UNIVERSITE D'ORLEANS

LABORATOIRE D'INFORMATIQUE FONDAMENTALE D'ORLEANS

LIFO

4 rue Léonard de Vinci
BP 6759
F-45067 Orléans Cedex 2
FRANCE
http://www.univ-orleans.fr/lifo

Rapport de Recherche

On parsimonious edge-colouring of graphs with maximum degree three

J.L. Fouquet and J.M. Vanherpe
LIFO, Université d'Orléans

Rapport n° **RR-2008-07**

On parsimonious edge-colouring of graphs with maximum degree three

J.L. Fouquet and J.M. Vanherpe

L.I.F.O., Faculté des Sciences, B.P. 6759
Université d'Orléans, 45067 Orléans Cedex 2, FR

Abstract

In a graph G of maximum degree Δ let γ denote the largest fraction of edges that can be Δ edge-coloured. Albertson and Haas showned that $\gamma \geq \frac{13}{15}$ when G is cubic [1]. We show here that this result can be extended to graphs with maximum degree 3 with the exception of a graph on 5 vertices. Moreover, there are exactly two graphs with maximum degree 3 (one being obviously Petersen's graph) for which $\gamma = \frac{13}{15}$. This extends a result given in [11]. These results are obtained in giving structural properties of the so called $\delta-$minimum edge colourings for graphs with maximum degree 3.

Key words: Cubic graph; Edge-colouring;

1 Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [12] that these graphs can be edge-coloured with 4 colours. Let ϕ $E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G. It is often of interest to try tu use one colour (say δ) as few as possible. When an edge colouring is optimal, following this constraints, we shall say that ϕ is $\delta - minimum$. In [3] we gave without proof (unfortunately in french) results on $\delta - minimum$ edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [10,11]. We give in section 2 a translation of these results and their proofs, they are contained in [4] (excepted for lemma 13 and Proposition 4).

An edge colouring of G with $\{\alpha, \beta, \gamma, \delta\}$ is said to be δ-*improper* whenever we only allow edges coloured with δ to be incident. It must be clear that a proper edge colouring (and hence a $\delta-$minimum edge-colouring) of G is a particular δ-improper edge colouring. For a proper or δ-improper edge colouring ϕ of G, it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$ the set of edges coloured with x by ϕ. For $x, y \in \{\alpha, \beta, \gamma, \delta\}$ $\phi(x, y)$ is the partial subgraph of G spanned by this two colours, that is $E_\phi(x) \cup E_\phi(y)$ (this subgraph being an union of paths and even cycles where the colours x and y alternate). Since any two $\delta-$minimum edge-colouring of G have the same number of edges coloured δ we shall denote by $s(G)$ this number (the *colour number* as defined in [10]).

As usually, for any undirected graph G, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we consider that $|V(G)| = n$ and $|E(G)| = m$. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2 On $\delta-$minimum edge-colouring

Our goal, in this section, is to give some structural properties of $\delta-$improper colourings and $\delta-$minimum edge-colourings . Graphs considered in the following serie of lemmas will have maximum degree 3.

Lemma 1 *Assume that G can be provide with a perfect matching then any 2-factor of G contains at least $s(G)$ disjoint odd cycles.*

Proof Assume that we can find a 2-factor of G with $k < s(G)$ odd cycles. Then let us colour the edges of this two factor with α and β, excepted one edge (coloured δ) on each odd cycle of our 2-factor and let us colour the remaining edges by γ (the edges of the $2-$factor). We get hence a new edge colouring ϕ with $E_\phi(\delta) < s(G)$, impossible. □

Lemma 2 *Let ϕ be a $\delta-$minimum edge-colouring of G. Any edge in $E_\phi(\delta)$ is incident to α, β and γ. Moreover this edge has one end of degree 2 and the other of degree 3 or the two ends of degree 3.*

Proof Any edge of $E_\phi(\delta)$ is certainly incident to α, β and γ. Otherwise this edge could be coloured with the missing colour and we should obtain an edge colouring ϕ' with $|E_{\phi'}(\delta)| < |E_\phi(\delta)|$. □

Lemma 3 *Let ϕ be $\delta-$improper colouring of G then there exists a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$*

Proof Let ϕ be a δ-improper edge colouring of G. If ϕ is a proper colouring, we are done. Hence, assume that uv and uw are coloured δ. If $d(u) = 2$ we can change the colour of uv in α, β or γ since v is incident to at most two colours in this set.

If $d(u) = 3$ assume that the third edge uz incident to u is also coloured δ, then we can change the colour of uv for the same reason as above.

If uz is coloured with α, β or γ, then v and w are incident to the two other colours otherwise one of them can be recoloured with the missing colour. W.l.o.g., consider that uz is coloured α then v and w are incident to β and γ. The path P of $\phi(\alpha, \beta)$ containing uz ends eventually in v or w (since these vertices have degree 1 in $\phi(\alpha, \beta)$). We can thus assume that v or w (say v) is not the other end vertex of P. Exchanging α and β along P does not change the colours incident to v. But now uz is coloured β and we can change the colour of uv with α.

In each case, we get hence a new δ-improper edge colouring ϕ_1 with $E_{\phi_1}(\delta) \subseteq E_{\phi}(\delta)$. Repeating this process leads us to construct a proper edge colouring of G with $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$ as claimed. □

Proposition 4 *Let $v_1, v_2 \ldots, v_k \in V(G)$ such that $G - \{v_1, v_2, \ldots v_k\}$ is 3-edge colourable. Then $s(G) \leq k$.*

Proof Let us consider a 3-edge colouring of $G - \{v_1, v_2, \ldots v_k\}$ with α, β and γ and let us colour the edges incident to $v_1, v_2 \ldots, v_k$ with δ. We get a δ-improper edge colouring ϕ of G. Lemma 3 gives a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$. Hence ϕ' has at most k edges coloured with δ and $s(G) \leq k$ □

Proposition 4 above has been obtained by Steffen [10] for cubic graphs.

Lemma 5 *Let ϕ be $\delta-$improper colouring of G then $|E_{\phi}(\delta)| \geq s(G)$*

Proof Applying Lemma 3, let ϕ' be a proper edge colouring of G such that $E_{\phi'}(\delta) \subseteq E_{\phi}(\delta)$. We clearly have $|E_{\phi}(\delta)| \geq |E_{\phi'}(\delta)| \geq s(G)$ □

Lemma 6 *Let ϕ be a $\delta-$minimum edge-colouring of G. For any edge $e = uv \in E_{\phi}(\delta)$ there are two colours x and y in $\{\alpha, \beta, \gamma\}$such that the connected component of $\phi(x, y)$ containing the two ends of e is an even path joining this two ends.*

Proof W.l.o.g. Assume that u is incident to α and β and v is incident to γ (see Lemma 2). In any case (v has degree 3 or degree 2) u and v are contained in paths of $\phi(\alpha, \gamma)$ or $\phi(\beta, \gamma)$. Assume that they are contained in paths of $\phi(\alpha, \gamma)$. If these paths are disjoint then we can exchange the two colours on the path containing u, e will be incident hence to only two colours β and γ in this new edge-colouring and e could be recoloured with α, a contradiction since we consider a $\delta-$minimum edge-colouring. □

An edge of $E_\phi(\delta)$ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$.

Lemma 7 *If G is a cubic graph then*

$$|A_\Phi| \equiv |B_\Phi| \equiv |C_\Phi| \equiv s(G) \pmod 2$$

.

Proof $\phi(\alpha, \beta)$ contains $2|A_\Phi| + |B_\Phi| + |C_\Phi|$ vertices of degree 1 and must be even. Hence we get $|B_\Phi| \equiv |C_\Phi| \pmod 2$. In the same way we get $|A_\Phi| \equiv |B_\Phi| \pmod 2$ leading to $|A_\Phi| \equiv |B_\Phi| \equiv |C_\Phi| \equiv C(G) \pmod 2$ □

Remark 8 It must be clear that A_ϕ, B_ϕ and C_ϕ are not necessarily pairwise disjoint since an edge of $E_\phi(\delta)$ with one end of degree 2 is contained into 2 such sets. Assume indeed that $e = uv \in E_\phi(\delta)$ with $d(u) = 3$ and $d(v) = 2$ then, if u is incident to α and β and v is incident to γ we have an alternating path whose ends are u and v in $\phi(\alpha, \gamma)$ as well as in $\phi(\beta, \gamma)$. Hence e is in $A_\phi \cap B_\phi$.

When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained in considering the path of $\phi(\alpha, \beta)$ together with e. We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ. In the following lemma we consider en edge in A_ϕ, an analogous result holds true whenever we consider edges in B_ϕ or C_ϕ as well.

Lemma 9 *Let ϕ be a $\delta-$minimum edge-colouring of G and let e be an edge in A_ϕ (B_ϕ, C_ϕ) then for any edge $e' \in C_{A_\phi}(e)$ there is a $\delta-$minimum edge-colouring ϕ' such that $E_{\phi'}(\delta) = E_\phi(\delta) - e + e'$, $e' \in A_{\phi'}$ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. Moreover, each edge outside $C_{A_\phi}(e)$ but incident with this cycle is coloured γ.*

Proof In exchanging colours δ and α and δ and β successively along the cycle $C_{A_\phi}(e)$, we are sure to obtain an edge colouring preserving the number of edges coloured δ. Since we have supposed that ϕ is δ-minimum, at each step, the resulting edge colouring is proper and δ-minimum (Lemma 3). Hence, there is no edge coloured δ incident with $C_{A_\phi}(e)$, which means that every such edge is coloured with γ.

We can perform these exchanges until e' is coloured δ. In the $\delta-$minimum edge-colouring ϕ' hence obtained, the two ends of e' are joined by a path of $\phi(\alpha, \beta)$. Which means that e' is in A_ϕ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. □

For each edge $e \in E_\phi(\delta)$ (where ϕ is a $\delta-$minimum edge-colouring of G) and we can associate one or two odd cycles following the fact that e is in one or two sets among A_ϕ, B_ϕ or C_ϕ. Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_\phi(\delta)$.

Remark 10 Since each edge coloured with δ in a $\delta-$minimum edge-colouring is contained in a cycle. These edges cannot be bridges of G.

Lemma 11 *For each cycle $C \in \mathcal{C}$, there are no two consecutive vertices with degree two.*

Proof Otherwise, we exchange colours along C in order to put the colour δ on the corresponding edge and, by Lemma 2, this is impossible in a $\delta-$minimum edge-colouring . □

Lemma 12 *Let $e_1, e_2 \in E_\phi(\delta)$ and let $C_1, C_2 \in \mathcal{C}$ such that $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ then C_1 and C_2 are disjoint.*

Proof If e_1 and e_2 are contained in the same set A_ϕ, B_ϕ or C_ϕ, we are done since their respective ends are joined by an alternating path of $\phi(x, y)$ for some two colours x and y in $\{\alpha, \beta, \gamma\}$.

W.l.o.g. assume that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Assume moreover that there exists an edge e such that $e \in C_1 \cap C_2$. We have hence an edge $f \in C_1$ with exactly one end on C_2. We can exchange colours on C_1 in order to put the colour δ on f. Which is impossible by Lemma 9. □

By Lemma 9 any two cycles in $\mathcal{C}$ corresponding to edges in distinct sets A_ϕ, B_ϕ or C_ϕ are at distance at least 2. Assume that $C_1 = C_{A_\phi}(e_1)$ and $C_2 = C_{A_\phi}(e_2)$ for some edges e_1 and e_2 in A_ϕ. Can we say something about the subgraph of G whose vertex set is $V(C_1) \cup V(C_2)$? In general, we have no answer to this problem. However, when G is cubic and any vertex of G lies on some cycle of $\mathcal{C}$ (we shall say that $\mathcal{C}$ is *spanning*), we have a property which will be useful after. Let us remark first that whenever $\mathcal{C}$ is spanning, we can consider that G is edge-coloured in such a way that the edges of the cycles of $\mathcal{C}$ are alternatively coloured with α and β (excepted one edge coloured δ) and the remaining perfect matching is coloured with γ. For this $\delta-$minimum edge-colouring of G we have $B_\phi = \emptyset$ as well as $C_\phi = \emptyset$.

Lemma 13 *Assume that G is cubic and $\mathcal{C}$ is spanning. Let $e_1, e_2 \in A_\phi$ and let $C_1, C_2 \in \mathcal{C}$ such that $C_1 = C_{A_\phi}(e_1)$ and $C_2 = C_{A_\phi}(e_2)$. Then at least one of the followings is true*

(i) *C_1 and C_2 are at distance at least 2*
(ii) *C_1 and C_2 are joined by at least 3 edges*
(iii) *C_1 and C_2 have at least two chords each*

Proof Let $C_1 = v_0v_1 \dots v_{2k_1}$ and $C_2 = w_0w_1 \dots w_{2k_2}$. Assume that C_1 and C_2 are joined by the edge v_0w_0. By lemma 6, we can consider a $\delta-$minimum edge-colouring ϕ such that $\phi(v_0v_1) = \phi(w_0w_1) = \delta$, $\phi(v_1v_2) = \phi(w_1w_2) = \beta$ and $\phi(v_0v_{2k_1}) = \phi(w_0w_{2k_2}) = \alpha$. Moreover each edge of G (in particular v_0w_0) incident with these cycles is coloured γ. We can change the colour of v_0w_0 in β. We obtain thus a new $\delta-$minimum edge-colouring ϕ'. Performing that exchange of colours on v_0w_0 transforms the edges coloured δ v_0v_1 and w_0w_1 in two edges of $C_{\phi'}$ lying on odd cycles C'_1 and C'_2 respectively. We get hence a new set $\mathcal{C}' = \mathcal{C} - \{C_1, C_2\} + \{C'_1, C'_2\}$ of odd cycles associated to $\delta-$coloured edges in ϕ'.

From Lemma 12 C'_1 (C'_2 respectively) is at distance at least 2 from any cycle in $\mathcal{C} - \{C_1, C_2\}$. Hence $V(C'_1) \cup V(C'_2) \subseteq V(C_1) \cup V(C_2)$. It is an easy task to check now that (ii) or (iii) above must be verified. □

Lemma 14 *Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G. Then v_1 is the only vertex in $N(u)$ of degree 2.*

Proof We have seen in Lemma 2 that uv has one end of degree 3 while the other has degree 2 or 3. Hence, we have $d(u) = 3$ and $d(v_1) = 2$. Let v_2 and v_3 the other neighbors of v. From Remark 8, we know that v_2 and v_3 are not pendent vertices. Assume that $d(v_2) = 2$ and uv_2 is coloured α, uv_3 is coloured β and, finally v_1 is incident to an edge coloured γ. The alternating path of $\phi(\beta, \gamma)$ using the edge uv_3 ends with the vertex v_1 (see Lemma 8), then, exchanging the colours along the component of $\phi(\beta, \gamma)$ containing v_2 allows us to colour uv_2 with γ and uv_1 with α. The new edge colouring ϕ' so obtained is such that $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, impossible. □

Lemma 15 *Let e_1 and e_2 two edges of $E_\phi(\delta)$ contained into two distinct subsets of A_ϕ, B_ϕ or C_ϕ. Then $\{e_1, e_2\}$ induces a $2K_2$.*

Proof W.l.o.g. assume that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Since $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$ are disjoint by Lemma 12, we can consider that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{e_1}(\phi)$ are coloured γ while those having exactly one end in $C_{e_2}(\phi)$ are coloured α. Hence there is no edge between e_1 and e_2 as claimed. □

Lemma 16 *Let e_1 and e_2 two edges of $E_\phi(\delta)$ contained into the same subset A_ϕ, B_ϕ or C_ϕ. Then $\{e_1, e_2\}$ are joined by at most one edge*

Proof W.l.o.g. assume that $e_1 = u_1v_1, e_2 = u_2v_2 \in A_\phi$. Since $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$ are disjoint by Lemma 12, we can consider that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{e_1}(\phi)$ (or $C_{e_2}(\phi)$) are coloured γ. Assume that u_1u_2 and v_1v_2 are edges of G. In exchanging eventually colours α and β along the path of $\phi(\alpha, \beta)$ joining the two ends of u_1u_2 we can consider that u_1 and u_2 are incident to α while v_1 and v_2 are incident to β. We know that u_1u_2 and v_1v_2 are coloured γ. Let us colour e_1 and e_2 with γ and u_1u_2 with β and v_1v_2 with α. We get a new edge colouring ϕ' where $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 2$, contradiction since ϕ is a $\delta-$minimum edge-colouring . □

Lemma 17 *Let e_1, e_2 and e_3 three edges of $E_\phi(\delta)$ contained into the same subset A_ϕ, B_ϕ or C_ϕ. Then $\{e_1, e_2, e_3\}$ contains at most one edge more.*

Proof W.l.o.g. assume that $e_1 = u_1v_1, e_2 = u_2v_2$ and $e_3 = u_3v_3 \in A_\phi$. From Lemma 16 we have just to suppose that (up to the names of vertices) $u_1u_3 \in E(G)$ and $v_1v_2 \in E(G)$. In exchanging eventually the colours α and β along the 3 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2 and e_3, we can suppose that u_1 and u_3 are incident to β while v_1 and v_2 are incident to α. Let ϕ' be obtained from ϕ when u_1u_3 is coloured with α, v_1v_2 with β and u_1v_1 with γ. It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a $\delta-$minimum edge-colouring . □

Lemma 18 *Let $e_1 = u_1v_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G. Then for any edge $e_2 = u_2v_2 \in E_\phi(\delta)$ $\{e_1, e_2\}$ induces a $2K_2$.*

Proof From Lemma 15 we have to consider that e_1 and e_2 are contained in the same subset A_ϕ, B_ϕ or C_ϕ. Assume w.l.o.g. that they are contained in A_ϕ. From Lemma 16 we have just to consider that there is a unique edge joining these two edges and we can suppose that $u_1u_2 \in E(G)$. In exchanging eventually the colours α and β along the 2 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2, we can suppose that u_1 and u_2 are incident to β while v_1 and v_2 are incident to α. We know that u_1u_2 is coloured γ. Let ϕ' be obtained from ϕ when u_1u_2 and u_2v_2 are coloured with α and u_1v_1 is coloured with γ. It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a $\delta-$minimum edge-colouring . □

3 Applications and problems

3.1 On a result by Payan

In [8] Payan showed that it is always possible to edge-colour a graph of maximum degree 3 with three maximal matchings (for the inclusion) and introduces henceforth a notion of *strong-edge colouring* where a strong edge-colouring means that one colour is a strong matching while the remaining colours are usual matchings. Payan conjectured that any $d-$regular graph has d pairwise disjoint maximal matchings and showed that this conjecture holds true for graphs with maximum degree 3.

Theorem 19 *Let G be a graph with maximum degree at most 3. Then G has $\delta-$minimum edge-colouring ϕ where $E_\phi(\delta)$ is a strong matching and, moreover, any edge in $E_\phi(\delta)$ has its two ends of degree 3 in G.*

Proof Let ϕ be a $\delta-$minimum edge-colouring of G. From Lemma 15, any two edges of $E_\phi(\delta)$ belonging to distinct subsets in A_ϕ, B_ϕ and C_ϕ induce a strong matching. Hence, we have to find a $\delta-$minimum edge-colouring where each subset A_ϕ, B_ϕ or C_ϕ induces a strong matching (with the supplementary property that the end vertices of these edges have degree 3). That means that we can work on each subset A_ϕ, B_ϕ and C_ϕ independently. W.l.o.g., we only consider A_ϕ here.

Assume that $A_\phi = \{e_1, e_2, \dots e_k\}$ and $A'_\phi = \{e_1, \dots e_i\}$ $(1 \le i \le k-1)$ is a strong matching and each edge of A'_ϕ has its two ends with degree 3 in G. Consider the edge e_{i+1} and let $C = C_{e_{i+1}}(\phi) = (u_0, u_1 \dots u_{2p})$ be the odd cycle associated to this edge (Lemma 6).

Let us mark any vertex v of degree 3 on C with a $+$ whenever the edge of colour γ incident to this vertex has its other end which is a vertex incident to an edge of A'_ϕ and let us mark v with $-$ otherwise. By Lemma 11 a vertex of degree 2 on C has its two neighbors of degree 3 and by Lemma 18 these two vertices are marked with a $-$. By Lemma 17 we cannot have two consecutive vertices marked with a $+$. Hence, C must have two consecutive vertices of degree 3 marked with $-$ whatever is the number of vertices of degree 2 on C.

Let u_j and u_{j+1} be two vertices of C of degree 3 marked with $-$ (j being taken module $2p+1$). We can transform the edge colouring ϕ by exchanging colours on C uniquely, in such a way that the edge of colour δ of this cycle is u_ju_{j+1}. In the resulting edge colouring ϕ_1 we have $A_{\phi_1} = A_\phi - e_{i+1} + u_ju_{j+1}$ and $A'_{\phi_1} = A'_\phi + u_ju_{j+1}$ is a strong matching where each edge has its two ends of degree 3. Repeating this process we are left with a new $\delta-$minimum colouring

ϕ' where $A_{\phi'}$ is a strong matching. $\square$

This result has been obtained first by Payan [8], but his technique does not exhibit explicitely the odd cycles associated to the edges of $E_\phi(\delta)$ and their properties. It appears in [4] with this proof.

Corollary 20 *Let G be a graph with maximum degree 3 then there are $s(G)$ vertices of degree 3 pairwise non-adjacent $v_1 \dots v_{s(G)}$ such that $G - \{v_1 \dots v_{s(G)}\}$ is 3-colourable.*

Proof Pick a vertex on each edge coloured δ in a δ-minimum colouring ϕ of G where $E_\phi(\delta)$ is a strong matching (Theorem 19). We get a subset S of vertices satisfying our corollary. $\square$

Steffen [10] obtained Corollary 20 for bridgeless cubic graphs.

3.2 Parsimonious edge colouring

Let $\chi'(G)$ be the classical chromatic index of G. For convenience let

$$c(G) = max\{|E(H)| : H \subseteq G \ \chi'(H) = 3\}$$

$$\gamma(G) = \frac{c(G)}{E(G)}$$

Staton [9] (and independently Locke [7]) showed that whenever G is a cubic graph distinct from K_4 then G contains a bipartite subgraph (and hence a 3-edge colourable graph, by König's theorem [6]) with at least $\frac{7}{9}$ of the edges of G. Bondy and Locke [2] obtained $\frac{4}{5}$ when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever G is a cubic graph, we have $\gamma(G) \geq \frac{13}{15}$ while for graphs with maximum degree 3 they obtained $\gamma(G) \geq \frac{26}{31}$. Our purpose here is to show that $\frac{13}{15}$ is a lower bound for $\gamma(G)$ when G has maximum degree 3, with the exception of the graph G_5 depicted in Figure 1 below.

Lemma 21 *Let G be a graph with maximum degree 3 then $\gamma(G) = 1 - \frac{s(G)}{m}$.*

Proof Let ϕ be a $\delta-$minimum edge-colouring we need to colour $s(G)$ edges of G with δ by Lemma 12. We have thus $m - s(G)$ edges coloured with α, β and γ. The results follows. $\square$

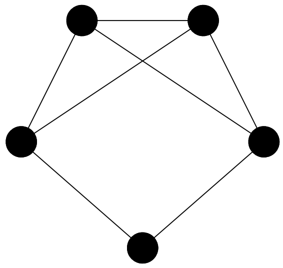

Fig. 1. G_5

Lemma 22 *[1] Let G be a graph with maximum degree* 3*. Assume that* $v \in V(G)$ *is such that* $d(v) = 1$ *then* $\gamma(G) > \gamma(G - v)$*.*

Proof We have certainly $s(G - v) = s(G)$ since v is not incident to an edge coloured δ in a $\delta-$minimum edge-colouring of G by Lemma 2. We have thus

$$\gamma(G) = 1 - \frac{s(G)}{m} > 1 - \frac{s(G - v)}{m - 1} = \gamma(G - v)$$

as claimed. □

A *triangle* $T = \{a, b, c\}$ is said to be *reducible* whenever its neighbors are distinct. When T is a reducible triangle in G (G having maximum degree 3) we can obtain a new graph G' with maximum degree 3 in shrinking this triangle into a single vertex and joining this new vertex to the neighbors of T in G.

Lemma 23 *[1] Let G be a graph with maximum degree* 3*. Assume that* $T = \{a, b, c\}$ *is a reducible triangle and let* G' *be the graph obtained by reduction of this triangle. Then* $\gamma(G) > \gamma(G')$*.*

Proof Assume that $s(G) = s(G')$ then

$$\gamma(G) = 1 - \frac{s(G)}{m} > 1 - \frac{s(G')}{m - 3} = \gamma(G')$$

and we are done.

Hence, we have just to prove that $s(G) = s(G')$. It is a classical task to transform an edge colouring of G' into an edge colouring of G without increasing the number of edges coloured δ. We get immediately that $s(G') \geq s(G)$. Let ϕ be a $\delta-$minimum edge-colouring of G. Using Theorem 19 we can suppose that we have at most one edge coloured δ involved in T and the incident edges to T. When considering G', we get eventually a a δ-improper colouring ϕ_1, where v is eventually the only vertex incident to 2 or 3 edges coloured δ. Leading to $s(G)$, $s(G) + 1$ or $s(G) + 2$ edges in $E_{\phi_1}(\delta)$. This colouring can be transformed in a proper colouring ϕ' with colouring ϕ' with Lemma 3 in order to eliminate the colour δ appearing on one or two edges incident to v if necessary. Since this operation does not modify the colour of the edges coloured δ which are not incident with v, we have $s(G) \geq s(G')$. □

Lemma 24 *Let G be a cubic graph which can be factored into $s(G)$ cycles of length 5 and without reducible triangle. The every 2-factor of G contains $s(G)$ cycles of length 5.*

Proof Since G has no reducible triangle, all cycles in a 2-factor have length at least 4. Let $\mathcal{C}$ be any 2-factor of G. When n_5 denotes the number of cycles of length 5 and $n_>$ denotes the number of cycles on at least 6 vertices in $\mathcal{C}$, there is an edge-coloring of G which uses at most $n_5 + n_>$ edges of color δ. In addition, since $\mathcal{C}$ is a spanning subgraph we have $5s(G) \geq 5n_5 + 6n_>$ and if $n_> \neq 0$, $s(G) > n_5 + n_>$, a contradiction. Thus $n_> = 0$ and $n_5 = s(G)$, the result follows. □

Theorem 25 *Let G be a graph with maximum degree 3 and $G \neq G_5$ then $\gamma(G) \geq \frac{13}{15}$.*

Proof From Lemma 22 and Lemma 21, we can consider that G has only vertices of degree 2 or 3 and that G contains no reducible triangle. Let V_2 be the set of vertices with degree 2 in G and V_3 those of degree 3.

Assume that we can associate a set P_e of at least 5 distinct vertices of V_3 for each edge $e \in E_\phi(\delta)$ in a $\delta-$minimum edge-colouring ϕ of G. Assume moreover that

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P'_e = \emptyset \tag{1}$$

Then

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \geq 1 - \frac{\frac{|V_3|}{5}}{\frac{3}{2}|V_3| + |V_2|}$$

Hence

$$\gamma(G) \geq 1 - \frac{\frac{2}{15}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

Which leads to $\gamma(G) \geq \frac{13}{15}$ as claimed.

It remains to see how to construct the sets P_e satisfying property (1). Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_\phi(\delta)$ (see Lemma 12). Let $e \in E_\phi(\delta)$, assume that e is contained in a cycle $C \in \mathcal{C}$ of length 3. By Lemma 12, the edges incident to that triangle have the same colour in α, β or γ. This triangle is hence reducible, impossible. We can thus consider that each cycle of $\mathcal{C}$ has length at least 5. By Lemma 2 and Lemma 14, we know that whenever such a cycle contains vertices of V_2, their distance on this cycle is at least 3. Which means that every cycle $C \in \mathcal{C}$ contains at least 5 vertices in V_3 as soon as C has length at least 7 or C has length 5 but does not contain a vertex of V_2. For each edge $e \in E_\phi(\delta)$ contained in such a cycle we associate P_e as any set of 5 vertices of V_3 contained in the cycle.

It remains eventually edges in $E_\phi(\delta)$ which are contained in a C_5 of $\mathcal{C}$ using a (unique) vertex of V_2. Let $C = a_1a_2a_3a_4a_5$ be such a cycle and assume that $a_1 \in V_2$. By Lemma 2 and Lemma 17, a_1 is the only vertex of degree 2 and in exchanging colours along this cycle, we can suppose that $a_1a_2 \in E_\phi(\delta)$. Since $a_1 \in V_2$, $e = a_1a_2$ is contained into a second cycle C' of $\mathcal{C}$ (see Remark 8). If C' contains a vertex $x \in V_3$ distinct from a_2, a_3, a_4 and a_5 then we set $P_e = \{a_2, a_3, a_4, a_5, x\}$. Otherwise $C' = a_1a_2a_4a_3a_5$ and G is isomorphic to G_5, impossible.

The sets $\{P_e | e \in E_\phi(\delta)\}$ are pairwise disjoint since any two cycle of $\mathcal{C}$ associated to distinct edges in $E_\phi(\delta)$ are disjoint. Hence property 1 holds and the proof is complete. □

Albertson and Haas [1] proved that $\gamma(G) \geq \frac{26}{31}$ when G is a graph with maximum degree 3. The bound given in Theorem 25 is better.

Corollary 26 *Let G be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. Then G is a cubic graph which can be factored into $s(G)$ cycles of length 5. Moreover every 2-factor of G has this property.*

Proof The optimal for $\gamma(G)$ in Theorem 25 is obtained whenever $s(G) = \frac{|V_3|}{5}$ and $|V_2| = 0$. That is, G is a cubic graph together with a 2-factor of $s(G)$ cycles of length 5. Moreover by Lemma 23 G has no reducible triangle, the result comes from Lemma 24. □

As pointed out by Albertson and Haas [1], Petersen's graph with $\gamma(G) = \frac{13}{15}$ supplies the extremal example for cubic graphs. Steffen [11] proved that the only cubic bridgeless graph with $\gamma(G) = \frac{13}{15}$ is the Petersen's graph. In fact, we can extend this result to graphs with maximum degree 3 where bridges are allowed (excluding the graph G_5).

Let P' be the cubic graph on 10 vertices obtained from two copies of G_5 (Figure 1) in joining by an edge the two vertices of degree 2.

Theorem 27 *Let G be a connected graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. Then G is isomorphic to Petersen's graph or to P'.*

Proof Let G be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. From Corollary 26, we can consider that G is cubic and G has a 2-factor of cycles of length 5. Let $\mathcal{C} = \{C_1 \dots C_{s(G)}\}$ be such a 2-factor ($\mathcal{C}$ is spanning). Let ϕ be a $\delta-$minimum edge-colouring of G induced by this $2-$factor.

W.l.o.g. consider two cycles in $\mathcal{C}$, namely C_1 and C_2, and let us denote $C_1 = v_1v_2v_3v_4v_5$ while $C_2 = u_1u_2u_3u_4u_5$ and assume that $v_1u_1 \in G$. From Lemma 13 C_1 and C_2 are joined by at least 3 edges or have 2 chords. If $S(G) > 2$ there is a cycle $C_3 \in \mathcal{C}$. W.l.o.g., G being connected, we can suppose that C_3 is joined to C_1 by an edge. Applying once more time Lemma 13, C_1 and C_3 have two chords or are joined by at least 3 edges, contradiction with the constraints imposed by C_1 and C_2. Hence $s(G) = 2$ and G has 10 vertices, which leads to a graph isomorphic to P' or Petersen's graph as claimed. □

We do not know example of 3-connected cubic graphs (excepted Petersen's graph) with a 2-factor of induced cycles of length 5 with chromatic index 4 and we propose the following conjecture.

Conjecture 28 *Let G be a 3-connected cubic graph distinct from Petersen's graph having a 2-factor of cycles of length 5 which are induced cycles of G, then G is 3-edge colourable.*

As a first step towards the resolution of this Conjecture we propose :

Theorem 29 *Let G be a cubic graph which can be factored into $s(G)$ induced cycles of length 5, then G is the Petersen's graph.*

Proof Let $\mathcal{F}$ be a 2-factor of $s(G)$ cycles of length 5 in G, every cycle of $\mathcal{F}$ being an induced cycle of G. We consider the $\delta-$minimum edge-colouring ϕ such that the edges of all cycles of $\mathcal{F}$ are alternatively coloured α and β excepted for exactly one edge per cycle which is coloured with δ, all the remaining edges of G being coloured γ. By construction we have $B_\phi = C_\phi = \emptyset$ and $A_\phi = \mathcal{F}$.

By Lemma 13 two cycles of $\mathcal{F}$ which are connected must be connected with at least three edges. Consequently a cycle of $\mathcal{F}$ cannot be connected by at least one edge to more than one other cycle of $\mathcal{C}$. It follows that $\mathcal{C}$ contains exactly two induced cycles of length 5. The only graph with 10 vertices and $s(G) = 2$ with such a 2-factor is the Petersen's graph. □

Comments: The index $s(G)$ used here is certainly less than $o(G)$ the *oddness* of G used by Huck and Kochol [5]. $o(G)$ is the minimum number of odd cycles in any 2-factor of a cubic graphs (assuming that we consider graphs with that property). Obviously $o(G)$ is an even number and it is an easy task to construct a cubic graph G with $s(G)$ odd which ensures that $0 < s(G) < o(G)$. We can even construct cyclically-5-edge-connected cubic graphs with that property with $s(G) = k$ for any integer $k \neq 1$ (see [4] and [11]). It can be pointed out that, using a parity argument (see 7, or [10]) a graph with oddness at least 2 needs certainly to be edge coloured with at least 2 edges in the less popular colour. In other words, $o(G) = 2 \Leftrightarrow s(G) = 2$.

When G is a cubic bridgeless planar graph, we know from the $4CT$ that G is $3-$edge colourable and hence $\gamma(G) = 1$. Albertson and Haas [1] gave $\gamma(G) \geq \frac{6}{7} - \frac{2}{35m}$ when G is a planar bridgeless graph with maximum degree 3. Our Theorem 25 improves this lower bound (allowing moreover bridges). On the other hand, they exhibit a family of planar graphs with maximum degree 3 (bridges are allowed) for which $\gamma(G) = \frac{8}{9} - \frac{2}{9n}$.

As in [11] we denote $g(\mathcal{F}) = \min\{V(C)| \quad C \quad \text{is an odd circuit of} \quad \mathcal{F}\}$ and $g^+(G) = \max\{g(\mathcal{F})| \quad \mathcal{F} \quad \text{is a} \quad 2-\text{factor of G}\}$. We suppose that $g^+(G)$ is defined, that is G has at least one 2-factor (when G is a cubic bridgeless graphs this condition is obviously fulfilled).

When G is cubic bridgeless, Steffen [11] showed that we have :

$$\gamma(G) \geq \quad \max\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\}$$

The difficult part being to show that $\gamma(G) \geq \frac{11}{12}$. We can remark that whenever $g^+(G) \geq 11$ we obtain the same result for graphs with maximum degree 3 and having at most $\frac{n}{3}$ vertices of degree 1 or 2.

Theorem 30 *Let G be a graph with maximum degree 3 and having at least one 2-factor. Then $\gamma(G) \geq 1 - \frac{2}{3g^+(G)}$.*

Proof We have $\gamma(G) = 1 - \frac{s(G)}{m}$ where $m = \frac{3n}{2}$ and, obviously, $s(G) \leq \frac{n}{g^+(G)}$. Hence $\gamma(G) \geq 1 - \frac{2}{3g^+(G)}$. □

Theorem 31 *Let G be a graph with maximum degree 3 and having at least one 2-factor. Let V_i $(i = 1..3)$ be the set of vertices of degree i. Assume that $|V_2| \leq \frac{n}{3}$ and $g^+(G) \geq 11$ then $\gamma(G) \geq \quad \max\{1 - \frac{2}{3g^+(G)}, \frac{11}{12}\}$.*

Proof Since, by Lemma 22 $\gamma(G) > \gamma(G - v)$ we can assume that $V_1 = \emptyset$. From Theorem 30 we have just to prove that $\gamma(G) \geq \frac{11}{12}$. Following the proof of Theorem 25, we try to associate a set P_e of at least 8 distinct vertices of V_3 for each edge $e \in E_\phi(\delta)$ in a $\delta-$minimum edge-colouring ϕ of G such that

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P'_e = \emptyset \qquad (2)$$

Indeed, let $\mathcal{F}$ be 2-factor of G where each odd cycle has length at least 11 and let $C_1, C_2 \dots C_{2k}$ be its set of odd cycles. We have, obviously $s(G) \leq 2k$. Let $V_3^{1\dots 2k}$ and $V_2^{1\dots 2k}$ be the sets of vertices of degree 3 and 2 respectively contained in the odd cycles . As soon as $|V_3^{1\dots 2k}| \geq 8s(G)$ we have

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{3}{2}|V_3| + |V_2|} \geq 1 - \frac{\frac{|V_3^{1\dots 2k}|}{8}}{\frac{3}{2}|V_3| + |V_2|} \tag{3}$$

which leads to

$$\gamma(G) \geq 1 - \frac{\frac{2|V_3^{1\dots 2k}|}{24|V_3|}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

Since $|V_3| \geq |V_3^{1\dots 2k}|$, we have

$$\gamma(G) \geq 1 - \frac{\frac{2}{24}}{1 + \frac{2}{3}\frac{|V_2|}{|V_3|}}$$

and

$$\gamma(G) \geq \frac{11}{12}$$

as claimed.

It remains the case where $|V_3^{1\dots 2k}| < 8s(G)$. Since each cycle has at least 11 vertices we have $|V_2^{1\dots 2k}| > 11 \times 2k - |V_3^{1\dots 2k}| > 3s(G)$.

$$\gamma(G) = \frac{m - s(G)}{m} \geq \frac{m - \frac{|V_2^{1\dots 2k}|}{3}}{m}$$

We have

$$\frac{m - \frac{|V_2^{1\dots 2k}|}{3}}{m} \geq \frac{11}{12}$$

when

$$m \geq 4|V_2^{1\dots 2k}| \tag{4}$$

Since $|V_2| \leq \frac{n}{3}$ we have $|V_3| \geq \frac{2n}{3}$ and

$$m = 3\frac{|V_3|}{2} + |V_2| = 3\frac{n - |V_2|}{2} + |V_2| = 3\frac{n}{2} - \frac{|V_2|}{2} \geq 4\frac{n}{3} \geq 4|V_2^{1\dots 2k}| \tag{5}$$

and the result holds. □

References

[1] M.O. Albertson and R. Haas. Parsimonious edge colouring. *Discrete Mathematics*, 148:1–7, 1996.

[2] J.A. Bondy and S. Locke. Largest bipartite subgraphs in triangle free graphs with maximum degree three. *J. Graph Theory*, 10:477–504, 1986.

[3] J.L. Fouquet. Graphes cubiques d'indice chromatique quatre. *Annals of Discrete Mathematics*, 9:23–28, 1980.

[4] J.L. Fouquet. *Contribution à l' étude des graphes cubiques et problèmes hamiltoniens dans les graphes orientés.* PhD thesis, Université Paris SUD, 1981.

[5] A. Huck and K. Kochol. Five cycle double covers of some cubic graphs. *J. Combin. Theory Ser. B*, 64:111–125, 1995.

[6] D. König. Über graphen und ihre anwendung auf determinantentheorie un mengenlehre. *Math.Ann*, 77:453–465, 1916.

[7] S.C. Locke. Maximum k-colourable subgraphs. *Graph Theory*, 6:123–132, 1982.

[8] C. Payan. Sur quelques problèmes de couverture et de couplage en combinatoire. Thèse d'état, 1977.

[9] W. Staton. Edge deletions and the chromatic number. *Ars Combin*, pages 103–106, 1980.

[10] E. Steffen. Classifications and characterizations of snarks. *Discrete Mathematics*, 188:183–203, 1998.

[11] E. Steffen. Measurements of edge-uncolorability. *Discrete Mathematics*, 280:191–214, 2004.

[12] V.G. Vizing. On an estimate of the chromatic class of p graph. *Discrete Analiz*, 3:25–30, 1964.