4 rue Léonard de Vinci
BP 6759
F-45067 Orléans Cedex 2
FRANCE
http://www.univ-orleans.fr/lifo

Rapport de Recherche

Kaiser-Raspaud Conjecture on Cubic Graphs with few vertices

J.L. Fouquet and J.M. Vanherpe
LIFO, Université d'Orléans

Rapport n^{o} **RR-2008-08**

Kaiser-Raspaud Conjecture on Cubic Graphs with few vertices.

J.L. Fouquet and J.M. Vanherpe

L.I.F.O., Faculté des Sciences, B.P. 6759
Université d'Orléans, 45067 Orléans Cedex 2, FR

Abstract

A conjecture of Kaiser and Raspaud [6] asserts (in a special form due to Máčajová and Škoviera) that every bridgeless cubic graph has two perfect matchings whose intersection does not contain any odd edge cut. We prove this conjecture for graphs with few vertices and we give a stronger result for traceable graphs.

Key words: Cubic graph; Edge-partition; Traceable graphs

1 Introduction

The following conjecture first appeared in [5] is known as The Berge-Fulkerson Conjecture, see [9].

Conjecture 1 *If G is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.*

Fan and Raspaud [3] conjectured that any bridgeless cubic graph can be provided with three perfect matchings with empty intersection (we shall say also *non intersecting perfect matchings*).

Conjecture 2 *[3] Every bridgeless cubic graph contains perfect matching M_1, M_2, M_3 such that*

$$M_1 \cap M_2 \cap M_3 = \emptyset$$

A *join* in a graph G is a set $J \subseteq E(G)$ such that the degree of every vertex in G has the same parity as its degree in the graph $(V(G), J)$. A perfect matching being a particular join in a cubic graph Kaiser and Raspaud conjectured in [6]

Preprint submitted to Elsevier *3 October 2008*

Conjecture 3 *[6] Every bridgeless cubic graph admits two perfect matching M_1, M_2 and a join J such that*

$$M_1 \cap M_2 \cap J = \emptyset$$

If true Conjecture 1 implies Conjecture 2 which itself implies Conjecture 3. Those conjectures being obviously true for cubic graphs with chromatic index 3, it is usefull to consider the following parameter for cubic graphs. The *oddness* of a cubic graph G is the minimum number of odd circuits in a 2-factor of G. In [6] Kaiser and Raspaud proved that Conjecture 3 holds true for bridgeless cubic graph of oddness two.

In this paper, we consider Conjecture 3 and an equivalent form due to Máčajová and Škoviera ([8], see Conjecture 5). We prove that a minimal counter example to Conjecture 5 must have at least 50 vertices.
The *cyclic edge connectivity* is the size of a smallest edge cut in a graph such that at least two of the connected components contain cycles. We prove that Conjecture 5 holds true while the order of the graph is less than a function of the cyclic edge connectivity.
In addition, we consider cubic bridgeless *traceable* graphs. A graph G is said to be traceable whenever G has an *Hamiltonian path* that is a path which visits each vertex exactly once. As we will see later the oddness of those graphs is either 0 or 2, thus Conjecture 3 holds true. We prove a stronger result for cubic bridgeless traceable graphs with chromatic index 4.

When A is a set of edges, $V(A)$ will denote the set of vertices which are an end-point of some edge in A. If M is a perfect matching of a cubic graph $G = (V, E)$, we denote G_M the 2-factor $G_M = (V, E - M)$. When X is a set of vertices, δX denotes the set of edges with precisely one end in X. An edge cut is a set of edges whose removal renders the graph disconnected and which is inclusion-wise minimal with this property. For basic graph-theoretic terms, we refer the reader to Bondy and Murty [1].

2 Preliminary results

2.1 Fractional perfect matchings

The following results belongs to folklore

Theorem 4 *Let G be a cubic bridgeless graph. G is 3-edge colorable if and only if there is a perfect matching in G that does not contain any odd edge cut.*

Proof Assume that G has a 3-edge coloring using colors α, β and γ. Let M_α be the set of edges colored with α, if M_α contains an odd edge-cut, there must be a partition (V_1, V_2) of $V(G)$ into two odd sets such that the edges of X have one end in V_1 and the other end in V_2. Since all the edges of X are colored with α, the set of edges in V_1 colored with β must be a perfect matching in V_1, a contradiction since V_1 has an odd number of vertices.

Conversely, consider a perfect matching M which does not contain any odd edge-cut. Suppose that the 2-factor G_M contains an odd cycle C, thus δC is an odd edge cut entirely contained in M, a contradiction. Consequently G_M does not contain any odd cycle, it follows that G is 3-edge colorable. □

This result is related to the following, an equivalent form of Conjecture 3 due to Máčajová and Škoviera.

Conjecture 5 *Every bridgeless cubic graph has two perfect matchings M_1, M_2 so that $M_1 \cap M_2$ does not contain an odd edge cut.*

Equivalence comes from the fact that a set of edges contains a join if and only if this set intersects all odd edge cuts([8]).

In order to prove this conjecture for bridgeless cubic graphs with few vertices, we will consider the notion of *fractional perfect matching* as used in [7].

For a graph $G = (V, E)$, a vector w of $\mathbb{R}^E$ is said to be a fractional perfect matching whenever w satisfies the following properties (the entry of w corresponding to $e \in E$ being denoted $w(e)$ and $w(A) = \Sigma_{e \in A} w(e)$, for $A \subseteq E$) :
- $0 \leq w(e) \leq 1$ for each edge e of G
- $w(\delta\{v\}) = 1$ for each vertex v of G
- $w(\delta X) \geq 1$ for each $X \subseteq V$ of odd cardinality.

The perfect matching polytope is the convex hull of the set of incidence vectors of perfect matchings of G. In [2] Edmonds showed that a vector $w \in \mathbb{R}^E$ belongs to the perfect matching polytope of G if and only if it is a fractional perfect matching

Moreover, when χ^A denotes the characteristic vector of the edge set A we will use the following tool :

Lemma 6 *[7] If w is a fractional perfect matching in a graph G = (V,E) and $c \in \mathbb{R}^E$, then G has a perfect matching M such that $c.\chi^M \geq c.w$ where . denotes the scalar product. Moreover, there exists such a perfect matching M that contains exactly one edge of each cut C with $w(C) = 1$.*

It is shown in [7], among other results, there must exist a perfect matching M_1 which intersects all edge cuts of size 3 into a single edge and a perfect matching M_2 such that $|M_2 - M_1| \geq \frac{4}{15}|E(G)|$. Since the size of a perfect matching is precisely $\frac{n}{2}$, it must be pointed out that $|M_1 \cap M_2| \leq \frac{n}{10}$.

Observe that there is an alternate proof of Theorem 4 in terms of fractional perfect matchings :
Consider a perfect matching M that does not contain any odd edge cut. We define a fractional perfect matching as follows : $w(e) = 0$ when $e \in M$ and $w(e) = \frac{1}{2}$ otherwise. Given an odd set of vertices, say X, δX is an odd edge cut which intersects M in a odd number of edges, since $\delta X \not\subseteq M$, w is a fractional perfect matching. By Lemma 6 there is a perfect matching M' such that

$$c.\chi^{M'} \geqslant c.w = \frac{1}{2} \times \frac{2}{3} \times |E| = \frac{n}{2}.$$

Since $c.\chi^{M'} = |M' \backslash M|$ we have $|M' \backslash M| = \frac{n}{2} = |M'|$ and thus $M \cap M' = \emptyset$. It follows that $\chi'(G) = 3$.

2.2 Balanced perfect matchings

Let M be a perfect matching of a cubic graph and let $\mathcal{C} = \{C_1, C_2 \dots C_k\}$ be the 2-factor G_M. $A \subseteq M$ is a *balanced* $M-$matching whenever there is a perfect matching M' such that $M \cap M' = A$. That means that each odd cycle of $\mathcal{C}$ is incident to at least one edge in A and the sub-paths determined by the ends of M' on the cycles of $\mathcal{C}$ incident to A have odd lengths.

Lemma 7 *Let M be a perfect matching of a cubic graph G. A matching A is a M-balanced matching if and only if the connected components of $G_M - V(A)$ are either odd paths or even cycles.*

Proof Since G_M is a 2-factor of G, the connected components of the subgraph $G_M - V(A)$ must be cycles or paths. Since A is a M-balanced matching, $M - A$ is a perfect matching of $G_M - V(A)$, thus the connected components of this graph must be even cycles or odd paths.

Conversely, assume that the connected components of $G_M - V(A)$ are paths or even cycles. Let A' be a perfect matching of $G_M - V(A)$, we set $M' = A \cup A'$ and we are done. □

3 On cubic graphs with few vertices

We first prove that Conjecture 5 holds true for bridgeless cubic graphs having less than 50 vertices

Theorem 8 *Let G be a bridgeless cubic graph of order $n < 50$. There is perfect matchings M and M' such that $M \cap M'$ does not contain any edge cut.*

Proof We know from [7]that there must exist a perfect matching M which intersects all edge cuts of size 3 into a single edge and a perfect matching M' such that $|M \cap M'| \leq \frac{n}{10}$. It is assumed $n < 50$, thus $|M \cap M'| < 5$. An odd edge cut, say C in $M \cap M'$ must be of size 3, but a such edge cut cannot exist since M intersects C in precisely one edge. □

Let us now consider cyclic edge connectivity in cubic graphs.

Theorem 9 *Let G be a cubic graph of order n with cyclic edge connectivity k. One of the following holds.*

(1) There are two perfect matchings M and M' such that $|M \cap M'| \leq \frac{n}{2(2\lfloor\frac{k}{2}\rfloor+3)}$.

(2) For all perfect matching M there is an edge cut of size $2\lfloor\frac{k}{2}\rfloor + 1$ entirely contained in M.

Proof For convenience we denote $s = 2\lfloor\frac{k}{2}\rfloor+3$. Let M be a perfect matching that does not contain any edge cut of size $s-2$. The graph being cyclically k-edge connected $s-2$ is the minimum size of an odd edge cut in G. We set $w(e) = \frac{1}{s}$ when $e \in M$ and $w(e) = \frac{s-1}{2s}$ otherwise. If X is an odd set of vertices, δX is an odd edge cut of size at least $s-2$. When $|\delta X| \geq s$, $w(\delta X) \geq 1$, when $|\delta X| = s-2$ there is at least 2 edges of δX that does not belong to M, thus $w(\delta X) \geq 1$ again and w is a fractional perfect matching.

Thus we apply Lemma 6 with $c = 1 - \chi^M$ and there is a perfect matching, say M' such that $c.\chi^{M'} \geq c.w = n \times \frac{s-1}{2s}$.

Since $c.\chi^{M'} = |M' - M|$ and $|M'| = \frac{n}{2}$ it follows that $|M \cap M'| \leq \frac{n}{2s}$. □

Theorem 10 *Let G be a cubic graph of order n with cyclic edge connectivity $k \geq 4$. If $n < 2(2\lfloor\frac{k}{2}\rfloor+3)(2\lfloor\frac{k}{2}\rfloor+1)$ then there are two perfect matchings whose intersection does not contain any odd edge cut.*

Proof Once again we denote $s = 2\lfloor\frac{k}{2}\rfloor + 3$. We can assume that there is an edge cut of size $s-2$ entirely contained in every perfect matching, otherwise, from Theorem 9 there is two perfect matchings whose intersection contains less than $s-2$ edges and we are done since any odd edge cut contains at least $s-2$ edges.

Let M be a perfect matching of G. We set $w(e) = \frac{1}{s-2}$ when $e \in M$ and $w(e) = \frac{s-3}{2(s-2)}$ otherwise. The weight of an edge being at least $\frac{1}{s-2}$, an odd edge cut having at least $s-2$ edges w is a fractional perfect matching. If $c = 1 - \chi^M$, by Lemma 6 there is a perfect matching M' which intersects in a single edge every edge cut C such that $w(C) = 1$.

In addition we know that $c.\chi^{M'} \geq c.w$, in other words $|M \cap M'| \geq \frac{2}{3} \times |E| \times \frac{s-3}{2(s-2)} = \frac{n}{2} \times \frac{s-3}{s-2}$. Consequently $|M \cap M'| \leq \frac{n}{2(s-2)}$. Since $n < 2s(s-2)$ we have that $|M \cap M'| \leq s$.

Assume that $M \cap M'$ contains an odd edge cut C. By the above relation $|C| = s-2$ and then $w(C) = 1$, a contradcition since M' intersects the edge cuts of size $s-2$ in a single edge. □

An example of consequence of Theorem 10 is that Conjecture 5 and therefore Conjecture 3 hold true for cyclically 4-edge-connected graphs having less than 70 vertices.

4 On cubic traceable graphs

In the following we prove a stronger result for cubic bridgeless traceable graphs.

4.1 Edge-coloring, notations

In this section we consider a cubic bridgeless traceable graph G which is not 3 edge-colorable, we provide a 4 edge-coloring of G which shows that the oddness of this graph is 2, we give some notations and describe the construction of an auxiliary graph H, a usefull tool that will be used in Theorem 11 in the next section.

Let $P = a_1 a_2 \ldots a_{n-1} a_n$ be an Hamiltonian path of G. For $i = 1 \ldots n-1$ we set $e_i = a_i a_{i+1}$. For convenience, a vertex will be denoted with it's index in the Hamiltonian path P and a chord of the path P connecting the vertex i to the vertex j will be denoted ij. Observe that the edge of the Hamiltonian path P connecting the vertex i to the vertex $i+1$ is denoted e_i.

An edge-coloring of G. We color the edge-set of G with a proper edge-coloring using four colors, namely α, β, γ, δ as follows :
• Let e_i be colored with α when e_i is odd and with β when e_i is even.
• Let $e_\delta \neq e_1$ be an edge adjacent to 1 and $e'_\delta \neq e_{n-1}$ be an edge adjacent to a_n, those two edges are colored with δ.
• All remaining edges of G being colored with γ.

Observe that the set M_α containing all the edges colored with α is a perfect matching. Let $C_1, C_2, \dots C_k$ be the connected components of the 2-factor G_{M_α}. The components of this 2-factor which do not contain the edge e_δ nor the edge e'_δ are even cycles since their edges are either colored with β or with γ. Since the 2-factor G_{M_α} contains an even number of odd cycles the edges e_δ and e'_δ must belong to distinct odd cycles of the 2-factor. Thus without loss of generality, we assume that e_δ is an edge of C_1 while e'_δ is an edge of C_k.

The above observation shows that those graphs have oddness 2.

The edges $e_{min(C)}$ and $e_{max(C)}$. For $C \in \{C_1, C_2 \dots C_k\}$ we denote $max(C)$ the greatest index i such that e_i is an edge of C, similarly $min(C)$ denotes the smallest index i such that e_i belongs to C. Observe that $max(C)$ and $min(C)$ are even numbers and that the corresponding edges are colored with β. Moreover the endpoints of $e_{min(C)}$ are $min(C)$ and $min(C) + 1$ as well as the end points of $e_{max(C)}$ are $max(C)$ and $max(C) + 1$

Observe that $min(C)$ and $max(C)$ are always defined and that $min(C) = max(C)$ if and only if C is a triangle.

The sequence $(\Gamma_j)_{j=1\dots h}$. We define a sequence $(\Gamma_j)_{j=1\dots h}$, $2 \leq h \leq k$ of members of $\{C_1, \dots C_k\}$ as follows :
• We set $\Gamma_1 = C_1$.
• If $max(\Gamma_j) < min(C_k)$, since the edge $e_{max(\Gamma_j)}$ is not a bridge there is a cycle C in G_{M_α} with $min(C) < max(\Gamma_j) < max(C)$. Among all such cycles, let us denote as Γ_{j+1} the cycle C for which $max(C)$ is maximum.
• If $max(\Gamma_j) > min(C_k)$, we set $h = j + 1$ and $\Gamma_h = C_k$.
Observe that by construction :
When $h = 2$, $1 < min(C_k) < max(C_1) = max(\Gamma_1) < n$.
When $h > 2$, for $1 < j < h$, $min(\Gamma_j) < max(\Gamma_{j-1}) < min(\Gamma_{j+1}) < max(\Gamma_j)$ and for $j = h$, $min(\Gamma_h) < min(C_k) < max(\Gamma_h) < n$.

An auxiliary graph H. We consider an auxiliary graph H, the vertex set of which being $V(G)$ we define the edge set of H from $E(G)$ as follows (see Figure 4.1) :

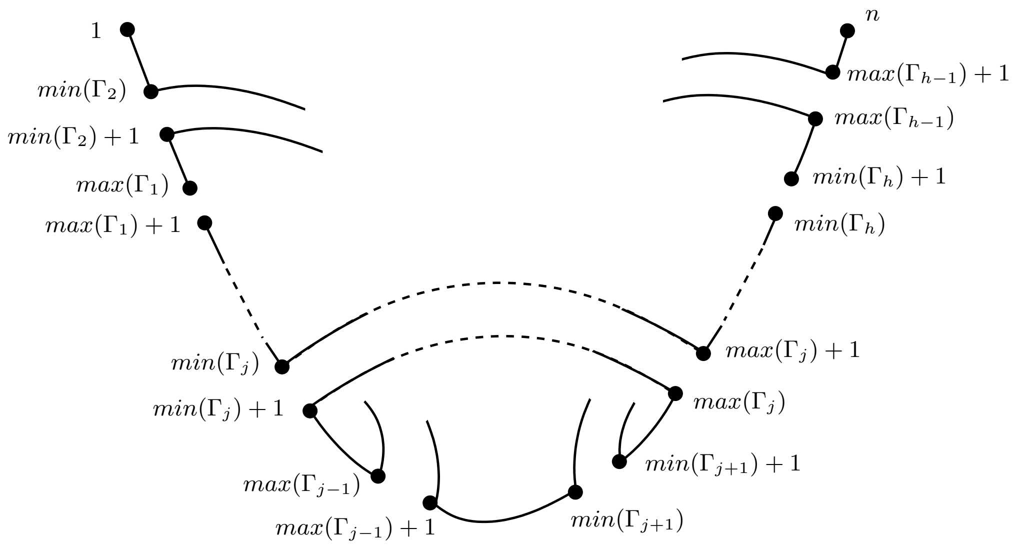

Fig. 1. An auxiliary graph (with $h > 2$)

- First, we delete the edges $e_{max(\Gamma_1)}$ and $e_{min(\Gamma_h)}$.
- For each cycle Γ_j $(2 < j < h)$ we delete the edges $e_{min(\Gamma_j)}$ and $e_{max(\Gamma_j)}$.
- Since Γ_j is an even cycle, when deleting the edges $e_{min(\Gamma_j)}$ and $e_{max(\Gamma_j)}$, we got two odd paths with one end in $\{min(\Gamma_j), min(\Gamma_j)+1\}$ and the other end in $\{max(\Gamma_j), max(\Gamma_j)+1\}$, namely P_j and P'_j. We put in $E(H)$ two new edges (denoted in the following as *additional* edges) one edge connecting the endpoints of P_j while the endpoints of the other edge are the endpoints of P'_j. We will say in the following that the first edge *represents* the path P_j while the other one *represents* the path P'_j
- Finally, we delete all the edges of G being colored with γ and δ (that is the chords of P).

All the vertices of H have degree 2 except 6 vertices, namely 1, $max(\Gamma_1)$, $max(\Gamma_1)+1$, $min(\Gamma_h)$, $min(\Gamma_h)+1$, n which have degree 1. Thus the connected components of H are precisely 3 paths whose end points are members of $\{1, max(\Gamma_1), max(\Gamma_1)+1, min(\Gamma_h), min(\Gamma_h)+1, n\}$

Notations $\prec_W$ and $W(z,t)$, concatenation of sub-walks. Let W be a walk of G. Writing $W = x \dots y$ induces a natural order on the vertices of W, let us denote $\prec_W$ this order. When $W = x \dots y$, W will be said to *start* with x and to *end* with y.When z and t are vertices of W such that $z \prec_W t$, the sub-walk $z \dots t$ of W whose endpoints are z and t will be denoted $W(z,t)$. When a walk W $(W = W(x,y))$ and a walk W' $(W' = W'(x',y'))$ have a common vertex, say a, we can *concatenate* the sub-walks $W(x,a)$ and $W'(a,y')$ in order to obtain another walk say W'', also denoted $W(x,a) + W'(a,y')$, such that $W''(x,a) = W(x,a)$ and $W''(a,y') = W'(a,y')$.

Conjecture 3 is known to be verified for bridgeless cubic graphs of oddness 2 (see [6]), Theorem 11 gives a stronger result for cubic bridgeless traceable graphs of chromatic index 4.

Theorem 11 *Let G be a cubic bridgeless traceable graph of chromatic index 4. Then there exists four perfect matchings M_α, M_1, M_2 and M_3 such that $M_\alpha \cap M_i$ does not contain any odd cut set, for $i \in \{1, 2?3\}$. Moreover, for $i \in \{1, 2, 3\}$ one can associate to M_i two joins J_i and J'_i such that $M_\alpha \cap M_i \cap J_i = M_\alpha \cap M_i \cap J'_i = \emptyset$.*

Proof We are in a position to use the notations and tools presented in section 4.1, in particular we consider a 4 edge-coloring of G with colors α, β, γ and δ ; the set M_α of edges colored with α is a perfect matching while $\{C_1, C_2 \dots C_k\}$ denotes the set of cycles of the 2-factor $G - M_\alpha$ together with the edges $min(C)$ and $max(C)$ defined for any cycle C of this 2-factor. Moreover we consider the sequence $(\Gamma_j)_{j=1\dots h}$, $2 \leq h \leq k$ of members of $\{C_1, \dots C_k\}$ as well as the auxiliary graph H described in Figure 4.1.

Claim 1 *The connected components of H are odd paths with one end in $\{1, max(\Gamma_1), max(\Gamma_1) + 1\}$ and the other end in $\{min(\Gamma_h), min(\Gamma_h) + 1, n\}$.*

Proof As a matter of fact, when $h = 2$, H is reduced to 3 sub-paths of P, namely: Q_1: $1 \dots min(\Gamma_2)$,Q_2: $min(\Gamma_2) + 1 \dots max(\Gamma_1)$ and Q_3:$max(\Gamma_1) + 1 \dots n$ We can thus suppose that $h > 2$

Let $Q = q_1 \dots q_r$ be a connected component of H with its endpoint q_1 in $\{1, max(\Gamma_1), max(\Gamma_1) + 1\}$, we will prove that the other endpoint q_r of Q belongs to $\{min(\Gamma_h), min(\Gamma_h)+1, n\}$. Let x be the maximum index in $\{1, \dots n\}$ of a vertex of Q.

Fact *$x > max(\Gamma_1)$ and for $1 < j < h - 1$, if $x > max(\Gamma_{j-1})$ then $x > max(\Gamma_j)$.*

Proof If not, the vertex x must be a vertex of one of the 2 sub-paths $max(\Gamma_{j-1})+1 \dots min(\Gamma_{j+1})$ or $min(\Gamma_{j+1}) \dots max(\Gamma_j)$ of P, thus $x = min(\Gamma_{j+1})$ or $x = max(\Gamma_j)$. In both cases, since $j < h - 1$ there must be in Q one vertex of $\{max(\Gamma_{j+1}), max(\Gamma_{j+1}) + 1\}$. But those vertices have an index greater than x, a contradiction. ∎

Thus $x > max(\Gamma_{h-2})$ and either $x = min(\Gamma_h) = q_r$ or $x = max(\Gamma_{h-1})$, in which case $q_r = min(\Gamma_h) + 1$ or $x = n = q_r$.

Consequently no connected component of H can be a path with both ends in $\{1, max(\Gamma_1), max(\Gamma_1)+1\}$, the Claim follows. □

To a path, say Q_s ($s \in \{1,2,3\}$), of H we can associate an odd walk R_s of G as follows :

• Let q_s be the last vertex of Q_s that belongs to C_1 when running on Q_s from its endpoint in $\{1, max(C_1), max(C_1)+1\}$. Similarly let q'_s be the last vertex of Q_s that belongs to C_k when running on Q_s from its endpoint in $\{n, min(C_k), min(C_k)+1\}$. Let R_s be the sub-path of Q_s whose endpoints are q_s and q'_s.

• Each *additionnal* edge of Q_s represents some odd sub-path (P_j or P'_j) of some cycle Γ_j. We replace this edge with the sub-path it represents.

The walks R_s ($s \in \{1,2,3\}$) defined above are α-disjoint walks (that is two distinct walks do not share any edge colored with α). Moreover those walks have one end which belongs to C_1 and the other end which belongs to C_k, have no edge of C_1 nor of C_k while their end-edges are colored with α.

When C denotes an even cycle of G_{M_α} and X denotes a set of vertices of C, X will be said a γ-chain (resp. a β-chain) when X induces on C a path whose end-edges are colored with γ (resp. β). A set Y of vertices of G will be said to *well-intersect* C if and only if $V(C) \cap Y$ determines on C a set of γ-chains. Observe that whenever a set Y *well-intersects* C, the set $V(C) - Y$ *well-intersects* C too.

We intend to apply Lemma 7 on some matching $A \subset M_\alpha$. Since we may have even paths among the connected components of $G_M - V(R_s)$, $s \in \{1,2,3\}$, in order to obtain a matching A for which Lemma 7 applies we will derive from R_1 or R_2 or R_3 a suitable walk.

Claim 2 *There is three α-disjoint walks S_1, S_2, S_3 such that :*

(1) when S denotes a walk of $\{S_1, S_2, S_3\}$, one end point of S belongs to $\{q_1, q_2, q_3\}$ while the other end is in $\{q'_1, q'_2, q'_3\}$;

(2) for a fixed $i_0 \in \{1,2,3\}$ and for all even cycle C of $\{\Gamma_2, \dots \Gamma_{h-1}\}$ such that $V(C) \cap V(S_{i_0}) \neq \emptyset$, $V(C) \cap V(S_{i_0})$ well-intersects C.

Proof Observe that the walks R_1, R_2 and R_3 are α-disjoints and verify Property (1) of Claim 2. Let us set, in a first stage, $R'_1 = R_1$, $R'_2 = R_2$ and $R'_3 = R_3$, in addition we fix w.l.o.g $i_0 = 1$.

Assume by induction that there is a (possibly empty) sequence $a_1, \dots, a_j$ of vertices of R'_1 such that :

• $a_1 \prec_{R'_1} \dots \prec_{R'_1} a_j$;

• Each vertex a of this sequence is the *first* vertex of R'_1 that meets some cycle, say C_a in $\{\Gamma_2, \dots, \Gamma_{h-1}\}$ (in other words $R'_1(q_1, a) \cap C_a = \{a\}$) ;

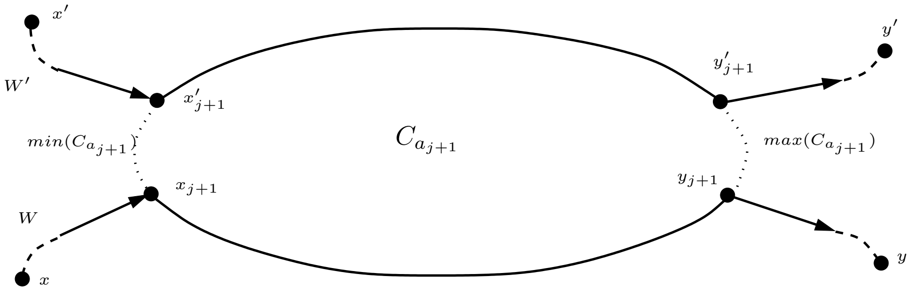

Fig. 2. The walks W and W' which intersect $C_{a_{j+1}}$.

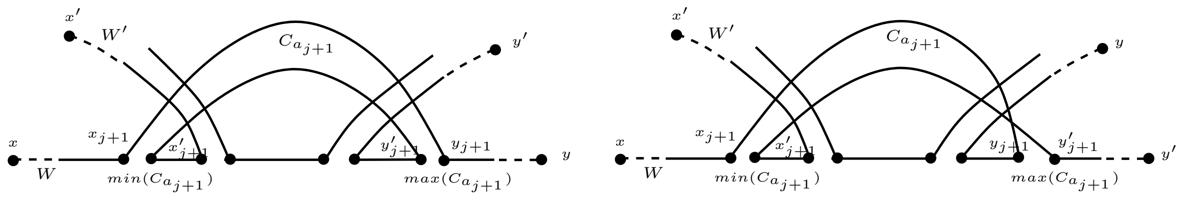

Fig. 3. Either $W(y_{j+1}, y)$ or $W'(y'_{j+1}, y')$ does not intersect $C_{a_{j+1}}$.

- R'_1 well-intersects the cycles $C_{a_1}, \ldots, C_{a_j}$.

Assume that there is a vertex in $R'_1(a_j, q'_1)$ say a_{j+1} and a cycle in $\{\Gamma_2, \ldots, \Gamma_{h-1}\}$, say $C_{a_{j+1}}$, such that $R'_1(q_1, a_{j+1}) \cap C_{a_{j+1}} = \{a_{j+1}\}$.
By construction $a_j \prec_{R'_1} a_{j+1}$ and a_{j+1} is the first vertex of R'_1 that meets $C_{a_{j+1}}$. We will show that, up to some modifications on $R'_1(a_{j+1}, q'_1)$ and on R'_2 or R'_3, the walk R'_1 well-intersects $C_{a_{j+1}}$ while R'_2 and R'_3 remain to verify Property (1) of Claim 2.

Observe that there is at least two walks in $\{R'_1, R'_2, R'_3\}$ that share vertices with $C_{a_{j+1}}$. We know that R'_1 intersects with $C_{a_{j+1}}$. Without loss of generality assume that R'_2 also shares vertices with $C_{a_{j+1}}$. In order to simplify our discussion we denote those walks W and W', moreover we write $W = W(x, y)$ and $W' = W'(x', y')$ where $\{x, x'\} = \{q_1, q_2\}$ and $\{y, y'\} = \{q'_1, q'_2\}$.

More precisely, by construction of W and W' we write (see Figure 2) :
$W = W(x, x_{j+1}) + W(x_{j+1}, y_{j+1}) + W(y_{j+1}, y)$ and
$W' = W(x', x'_{j+1}) + W('x_{j+1}, y'_{j+1}) + W(y'_{j+1}, y')$ where x_{j+1} and x'_{j+1} are the end-points of the edge $min(C_{a_{j+1}})$ while y_{j+1} and y'_{j+1} are the end-points of the edge $max(C_{a_{j+1}})$. Observe that at least one walk of $W(y_{j+1}, y)$ or $W'(y'_{j+1}, y')$ does not intersect $C_{a_{j+1}}$, see Figure 3. Suppose w.l.o.g that the vertex a_{j+1} belongs to $W(x_{j+1}, y_{j+1})$. Let b be the neighbor of a_{j+1} such that the edge $a_{j+1}b$ is colored with β. From now on we distinguish two cases.

Case 1 : $b \prec_W a_{j+1}$.

Case 1.1 If $W(y_{j+1}, y)$ does not intersect $C_{a_{j+1}}$ we set :
$R'_1 = R'_1(q_1, a_{j+1}) + W(a_{j+1}, y_{j+1}) + W(y_{j+1}, y)$ and $R'_2 = W'$ or $R'_2 =$

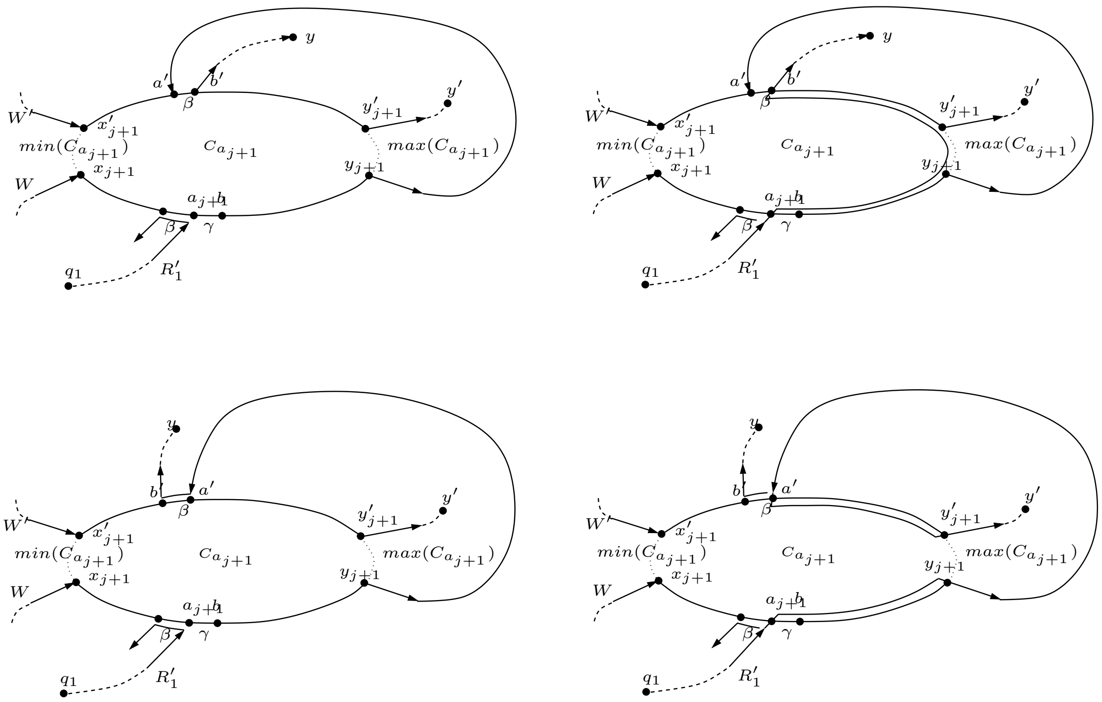

Fig. 4. Modifications of R_1', Case 1.

$W(x, x_{j+1}) + x_{j+1}x'_{j+1} + W'(x'_{j+1}, y')$ according to the fact that $q_2 = x'$ or $q_2 = x$.

Case 1.2 When $W(y_{j+1}, y)$ intersects $C_{a_{j+1}}$ we have to consider an edge of $W(y_{j+1}, y)$ colored with β that belongs to the sub-path of $C_{a_{j+1}}$ with a_{j+1} and y_{j+1} as end-points which contains b (see Figure 4). Among such edges let $a'b'$ ($a' \prec_W b'$) be such that the sub-path $a_{j+1} \ldots a'$ which does not contain b has minimum length.

If $a' \prec_{W'} b'$, let P be the subpath of $C_{a_{j+1}}$ whose end-points are a_{j+1} and b' which does not contain b. We set $R_1' = R_1'(q_1, a_{j+1}) + P + W(b', y)$ and $R_2' = W'$ or $R_2' = W(x, x_{j+1}) + x_{j+1}x'_{j+1} + W'(x'_{j+1}, y')$ according to the fact that $q_2 = x'$ or $q_2 = x$.

If $b' \prec_{W'} a'$ we set $R_1' = R_1'(q_1, a_{j+1}) + W(a_{j+1}, y_{j+1}) + W(y_{j+1}, a' = +W'(a', y')$ and $R_2' = W'(x', b') + W(b', y)$ or $R_2' = W(x, x_{j+1}) + x_{j+1}x'_{j+1} + W'(x'_{j+1}, b') + W(b', y)$ according to the fact that $q_2 = x'$ or $q_2 = x$.

Case 2 : $a_{j+1} \prec_W b$. Let $P = a_{j+1} \ldots y'_{j+1}$ be the sub-path of $C_{a_{j+1}}$ whose end-points are a_{j+1} and y'_{j+1} which avoids b.

Case 2.1 If $W'(y'_{j+1}, y')$ does not intersect $C_{a_{j+1}}$ we set :
$R_1' = R_1'(q_1, a_{j+1}) + P + W'(y'_{j+1}, y')$ and $R_2' = W$ or $R_2' = W'(x', x'_{j+1}) + x'_{j+1}x_{j+1} + W(x_{j+1}, y)$ according to the fact that $q_2 = x$ or $q_2 = x'$.

Case 2.2 When $W'(y'_{j+1}, y')$ intersects $C_{a_{j+1}}$ we have to consider an edge of $W'(y'_{j+1}, y')$ colored with β that belongs to $W(b, y_{j+1})$ (see Figure 5). Among such edges let $a'b'$ ($a' \prec_{W'} b'$) be such that the sub-path $a_{j+1} \ldots a'$ which contains b has minimum length.

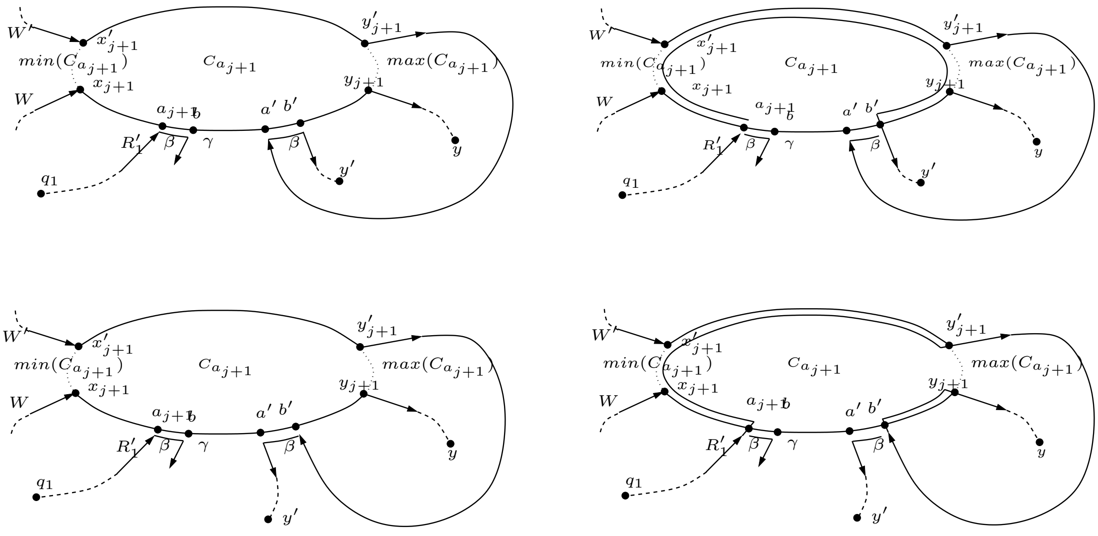

Fig. 5. Modifications of R'_1, Case 2.

If $a' \prec_W b'$, we write $R'_1 = R'_1(q_1, a_{j+1}) + P + y'_{j+1}y_{j+1} + W(y_{j+1}, b') + W'(b', y')$ and $R'_2 = W$ or $R'_2 = W'(x', x'_{j+1}) + x'_{j+1}x_{j+1} + W(x_{j+1}, y)$ according to the fact that $q_2 = x$ or $q_2 = x'$.
If $b' \prec_W a'$ we write $R'_1 = R'_1(q_1, a_{j+1}) + P + W'(y'_{j+1}, a') + W(a', y)$ and $R'_2 = W(x, x_{j+1}) + W(x_{j+1}, b') + W'(b', y')$ or $R'_2 = W'(x', x'_{j+1}) + x'_{j+1}x_{j+1} + W(x_{j+1}, b') + W'(b', y")$ according to the fact that $q_2 = x$ or $q_2 = x'$.

It is easy to see that up to the modifications described above on R'_1, on R'_2 and a renaming of the vertices y or y' as q'_1 or q'_2, the walk $R'_1 = R_1(q_1, q'_1)$ well-intersects the cycle $C_{a_{j+1}}$. In addition R'_1 and R'_2 remains to be α-disjoints and R'_2 has one end in C_1 and the other end in C_k. The Claim follows by induction. □

Due to Lemma 7, the set of edges $A = M_\alpha \cap S_1$ is a M_α-balanced matching, that is there is a perfect matching M_1 such that $M_\alpha \cap M_1 = A$.
But now, if $M_\alpha \cap M_1$ contains an odd cut set, say X, there must be a partition (V_1, V_2) of $V(G)$ into two odd sets such that the edges of X have one end in V_1 and the other end in V_2. Moreover $X \subset M_\alpha$ the edges of X are colored with α, V_1 and V_2 being odd, each of those sets precisely contains exactly one odd cycle of M_α. Since S_2 and S_3 are both connecting a vertex of C_1 to a vertex of C_k, there must be an edge of S_1 and an edge of S_3 in X, a contradiction since S_1, S_2 and S_3 are α-disjoint.

Moreover, the set of vertices $S_1 \cup S_2 - S_1 \cap S_2$ together with a sub-path of C_1 whose endpoints are q_1 and q_2 and a sub-path of C_k whose endpoints are q'_1 and q'_2 form a set X of vertices which induce cycles of G. Thus the edgeset J_1 of the subgraph induced with $V(G) - X$ is a join which avoids the edges of S_1, in other word $M_\alpha \cap M_1 \cap J_1 = \emptyset$. Similarly we can derive from $S_1 \cup S_3 - S_1 \cap S_3$ another join J'_1 with the same property. □

A direct consequence of Theorem 11 is :

Corollary 12 *Conjecture 5 holds true for cubic bridgeless traceable graphs.*

5 Conclusion

As far as we know the technics developed in Theorem 11 do not lead to a proof of Conjecture 2 for cubic bridgeless traceable graphs.

In a forthcoming paper ([4]), we prove that a minimal counter-example to Conjecture 2 must have at least 42 vertices.

References

[1] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications.* Elsevier, North-Holland, 1976.

[2] J Edmonds. Maximum matching and a polyhedron with $(0,1)$ vertices. *J. Res. Nat. Bur. Standards*, B(69):125–130, 1965.

[3] G. Fan and A. Raspaud. Fulkerson's conjecture and circuit covers. *J. Comb. Theory Ser. B*, 61:133–138, 1994.

[4] J.L Fouquet and J.M Vanherpe. On Fan Raspaud Conjecture. manuscript, 2008.

[5] D.R. Fulkerson. Blocking and anti-blocking pairs of polyhedra. *Math. Programming*, 1:168–194, 1971.

[6] T. Kaiser and A. Raspaud. Non-intersecting perfect matchings in cubic graphs. *Electronic Notes in Discrete Mathematics*, 28:293–299, 2007.

[7] Tomás Kaiser, Daniel Král, and Serguei Norine. Unions of perfect matchings in cubic graphs. *Electronic Notes in DISCRETE MATHEMATICS*, 22:341–345, 2005.

[8] E. Màčajová and M. Škoviera. http://garden.irmacs.sfu.ca/?q=op/intersecting two perfect matchings, 2007.

[9] P. Seymour. On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc.(3)*, 38:423–460, 1979.