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Rapport de Recherche

Tools for parsimonious edge-colouring of graphs with maximum degree three

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Abstract

In a graph G of maximum degree Δ let γ denote the largest fraction of edges that can be Δ edge-coloured. Albertson and Haas showed that $\gamma \geq \frac{13}{15}$ when G is cubic [1]. The notion of $\delta-$minimum edge colouring was introduced in [3] in order to extend the so called *parcimonious edge-colouring* to graphs with maximum degree 3. We propose here an english translation of some structural properties already present in [2, 3] (in French) for $\delta-$minimum edge colourings of graphs with maximum degree 3.
Keywords : Cubic graph; Edge-colouring;

1 Introduction

Throughout this note, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [7] that these graphs can be edge-coloured with 4 colours. Let $\phi :\ E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G. It is often of interest to try to use one colour (say δ) as few as possible. When an edge colouring is optimal, following this constraint, we shall say that ϕ is $\delta - minimum$. In [2] we gave without proof (in French) results on $\delta - minimum$ edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [5] and [6]. The purpose of Section 2 is to give with their proofs those results as structural properties of $\delta-$minimum edge-colourings .

An edge colouring of G using colours $\alpha, \beta, \gamma, \delta$ is said to be *δ-improper* provided that adjacent edges having the same colours (if any) are coloured with δ. It is clear that a proper edge colouring (and hence a $\delta-$minimum edge-colouring) of G is a particular δ-improper edge colouring. For a proper or δ-improper edge colouring ϕ of G , it will be convenient to denote $E_{\phi}(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges coloured with x by ϕ. For $x, y \in \{\alpha, \beta, \gamma, \delta\}$,$x \neq y$, $\phi(x, y)$ is the partial subgraph of G spanned by these two colours, that is $E_{\phi}(x) \cup E_{\phi}(y)$ (this subgraph being a union of paths and even cycles where the colours x and y alternate). Since any two $\delta-$minimum edge-colourings of G have the same number of edges coloured δ we shall denote by $s(G)$ this number (the *colour number* as defined in [5]).

As usual, for any undirected graph G, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we suppose that $|V(G)| = n$ and

$|E(G)| = m$. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C, or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2 Structural properties of $\delta-$minimum edge-colourings

The graph G considered in the following series of Lemmas will have maximum degree 3.

Lemma 1 *[2, 3] Any 2-factor of G contains at least $s(G)$ disjoint odd cycles.*

Proof Assume that we can find a 2-factor of G with $k < s(G)$ odd cycles. Then let us colour the edges of this 2-factor with α and β, except one edge (coloured δ) on each odd cycle of our 2-factor and let us colour the remaining edges by γ. We get hence a new edge colouring ϕ with $E_\phi(\delta) < s(G)$, impossible. □

Lemma 2 *[2, 3] Let ϕ be a $\delta-$minimum edge-colouring of G. Any edge in $E_\phi(\delta)$ is incident to α, β and γ. Moreover each such edge has one end of degree 2 and the other of degree 3 or the two ends of degree 3.*

Proof Any edge of $E_\phi(\delta)$ is certainly adjacent to α, β and γ. Otherwise this edge could be coloured with the missing colour and we should obtain an edge colouring ϕ' with $|E_{\phi'}(\delta)| < |E_\phi(\delta)|$. □

Lemma 3 below was proven in [4], we give its proof for sake of completeness.

Lemma 3 *[4] Let ϕ be a $\delta-$improper colouring of G then there exists a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$*

Proof Let ϕ be a δ-improper edge colouring of G. If ϕ is a proper colouring, we are done. Hence, assume that uv and uw are coloured δ. If $d(u) = 2$ we can change the colour of uv to α, β or γ since v is incident to at most two colours in this set.

If $d(u) = 3$ assume that the third edge uz incident to u is also coloured δ, then we can change the colour of uv for the same reason as above.

If uz is coloured with α, β or γ, then v and w are incident to the two remaining colours of the set $\{\alpha, \beta, \gamma\}$ otherwise one of the edges uv, uw can be recoloured with the missing colour. W.l.o.g., consider that uz is coloured α then v and w are incident to β and γ.Since u has degree 1 in $\phi(\alpha, \beta)$ let P be the path of $\phi(\alpha, \beta)$ which ends on u. We can assume that v or w (say v) is not the other end vertex of P. Exchanging α and β along P does not change the colours incident to v. But now uz is coloured α and we can change the colour of uv with β.

In each case, we get hence a new δ-improper edge colouring ϕ_1 with $E_{\phi_1}(\delta) \subsetneq E_\phi(\delta)$. Repeating this process leads us to construct a proper edge colouring of G with $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$ as claimed. □

Lemma 4 *[2, 3] Let ϕ be a $\delta-$minimum edge-colouring of G. For any edge $e = uv \in E_\phi(\delta)$ there are two colours x and y in $\{\alpha, \beta, \gamma\}$such that the connected component of $\phi(x, y)$ containing the two ends of e is an even path joining these two ends.*

Proof Without loss of generality assume that u is incident to α and β and v is incident to γ (see Lemma 2). In any case (v has degree 3 or degree 2) u and v are contained in paths of $\phi(\alpha, \gamma)$ or $\phi(\beta, \gamma)$. Assume that they are contained in paths of $\phi(\alpha, \gamma)$. If these paths are disjoint then we can exchange the two colours on the path containing u, e will be incident hence to only two colours β and γ in this new edge-colouring and e could be recoloured with α, a contradiction since we consider a $\delta-$minimum edge-colouring. □

Lemma 5 *[2, 3] If G is a cubic graph then $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G) \pmod 2$.*

Proof $\phi(\alpha, \beta)$ contains $2|A_\phi| + |B_\phi| + |C_\phi|$ vertices of degree 1 and must be even. Hence we get $|B_\phi| \equiv |C_\phi| \pmod 2$. In the same way we get $|A_\phi| \equiv |B_\phi| \pmod 2$ leading to $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G) \pmod 2$ □

Remark 6 An edge of $E_\phi(\delta)$ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$. It is clear that A_ϕ, B_ϕ and C_ϕ are not necessarily pairwise disjoint since an edge of $E_\phi(\delta)$ with one end of degree 2 is contained in 2 such sets. Assume indeed that $e = uv \in E_\phi(\delta)$ with $d(u) = 3$ and $d(v) = 2$ then, if u is incident to α and β and v is incident to γ we have an alternating path whose ends are u and v in $\phi(\alpha, \gamma)$ as well as in $\phi(\beta, \gamma)$. Hence e is in $A_\phi \cap B_\phi$. When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with e. We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ. In the following lemma we consider an edge in A_ϕ, an analogous result holds true whenever we consider edges in B_ϕ or C_ϕ as well.

Lemma 7 *[2, 3] Let ϕ be a $\delta-$minimum edge-colouring of G and let e be an edge in A_ϕ then for any edge $e' \in C_{A_\phi}(e)$ there is a $\delta-$minimum edge-colouring ϕ' such that $E_{\phi'}(\delta) = E_\phi(\delta) - \{e\} \cup \{e'\}$, $e' \in A_{\phi'}$ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. Moreover, each edge outside $C_{A_\phi}(e)$ but incident with this cycle is coloured γ, ϕ and ϕ' only differ on the edges of $C_{A_\phi}(e)$.*

Proof By exchanging colours δ and α and δ and β successively along the cycle $C_{A_\phi}(e)$, we are sure to obtain an edge colouring preserving the number of edges coloured δ. Since we have supposed that ϕ is δ-minimum, at each step, the resulting edge colouring is proper and δ-minimum (Lemma 3). Hence, there is no edge coloured δ incident with $C_{A_\phi}(e)$, which means that every such edge is coloured with γ.

We can perform these exchanges until e' is coloured δ. In the $\delta-$minimum edge-colouring ϕ' hence obtained, the two ends of e' are joined by a path of $\phi(\alpha, \beta)$. Which means that e' is in A_ϕ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. □

For each edge $e \in E_\phi(\delta)$ (where ϕ is a $\delta-$minimum edge-colouring of G) we can associate one or two odd cycles following the fact that e is in one or two sets among A_ϕ, B_ϕ or C_ϕ. Let $\mathcal{C}$ be the set of odd cycles associated to edges in $E_\phi(\delta)$.

Lemma 8 *[2, 3] For each cycle $C \in \mathcal{C}$, there are no two consecutive vertices with degree two.*

Proof Otherwise, we exchange colours along C in order to put the colour δ on the corresponding edge and, by Lemma 2, this is impossible in a $\delta-$minimum edge-colouring . □

Lemma 9 *[2, 3] Let $e_1, e_2 \in E_\phi(\delta)$ and let $C_1, C_2 \in \mathcal{C}$ be such that $C_1 \neq C_2$, $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ then C_1 and C_2 are (vertex) disjoint.*

Proof If e_1 and e_2 are contained in the same set A_ϕ, B_ϕ or C_ϕ, we are done since their respective ends are joined by an alternating path of $\phi(x, y)$ for some two colours x and y in$\{\alpha, \beta, \gamma\}$.

Without loss of generality assume that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Assume moreover that there exists an edge e such that $e \in C_1 \cap C_2$. We have hence an edge $f \in C_1$ with exactly one end on C_2. We can exchange colours on C_1 in order to put the colour δ on f. Which is impossible by Lemma 7. □

Lemma 10 *[2, 3] Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G. Then v_1 is the only vertex in $N(u)$ of degree 2 and for any edge $e_2 = u_2v_2 \in E_\phi(\delta)$, $\{e_1, e_2\}$ induces a $2K_2$.*

Proof We have seen in Lemma 2 that uv_1 has one end of degree 3 while the other has degree 2 or 3. Hence, we have $d(u) = 3$ and $d(v_1) = 2$. Let v_2 and v_3 the other neighbours of u. From Remark 6, we know that v_2 and v_3 are not pendant vertices. Assume that $d(v_2) = 2$ and uv_2 is coloured α, uv_3 is coloured β and, finally v_1 is incident to an edge coloured γ. The alternating path of $\phi(\beta, \gamma)$ using the edge uv_3 ends with the vertex v_1 (see Lemma 4), then, exchanging the colours along the component of $\phi(\beta, \gamma)$ containing v_2 allows us to colour uv_2 with γ and uv_1 with α. The new edge colouring ϕ' so obtained is such that $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, impossible. □

Lemma 11 *[2, 3] Let e_1 and e_2 be two edges of $E_\phi(\delta)$. If e_1 and e_2 are contained in two distinct sets of A_ϕ, B_ϕ or C_ϕ then $\{e_1, e_2\}$ induces a $2K_2$ otherwise e_1, e_2 are joined by at most one edge.*

Proof Assume in a first stage that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Since $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$ are disjoint by Lemma 9, we know that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{e_1}(\phi)$ are coloured γ while those having exactly one end in $C_{e_2}(\phi)$ are coloured α. Hence there is no edge between e_1 and e_2 as claimed.

Assume in a second stage that $e_1 = u_1v_1, e_2 = u_2v_2 \in A_\phi$.Since $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$ are disjoint by Lemma 9, we can consider that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{e_1}(\phi)$ (or $C_{e_2}(\phi)$) are coloured γ. Assume that u_1u_2 and v_1v_2 are edges of G. We may suppose without loss of generality that u_1 ans u_2 are incident to α while v_1 and v_2 are incident to β (if necessary, colours α and β can be exchanded on $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$). We know that u_1u_2 and v_1v_2 are coloured γ. Let us colour e_1 and e_2 with γ and u_1u_2 with β and v_1v_2 with α. We get a new edge colouring ϕ' where $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 2$, contradiction since ϕ is a $\delta-$minimum edge-colouring . □

Lemma 12 *[2, 3] Let e_1, e_2 and e_3 be three distinct edges of $E_\phi(\delta)$ contained in the same set A_ϕ, B_ϕ or C_ϕ. Then $\{e_1, e_2, e_3\}$ induces a subgraph with at most four edges.*

Proof Without loss of generality assume that $e_1 = u_1v_1, e_2 = u_2v_2$ and $e_3 = u_3v_3 \in A_\phi$. From Lemma 11 we have just to suppose that (up to the names of vertices) $u_1u_3 \in E(G)$ and $v_1v_2 \in E(G)$. Possibly, by exchanging the colours α and β along the 3 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2 and e_3, we can suppose that u_1 and u_3 are incident to β while v_1 and v_2 are incident to α. Let ϕ' be obtained from ϕ when u_1u_3 is coloured with α, v_1v_2 with β and u_1v_1 with γ. It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a $\delta-$minimum edge-colouring . □

Lemma 13 *[2, 3] Let $e_1 = u_1v_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G. Then for any edge $e_2 = u_2v_2 \in E_\phi(\delta)$ $\{e_1, e_2\}$ induces a $2K_2$.*

Proof From Lemma 11 we have to consider that e_1 and e_2 are contained in the same set A_ϕ, B_ϕ or C_ϕ. Assume without loss of generality that they are contained in A_ϕ. From Lemma 11 again we have just to consider that there is a unique edge joining these two edges and we can suppose that $u_1u_2 \in E(G)$. Possibly, by exchanging the colours α and β along the 2 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2, we can suppose that u_1 and u_2 are incident to β while v_1 and v_2 are incident to α. We know that u_1u_2 is coloured γ. Let ϕ' be obtained from ϕ when u_1u_2 is coloured with α and u_1v_1 is coloured with γ. It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a $\delta-$minimum edge-colouring . □

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