R has an even index whenever this edge is a junction edge of index C_i with i even. A junction edge of odd index is defined in the same way. Let A_R be the set of junction edge of even index of R and B_R the set of junction edge of odd index. We let $A = \bigcup_{R \in \mathcal{R}} A_R$ and $B = \bigcup_{R \in \mathcal{R}} B_R$.

For each star $S \in \mathcal{S}$, assume that each vertex of the center is coloured with the name of the odd cycle of S to whom this vertex is adjacent. Let T_S and T'_S be two disjoint balanced triples (Lemma 8) of the center of S. Let N_S and N'_S be the sets of three edges joining the center of S to the other cycles of S, determined by T_S and T'_S. Let $A' = \bigcup_{S \in \mathcal{S}} N_S$ and $B' = \bigcup_{S \in \mathcal{S}} N'_S$.

It is an easy task to check that $A + A'$ and $B + B'$ are two disjoint balanced $M-$matchings. Hence, the result holds from Lemma 5. $\square$

A particular case of the above result is given by E. Màčajová and M. Škoviera. The length of a ring is the number of jonction edges. A ring of length 2 is merely a set of two odd cycles joined by two edges.

Corollary 16 *[7]Assume that G has a perfect matching M where the odd cycles of the $2-$factor $\mathcal{C}$ can be arranged into rings of length 2. Then G has three perfect matchings with an empty intersection.*

It can be pointed out that this technique of rings of length 2 was used in [4] for the $5-$ flow problem when dealing with graphs of small order and graphs with low genus. This technique has been developped independently by Steffen in [9].

4 On graphs with at most 32 vertices

Determining the structure of a minimal counterexample to a conjecture is one of the most typical methods in Graph Theory. In this section we investigate some basic structures of minimal counterexamples to Conjecture 1.

The *girth* of a graph is the length of shortest cycle. Màčajová and Škoviera [7] proved that the girth of a minimal counterexample is at least 5.

Lemma 17 *[7] If G is a smallest bridgeless cubic graph with no 3 non-intersecting perfect matchings, then the girth of G is at least* 5

Lemma 18 *If G is a smallest bridgeless cubic graph with no 3 non-intersecting perfect matchings, then G does not contain a subgraph isomorphic to G_8 (see Figure 3).*

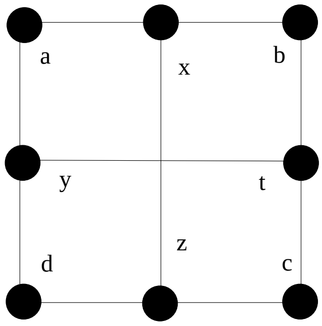

Fig. 3. G_8

Proof Assume that G contains G_8. Let a', b', c' and d' be the vertices of $G - G_8$ adjacent to, respectively a, b, c and d. Let G' be the graph obtained in deleting G_8 and joining a' to c' and b' to d'. It is an easy task to verify that G' has chromatic index 3 if and only if G itself has chromatic index 3. We do not know whether this graph is connected or not but each component is smaller than G and contains thus 3 non-intersecting perfect matchings leading to 3 non-intersecting perfect matchings for G'. Let P_1 P_2 and P_3 these perfect matchings. Our goal is to construct 3 non-intersecting perfect matchings for G M_1, M_2 and M_3 from those of G'. We have thus to delete the edge $a'c'$ and $b'd'$ from P_1, P_2 and P_3 whenever they belong to these sets and add some edges of G_8 in order to obtain the perfect matchings for G.

Let us now consider the number of edges in $\{a'c', b'd'\}$ which are contained in $P_1 \cap P_2$ or in $P_1 \cap P_3$ or in $P_2 \cap P_3$.

When none of $P_1 \cap P_2$,$P_1 \cap P_3$ or $P_2 \cap P_3$ contain $a'c'$ nor $b'd'$ we set $M_1 = P_1 + \{ax, bt, cz, dy\}$, $M_2 = P_2 + \{ay, dz, ct, bx\}$ and $M_3 = P_3 + \{ax, bt, cz, dy\}$.

Assume that the edges $a'c'$ and $b'd'$ both belong to some $P_i \cap P_j$ $(i \neq j \in \{1, 2, 3\})$, say $P_1 \cap P_2$. In this case P_3 cannot contain one of those edges. Thus we write $M_1 = P_1 - \{a'c', b'd'\} + \{a'a, c'c, b'b, d'd\} + \{xz, ut\}$,$M_2 = P_2 - \{a'c', b'd'\} + \{a'a, c'c, b'b, d'd\} + \{xz, ut\}$ and $M_3 = P_3 + \{ax, bt, cz, dy\}$.

Finally assume w.l.o.g that $P_1 \cap P_2 = \{a'c'\}$. When $P_2 \cap P_3 = P_1 \cap P_3 = set$ we set $M_1 = P_1 - \{a'c'\} + \{a'a, c'c\} + \{yt, xb, dz\}$, $M_2 = P_2 - \{a'c'\} + \{a'a, c'c\} + \{bt, xz, dy\}$ and $M_3 = P_3 + \{ax, bt, cz, dy\}$. On the last hand, if one of the sets $P_2 \cap P_3$ or $P_1 \cap P_3$ (say $P_2 \cap P_3$) contain the edge $b'd'$, we write $M_1 = P_1 - \{a'c'\} + \{a'a, c'c\} + \{yt, xb, dz\}$,$M_2 = P_2 - \{a'c', b'd'\} + \{a'a, b'b, c'c, d'd\} + \{xz, yt\}$ and $M_3 = P_3 - \{b'd'\} + \{b'b, d'd\} + \{ay, xz, ct\}$.

In all cases, since $P_1 \cap P_2 \cap P_3 = \emptyset$ we have $M_1 \cap M_2 \cap M_3 = \emptyset$.

□

Lemma 19 *If G is a smallest bridgeless cubic graph with no 3 non-intersecting perfect matchings, then G does not contain a subgraph isomorphic to the Petersen graph with one vertex deleted.*

Proof Let P be a graph isomorphic to the Petersen graph whose vertex set is $\{a, b, c, d, e, x, y, z, t, u\}$ and such that $abcde$ and $xyztu$ are the two odd cycles of the 2-factor associated to the perfect matching $\{ax, bt, cy, du, ez\}$. Assume that $H = P - a$ is a subgraph of G. Let x', b' and c' be respectively the neighbors of x, b and c in $G - H$. Let G' be the graph whose vertex set is $V(G - H) \cup \{v\}$ where $v \notin V(G)$ is a new vertex and whose edge set is $E(G - H) \cup \{vx', ve', vb'\}$. Since G' is smaller than G, G' contains 3 non-intersecting perfect matchings P_1, P_2, P_3.

For $i \in \{1, 2, 3\}$ we can associate to P_i two perfect matchings of G , namely M_i and M'_i, as follows (observe that exactly one of the edges vx', vb' or vc' belongs to P_i) :

When $vx' \in P_i$ we set $M_i = P_i - \{vx'\} \cup \{xx', bt, cy, du, ez\}$
and $M'_i = P_i - \{vx'\} \cup \{xx', tu, bc, yz, ed\}$.
When $vb' \in P_i$ we set $M_i = P_i - \{vb'\} \cup \{bb', cy, xu, de, zt\}$
and $M'_i = P_i - \{vb'\} \cup \{bb', cd, ut, ez, xy\}$.
When $ve' \in P_i$ we set $M_i = P_i - \{ve'\} \cup \{ee', cd, bt, zy, xu\}$
and $M'_i = P_i - \{ve'\} \cup \{ee', du, xy, zt, bc\}$.

But now, if on one hand $P_i \cap P_j$ contains one of the edges in $\{vx', vb', ve'\}$ for some $i \neq j \in \{1, 2, 3\}$ and for $k \in \{1, 2, 3\}$ distinct from i and j, $M_i \cap M'_j \cap M_k = M_i \cap M'_j \cap M'_k = P_1 \cap P_2 \cap P_3 = \emptyset$,a contradiction. If, on the other hand, each of P_i, P_j and P_k (for i, j, k distinct members of $\{1, 2, 3\}$) contains exactly one edge of $\{vx', vb', vc'\}$ we also have $M_i \cap M_j \cap M_k = P_i \cap P_j \cap P_k = \emptyset$, a contradiction. □

Theorem 20 *If G is a smallest bridgeless cubic graph with no 3 non-intersecting perfect matchings, then G has at least 32 vertices*

Proof Assume to the contrary that G is a counterexample with at most 30 vertices. We can obviously suppose that G is connected. Let M be a perfect matching and let $\mathcal{C}$ be the 2−factor of $G - M$. Assume that the number of odd cycles of $\mathcal{C}$ is the oddness of G. Since G has girth at least 5 by Lemma 17, the oddness of G is 2, 4 or 6.

Claim 1 *G has oddness 2 or 4.*

Proof Assume that G has oddness 6. We have $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ and each cycle C_i ($i = 1 \ldots 6$) is chordless and has length 5. Each cycle C_i is joined to at least two other cycles of $\mathcal{C}$. Otherwise, if C_i is joined to only one cycle C_j $(i \neq j)$, these two cycles would form a connected component of G and G would not be connected, impossible. It is an easy task to see that we can thus partition $\mathcal{C}$ into good rings and the results comes from Theorem 15. ∎

Assume now that G has oddness 4. Hence $\mathcal{C}$ contains 4 odd cycles C_1, C_2, C_3 and C_4. Since these cycles have length at least 5, $\mathcal{C}$ contains eventually an even cycle C_5. From Lemmas 17 and 18 if C_5 exists, C_5 is a chordless cycle of length 6 or C_5 has length 8 (with at most one chord) or 10. When C_5 has length 10, C_1, C_2, C_3 and C_4 are chordless cycles of length 5. When C_5 has length 8, C_1, C_2, C_3 and C_4 are chordless cycles of length 5 or 3 of them have length 5 while the last one has length 7.

Theorem 13 says that we are done as soon as we can find two edges allowing to arrange by pairs C_1, C_2, C_3 and C_4 (say for example C_1 joined to C_2 and C_3 to C_4) and Theorem 15 says that we are done whenever these 4 odd cycles induce a good star. That means that the subgraph H induced by the four odd cycles is of one of the two following types:

Type 1 One odd cycle (say C_4) has all its neighbors in C_5 and the 3 other odd cycles induce a connected subgraph

Type 2 One cycle (say C_4) is joined to the other by at least one edge while the others are not adjacent.

Claim 2 *C_5 has length at least 8.*

Proof Assume that $|C_5| = 6$, the girth of G being at least 5 (Lemma 17) we can suppose that C_5 has no chord. H is not of type 1, otherwise C_4 having its neighbors in C_5, C_5 is connected to the remaining part of G with one edge only, impossible since G is bridgeless. Assume thus that H is of type 2. Then, there are 6 edges between C_5 and H. Since there are at least 15 edges going out C_1, C_2 and C_3 that means that there are at least 9 edges between C_0 and the other odd cycles. Hence, C_0 must have length 9 and can not be adjacent to C_5. G is then partitioned into a good star and an even cycle and the result comes from Theorem 15. ∎

Claim 3 *If C_5 has length 8 then it has no chord.*

Proof If C_5 has a chord then there are at most 6 edges joining C_5 to H. If H is of type 1 then C_4 has at least 5 neighbors in C_5. Hence there is at most one edge between H and C_5, impossible. If H is of type 2, then the the three cycles C_1, C_2 and C_3 have at least 9 neighbors in C_4, impossible since G has at most 30 vertices. ■

Claim 4 *If C_5 exists then H is not of type 1.*

Proof If H is of type 1, then C_4 has its neighbors (at least 5) in C_5 and there are 3 or 5 edges between H and C_5.

Whenever there are 5 edges between H and C_5, C_5 has length 10 and C_1, C_2, C_3 have length 5 (as well as C_4). In that case w.l.o.g., we can consider that C_3 is joined by exactly one edge to C_5 and joined by 4 edges to C_2. The las t neighbor of C_2 cannote be on C_5, otherwise the 5 neighbors of C_1 are on C_5 and C_5 must have length 12, impossible. Hence, C_2 is joined to C_1 by exactly one edge and C_1 is joined to C_5 by 4 edges. Let us colour each vertex v of C_5 with $1, 3$ or 4 when v is adjacent to C_i $(i = 1, 3, 4)$. From Lemma 9, we can find 2 disjoint balanced pairs on C_5 $P = \{u, v\}$ and $P' = \{u', v'\}$ with u and u' coloured with 4, v and v' coloured with 1. These two pairs determine two disjoint set of edges $N' = \{e, f\}$ and $N" = \{h, i\}$ in M and allow us to construct two disjoint balanced $M-$matchings $M' = \{e, f, g\}$ and $M" = \{h, i, j\}$ in choosing two distinct edges g and j between C_2 and C_3. The result follows from Lemma 5

Whenever there are 3 edges between H and C_5, C_5 has length 8 or 10, any two cycles in $\{C_1, C_2, C_3\}$ are joined by at least two edges and each of them is joined to C_5 by exactly one edge. Let A be the three vertices of C_5 which are the neighbors of $C_1 \cup C_2 \cup C_3$. Let B be the neighbors of C_4 on C_5. When C_5 has length 10 this cycle induces a chord xy. In that case, Lemma 10 says that we can find 2 disjoint balanced pairs $P = \{u, v\}$ and $P' = \{u', v'\}$ with $u, u' \in A$ and $v, v' \in B$. These two pairs determine two disjoint set of edges $N' = \{e, f\}$ and $N" = \{h, i\}$ in M and allow us to construct two disjoint balanced $M-$matchings $M' = \{e, f, g\}$ and $M" = \{h, i, j\}$ in choosing two suitable distinct edges g and j joining two of the cycles in $\{C_1, C_2, C_3\}$. When C_5 has no chord, we can apply the same technique in choosing x and y in B.

The result follows from Lemma 5.

■

Claim 5 *if H is of type 2 then C_5 has 8 vertices.*

Proof When C_5 has length 10, this cycle has no chord. Otherwise, we have at most 8 edges between H and C_5. Hence C_1, C_2 and C_3 are joined to C_4 with at least 7 edges, impossible since G hat at most 30 vertices. Assume thus that C_5 is a chordless cycle of length 10 then there are 15 edges going out $C_1 \cup C_2 \cup C_3$

and at most 5 of them are incident to C_4. Hence there are 10 edges between $C_1 \cup C_2 \cup C_3$ and C_5, 5 edges between $C_1 \cup C_2 \cup C_3$ and C_4 and henceforth no edge between C_4 and C_5. One cycle in $\{C_1, C_2, C_3\}$ has exactly one neighbor in C_5 (say C_1) or two of them (say C_1 and C_2) have this property .

It is an easy task to find a balanced triple u, v, w on C_4 where u is a neighbor of C_1, v a neighbor of C_2 and w a neighbor of C_3. This balanced triple determine a balanced $M-$matching A. We can construct a balanced $M-$matching B disjoint from A in choosing two edges e end f connecting $C_1 \cup C_2$ to C_5 whose ends are adjacent on C_5 (since 7 or 8 edges are involved between these two sets) and an edge $h \notin A$ between C_3 and C_4. The result follows from Lemma 5

■

Claim 6 *If C_5 exists then H is not of type 2.*

Proof From claim 5, it remains to assume that C_5 has length 8. Then $C_1 \cup C_2 \cup C_3$ is joined to C_4 by at least 7 edges. C_4 has then no neighbor in C_5 and G is partitioned into a good star centered on C_4 and an even cycle as soon as C_1, C_2 and C_3 have two neighbors at least in C_4. In that case, the result follows from Theorem 15.

Assume thus that C_1 has only one neighbor in C_4 (and then 4 neighbors in C_5). Assume that C_2 has more neighbors in C_5 than C_3. Hence C_2 has at least 2 neighbors in C_5. Let us colour each vertex v of C_5 with $1, 2$ or 3 when v is adjacent to C_1, C_2 or C_3. With that colouring C_5 is a good even cycle. We can find 2 disjoint balanced pairs intersecting the colour 1 exactly once each. Let $\{e, f\}$ and $\{g, h\}$ the two pairs of edges of M so determined. We can complete these two pairs with a third edge i (j respectively) connecting C_3 to C_4 or C_2 to C_3,following the cases, in such a way that $A = \{e, g, i\}$ and $B = \{f, h, j\}$ are two disjoint balanced $M-$matchings. The result follows from Lemma 5

■

Claim 7 *The oddness of G is at most* 2.

Proof In view of the previous claims, it remains to consider the case were $\mathcal{C}$ is reduced to a set of four odd cycles $\{C_1, C_2, C_3, C_4\}$. Once again, Theorem 13, says that, up to the name of cycles, C_1, C_2 and C_3 are joined to the last cycle C_4 and have no other neighboring cycle. That means that C_1, C_2, C_3 have length 5 and C_4 has length 15. These 4 cycles are chordless and the result comes from Theorem 14. ∎

Hence, we can assume that $\mathcal{C}$ contains only two odd cycles C_1 and C_2. Since we consider graphs with at most 30 vertices and since the even cycles of $\mathcal{C}$ have length at least 6, $\mathcal{C}$ contains only one even cycle C_3 or two even cycles C_3 and C_4 or three even cycles C_3, C_4 and C_5. From Theorem 11, C_1 and C_2 are at distance at least 4. That means that the only neighbors of these two cycles are vertices of the remaining even cycles.

It will be convenient, in the sequel, to consider that the vertices of the even cycles are coloured alternately in red and blue.

Claim 8 *If C_1 and C_2 are joined to an even cycle in $\mathcal{C}$, then their neighbors in that even cycle have the same colour*

Proof Assume that C_1 is joined to a blue vertex of an even cycle of $\mathcal{C}$ by the edge e and C_2 is joined to a red vertex of this same cycle by the edge e'. $A = \{e, e'\}$ is then a balanced $M-$matching. Let M_2 be the perfect matching of G such that $M \cap M_2 = A$ and let M_3 be a perfect matching avoiding e and e' (Lemma 7). then M, M_2 and M_3 are two non intersecting perfect matchings, a contradiction. ∎

Hence, for any even cycle of $\mathcal{C}$ joined to the two cycles C_1 and C_2, we can consider that, after a possible permutation of colours for some even cycle, the vertices adjacent to C_1 or C_2 have the same colour (say red).

Claim 9 *$\mathcal{C}$ contains 2 even cycles*

Proof Assume that $\mathcal{C}$ contains 3 even cycles C_3, C_4 and C_5. We certainly have, up to isomorphism, C_3 and C_4 with length 6 and C_5 of length 6 or 8 while the lengths of C_1 and C_2 are bounded above by 7. In view of Claim 8 $C_1 \cup C_2$ has at most 3 neighbors in C_3 and in C_4 and at most 4 neighbors in C_5. Since C_1 and C_2 have at least 10 neighbors, that means that all the red vertices of $C_3 \cup C_4 \cup C_5$ are adjacent to some vertex in C_1 or C_2. It is then easy to see that two even cycles are joined by two distinct edges (i and j) whose ends are blue and each of them is connected to both C_1 and C_2 (say e and f connecting C_1 and g and h connecting C_2). Then $A = \{e, g, i\}$ and $B = \{f, h, j\}$ are two disjoint balanced $M-$matchings and the result follows.

Assume now that $\mathcal{C} = \{C_1, C_2, C_3\}$. Since C_1 and C_2 have at least 5 neighbors each in C_3, C_3 must have 10 red vertices. Hence C_1 and C_2 have length 5 and C_3 has length 20. The 10 blue vertices of C_3 are matched by 5 edges of M. For any chord of C_3, we can find a red vertex in each path determined by this chord on C_3, one being adjacent to C_1 and the other to C_2. Let A be the three edges so determined. A is a balanced $M-$matching. By systematic inspection we can check that it is always possible to find two disjoint balanced $M-$matchings so constructed. The result follows from Lemma 5 ∎

We shall say that G is a graph of *type 3* when

Type 3 $\mathcal{C}$ contains two even cycles C_3 and C_4, the neighborhood of C_1 is contained in C_3, the neighborhood of C_2 is contained in C_4, and C_3 and C_4 are joined by 3 or 5 edges.

Claim 10 *C_1 and C_2 have length 5 or 7 or one of them has length 9. In the latter case G is a graph of type 3*

Proof G being connected and bridgeless, C_1 and C_2 are joined to the remaining cycles of $\mathcal{C}$ by an odd number of edges (at least 3).

Assume that C_1 has length at least 11, then there at least 16 vertices involved in $C_1 \cup C_2$. Hence, $\mathcal{C}$ contains exactly one even cycle. From Claim 9 this is impossible.

Assume that C_1 has length 9, then if C_1 is connected to the remaining part of G with 3 edges, that means that C_1 has 3 chords. Since G has girth at least 5, C_1 induces a subgraph isomorphic to the Petersen graph where a vertex is deleted. This is impossible in view of Lemma 19.

Hence C_1 is connected to the even cycles of $\mathcal{C}$ with 5 edges. If $\mathcal{C}$ has only one cycle C_3, then, in view of claim this cycle must has length at least 20, impossible. We can thus assume that $\mathcal{C}$ contains two cycles C_3 and C_4. Since $C_1 \cup C_2$ contains at least 14 vertices, C_3 and C_4 have length 8. If C_1 and C_2 have both some neighbors in C_3, there are at most 4 such vertices in view of Claim 8. In that case, the remaining (at least 6)neighbors are in C_4, impossible since this forces C_4 to have length at least 10.

Hence C_1 has all its neighbors in C_3 and C_2 all its neighbors in C_4. The perfect matching M forces C_3 and C_4 to be connected with an odd number (3 or 5) of edges and G is a graph of type 3, as claimed. ∎

From now on, we have $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$

Claim 11 *G is not a graph of type 3*

Proof

Let $f = ab$ and $g = cd$ two edges joining C_3 to C_4 with a and c in C_3. Whatever is the colour of a and c we can choose two distinct vertices u and v in the neighboring vertices of C_1 on C_3 such that u and a have distinct colours as well as v and c. Let g be the edge joining C_1 to u and h the edge joining C_1 to v. In the same way, we can find two distinct vertices w and x in the neighboring vertices of C_2 on C_4 with the same property relatively to b and d leading to the edges g' and h'.

We can check that $M' = \{f, g, h\}$ and $M" = \{f', g', h'\}$ are two disjoint balanced $M-$matchings. The result follows from Lemma 5

■

CLAIM 12 *One of C_1 or C_2 has its neighborhood included in C_3 or C_4*

Proof If C_1 and C_2 have neighbors in C_3 and C_4 each, then, from Claim 8, there are at least 20 vertices involved in $C_3 \cup C_4$. Hence C_1 and C_2 have length 5 and $C_3 \cup C_4$ contains exactly 20 vertices. The 10 red vertices of $C_3 \cup C_4$ are adjacent to C_1 or C_2 and the blue vertices are connected together.

Let f be a chord for C_3 and f' be a chord for C_4 (whenever these two chord exist). We can find two red vertices in C_3 separated by f, one being adjacent to C_1 by an edge g while the other is adjacent to C_2 by an edge h. Let $M' = \{f, g, h\}$ be the set of three edges so constructed.In the same way we get $M" = \{f', g', h'\}$ when considering C_4. M' and $M"$ are two disjoint balanced $M-$matchings. The result follows from Lemma 5

Assume thus that C_1 has no chord. That means that we can find two distinct edges e and f connecting C_3 to C_4. Let g be an edge connecting C_1 to C_3, h be an edge connecting C_2 to C_4, i an edge connecting C_1 to C_4 and j an edge connecting C_2 to C_3. Then $M' = \{e, g, h\}$ and $M" = \{f, i, j\}$ are two disjoint balanced $M-$matchings. The result follows from Lemma 5 ■

We can assume now that C_2 has its neighbors contained in C_4. Since G is not of type 3 by Claim 11, C_1 has some neighbor in C_4. C_4 must have length 12 at least from Claim 8. This forces C_3 to have length 8, C_4 length 12 and C_1 and C_2 lengths 5. Moreover, there is one edge exactly between C_1 and C_4 and 2 or 4 edges between C_3 and C_4. It is then an easy task to find $M' = \{e, f, g\}$ and $M" = \{h, i, j\}$ with e and h connecting C_1 and C_3, f and i connecting C_3 and C_4, g and j connecting C_4 and C_2 such that M' and $M"$ are two disjoint balanced $M-$matchings. The result follows from Lemma 5 □

5 Conclusion

A Fano colouring of G is any assignment of points of the Fano plane $\mathcal{F}_7$ (see, e.g., [7]) to edges of G such that the three edges incident with each vertex of G are mapped to three distinct collinear points of $\mathcal{F}_7$. The following conjecture appears in [7]

Conjecture 21 *[7] Every bridgeless cubic graph admits a Fano colouring which uses at most four lines.*

In fact, Màčajová and Škoviera proved in [7] that conjecture 1 and Conjecture 21 are equivalent. Hence, our results can be immediately translated in terms of the Màčajová and Škoviera conjecture.

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