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**VALUATION OF INTEREST RATE OPTIONS IN A
TWO-FACTOR MODEL OF THE TERM STRUCTURE
OF INTEREST RATES**

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Valuation of Interest Rate Options in a Two-Factor Model of the Term Structure of Interest Rates

Résumé

Nous présentons un modèle de la structure par terme des taux d'intérêt à deux variables d'état : le taux d'intérêt instantané et sa valeur moyenne observée sur une courte période. Le choix des facteurs est fondé sur les résultats de tests empiriques de la gamme des taux et tente de pallier les faiblesses théoriques des modèles existants. Dans le cadre de ce modèle, nous évaluons les options sur obligations et sur contrats forward et futures d'obligations.

Abstract

We present a two-factor model of the term structure of interest rates in which both the short rate and its short term mean are assumed to be stochastic. The choice of the two factors is based on empirical evidence and tries to remedy the theoretical weaknesses of existing models. In this framework, we evaluate options on bond prices and options on bond forward and futures contracts.

Valuation of Interest Rate Options in a Two-Factor Model of the Term Structure of Interest Rates

1. Introduction

Two main approaches have been followed in the academic literature to model the term structure of interest rates. Whatever the approach, models try to explain some features of the behaviour of the term structure and to identify the sources of risk associated with the dynamics of the yield curve. In the first approach, models developed, among others, by Vasicek (1977), Dothan (1978), Brennan and Schwartz (1979, 1982), Courtadon (1982), Schaefer and Schwartz (1984), Cox, Ingersoll and Ross (1985 a, b) and Longstaff and Schwartz (1992), explain the evolution of the term structure through the stochastic evolution of one or two state variables (the spot interest rate and the long-term interest rate or the volatility of interest rates). In these models, the stochastic process for the state variables is either specified exogenously (no-arbitrage term structure models) or determined endogenously from assumptions on investor preferences and technologies (intertemporal term structure models). Two-factor models overcome the limitations imposed by single-factor models in that they allow the returns on bonds of all maturities to be imperfectly correlated. They also allow the model to fit many observed shapes of the term structure. Although these models are useful for valuing interest rate sensitive claims in a consistent way, they require assumptions about investors

preferences and the resulting term structure does not match the initial yield curve.

In the second approach, models are preference-free and consistent with the current term structure of interest rates. They fall into three categories. In the first one, the models imply modelling the dynamics of discount bond prices. Ho and Lee (1986) were the first to build a model, by using a binomial tree to describe the evolution of discount bond prices, that provides a perfect fit to the current yield curve. The models of the second category belongs to the class of Heath, Jarrow and Morton (1992) (henceforth the HJM model) models. The continuous-time HJM model subsequently generalises the Ho-Lee model. They let the forward rate follow a stochastic process governed by a finite number of sources of uncertainty. In the third one, models specify the short-rate process like Hull and White (1990, 1993). They extend the Vasicek and CIR models by letting the parameters in the stochastic process of the instantaneous rate to be deterministic functions of time in order to match the model to the initial term structure.

In this paper a two-factor model of the term structure is proposed in which both the short rate and its short term mean are assumed to be stochastic. This assumption is based on empirical studies which have investigated interest rate behaviour. Chan et al. (1992) and Tse (1995) have estimated various continuous-time models of the short rate. They have found only weak evidence that the short rate reverts to a long term mean value. This weak evidence suggests that the short rate is mean reverting about a short term

mean. Therefore, we suppose that the short rate reverts to a short run mean which follows itself a mean-reverting process. The assumption of stochastic mean is also supported by empirical evidence. In October 1979 the Federal Reserve (Fed) changed its monetary policy. Pearson and Sun (1991) have shown that the mean parameter of the short rate process are different in the periods before and after this date. Recently regime-switching models of interest rates are developed (see, for instance, Hamilton (1988), Cai (1994), Gray (1996) and Tice and Webber (1997)). In these models a spot interest rate process can shift randomly between regimes of low mean rates to high mean rates (e.g., the Fed experiment of 1979 to 1982). Tice and Webber give an economic interpretation to a broad class of two-factor models including stochastic mean models. In particular, the dynamic behaviour of the mean is related to fiscal and monetary economic policies.

This paper aims at pricing European-type interest rate contingent claims under a two-factor model of the term structure of interest rates. The choice of the factors (the short rate and short term mean) is based on empirical evidence and each of the factors is assumed to follow an Ornstein-Uhlenbeck (O-U hereafter) process. Other stochastic specifications for the mean have been proposed. For example, Brennan and Schwartz (1979) use two factors to describe movements of the term structure: the short rate and the yield to maturity on a perpetual coupon bond. Hogan (1993) demonstrates that this model admits arbitrage opportunities. Longstaff and Schwartz (1992), in a general equilibrium framework, build a two-

factor model which can be interpreted as a random volatility specification because the volatility of the instantaneous interest rate is a function of the two factors. They obtain closed-form formulas for discount bond options which depend on investors' preferences. Chen (1994) develops a three-factor model of the term structure of interest rates. In this model the current short rate, the short term mean and the current volatility of the short rate follow a square-root process. Chen obtains a general formula for valuing interest rate derivatives that requires the computation of high-dimensional integrals. Two-factor models developed, for instance, by Richard (1978) who argues that the instantaneous interest rate is the sum of the real rate of interest and the inflation rate. Chen and Scott (1993) decompose the instantaneous interest rate into two unspecified factors each of which follows a square root process. A common characteristic of these two models is that there is little theoretical support to the choice of the factors¹. Note that the models of the first approach mentioned above are all preference-dependent. In the framework of the second approach, we derive simple formulas for interest rate contingent claims.

The remainder of the paper is organised as follows. In the second section the two-factor model is presented and the fundamental partial differential equation (PDE) for the discount bond price is derived. Sections three and four are devoted to the pricing of options on bonds, on bond forward contract and bond futures contract. The fifth section offers some remarks and conclusions. An appendix includes all proofs.

2. The model

In this section we describe a dynamic mean interest rate model and derive the discount bond prices.

The instantaneous interest rate $r(t)$, is assumed to follow a mean-reverting process of the form :

$$dr(t) = \mathbf{a}(q(t) - r(t))dt + \mathbf{s}_r dW_r(t) \quad (1)$$

$\theta(t)$ represents the short term mean of the short rate which follows an O-U process given by :

$$dq(t) = \mathbf{m}(b - q(t))dt + \mathbf{s}_q dW_q(t) \quad (2)$$

where \mathbf{a} , μ , \mathbf{b} , σ_r and σ_θ are positive constants. $dW_r(t)$ and $dW_\theta(t)$ are independent Wiener processes under the historical probability measure P .

The O-U process was first used by Vasicek (1977) to model the term structure. The short rate has a tendency to revert to a random short term mean value. The latter follows a stochastic differential equation (SDE) which has a mean reverting drift pulling $\theta(t)$ towards its constant long term mean.

In their seminal work, Harrison-Kreps (1979) and Harrison-Pliska (1981) have shown that, in order to avoid arbitrage opportunities, for any non dividend paying asset chosen as

numeraire, there exists a unique (complete markets) probability (risk-neutral) measure Q equivalent to the true historical probability P , such that the relative price of any security is a Q -martingale. Under Q , the expectation of the instantaneous return of any financial security equals the riskless rate². In this framework, the current value of a financial claim is equal to the conditional expectation, under the risk-neutral probability, of the discounted final payoff.

Let λ_r and λ_θ be fixed real-valued constants that can be interpreted as the market prices of risk associated with $r(t)$ and $\theta(t)$ respectively. Define $d\tilde{W}_r(t) \equiv dW_r(t) + \mathbf{I}_r dt$ and $d\tilde{W}_q(t) \equiv dW_q(t) + \mathbf{I}_q dt$. Under Q , according to Girsanov's theorem $d\tilde{W}_r(t)$ and $d\tilde{W}_q(t)$ are standard Brownian motions and $r(t)$ and $\theta(t)$ satisfy the SDE's:

$$dr(t) = [\mathbf{a}(q(t) - r(t)) - \mathbf{I}_r \mathbf{s}_r] dt + \mathbf{s}_r d\tilde{W}_r(t) \quad (1')$$

$$dq(t) = [\mathbf{m}(b - q(t)) - \mathbf{I}_q \mathbf{s}_q] dt + \mathbf{s}_q d\tilde{W}_q(t) \quad (2)$$

The market price at $t < T$ of a discount bond delivering one monetary unit at maturity T is noted $B(r, \theta, t, T) \equiv B(t, T)$. Within this framework, using the standard arbitrage argument in Brennan and Schwartz (1979), the price of a discount bond with two state variables satisfies the following PDE :

$$\frac{1}{2}B_{rr}\mathbf{s}_r^2 + \frac{1}{2}B_{qq}\mathbf{s}_q^2 + [\mathbf{a}(\mathbf{q}(t) - r(t)) - I_r\mathbf{s}_r]B_r + [\mathbf{m}(b - \mathbf{q}(t)) - I_q\mathbf{s}_q]B_q + B_t - r(t)B = 0 \quad (3)$$

subject to the maturity condition $B(T, T) = 1$. Subscripts of B denote partial derivatives with respect to the state variables and time. The solution to equation (3) is given by:

$$B(t, T) = E^Q \left[\exp \left(- \int_t^T r(s) ds \right) F_t \right]$$

where E^Q is the expectation under the equivalent probability measure with respect to the risk-adjusted processes (1') and (2') for the instantaneous interest rate and the short term mean. F_t is the information available at date t .

We use the standard separation of variables method and consider a discount bond price function of the form :

$$B(t, T) = \exp(-D(\mathbf{t})r(t) + A(\mathbf{t})\mathbf{q}(t) + C(\mathbf{t})) \quad (4)$$

where $D(\tau)$, $A(\tau)$ and $C(\tau)$ are functions of time to maturity, $\tau = T - t$, that are assumed to be twice continuously differentiable. These functions satisfy the terminal conditions: $D(0) = A(0) = C(0) = 0$. Differentiating (4) with respect to $r(t)$, $\theta(t)$ and τ yields : $B_r = B(-D_\tau(\tau)r(t) + A_\tau(\tau)\theta(t) + C_\tau(\tau))$, $B_\tau = -D(\tau)B$, $B_\theta = A(\tau)B$, $B_{\tau\tau} = D^2(\tau)B$ and $B_{\theta\theta}(\tau) = A^2(\tau)B$. Substituting

these partial derivatives into the PDE (3) and rearranging terms, we obtain :

$$\begin{aligned} & \frac{1}{2} \mathbf{s}_r^2 D^2(\mathbf{t}) + \frac{1}{2} \mathbf{s}_q^2 A^2(\mathbf{t}) - [\mathbf{a}(\mathbf{q}(t) - r(t)) - \mathbf{I}_r \mathbf{s}_r] D(\mathbf{t}) \\ & + [\mathbf{m}(b - \mathbf{q}(t)) - \mathbf{I}_q \mathbf{s}_q] A(\mathbf{t}) + D_t(\mathbf{t}) - A_t(\mathbf{t}) - C_t(\mathbf{t}) - r(t) = 0 \quad (3') \end{aligned}$$

The left-hand side of (3') is linear in $r(t)$ and $\theta(t)$. Collecting terms in $r(t)$, $\theta(t)$ and terms independent of $r(t)$ and $\theta(t)$ gives the following differential equations (ODE) subject to the terminal conditions :

a) $D_\tau(\tau) + \alpha D(\tau) - 1 = 0$

This equation has the solution : $D(\tau) \circ D_\alpha(\tau) = \frac{1 - e^{-\alpha\tau}}{\alpha}$

b) $\alpha D(\tau) + \mu A(\tau) + A_\tau(\tau) = 0$

The solution of $A(\tau)$ is given by the following expression:

$$A(\tau) = \frac{\alpha}{\alpha - \mu} [D_\alpha(\tau) - D_\mu(\tau)]$$

c) $\frac{1}{2} \mathbf{s}_r^2 D^2(\mathbf{t}) + \frac{1}{2} \mathbf{s}_q^2 A^2(\mathbf{t}) + \mathbf{I}_r \mathbf{s}_r D(\mathbf{t}) + b^* A(\mathbf{t}) - C_t(\mathbf{t}) = 0$

Integrating for $C(\tau)$ yields:

$$\begin{aligned}
C(t) = & \left(\frac{1}{2} \frac{s_r^2}{a^2} + \frac{1}{2} \frac{s_q^2}{m^2} + \frac{I_r s_r}{a} - \frac{b^*}{\mu} \right) t - \left(\frac{1}{2} \frac{s_r^2}{a^2} + \frac{1}{2} \frac{s_q^2}{(a-m)^2} - \frac{as_q^2}{m(a-m)} \right. \\
& \left. + \frac{I_r s_r}{a} - \frac{b^*}{a-m} \right) D_a(t) - \frac{a}{m(a-m)} \left(\frac{1}{2} \frac{as_q^2}{m(a-m)} - \frac{s_q^2}{(a-m)} - b^* \right) \\
& D_m(t) - \frac{1}{4} \left(\frac{s_r^2}{a} + \frac{as_q^2}{(a-m)^2} \right) D_a^2(t) - \frac{1}{4} \frac{1}{m} \left(\frac{as_q}{a-m} \right)^2 D_m^2(t) \\
& - \frac{as_q^2}{m(a-m)^2} D_{a+m}(t)
\end{aligned}$$

$$\text{where: } D_m(t) = \frac{1 - e^{-mt}}{m}, D_{a+m}(t) = \frac{1 - e^{-(a+m)t}}{a+m} \text{ and } b^* = mb - I_q s_q.$$

The model provides a closed-form solution for discount bond prices which are functions of the interest rate and its stochastic short term mean. Note that the bond price depends on the market prices of risk and therefore is not arbitrage-free. $D_\alpha(\tau)$ is the duration which measures the sensitivity of a bond to changes in the level of interest rates. It is identical to that obtained by Vasicek's model. $A(\tau)$ is a measure that assesses the sensitivity of a bond to changes in the short run mean of interest rates. If $\mu = \sigma_\theta = 0$, the second factor (the short mean) disappears. Furthermore, if $\theta(t) = b = \text{constant}$, the above solution reduces to Vasicek's formula.

The resulting bond price satisfies the maturity condition $B(r, \theta, T, T) = 1$. The yield to maturity given by the following equation :

$$R(r, \mathbf{q}, \mathbf{t}) = -\frac{\text{Ln}B(r, \mathbf{q}, \mathbf{t})}{\mathbf{t}} = \frac{1}{\mathbf{t}}(D(\mathbf{t})r(t) - A(\mathbf{t}) - C(\mathbf{t}))$$

is linear in $r(t)$ and $\theta(t)$. Because of its dependence on two factors, the yield curve can attain more complex and realistic shapes than is possible for one-factor models of the term structure. Furthermore, different maturity discount bond prices are imperfectly correlated, a property which is consistent with reality.

Since the discount bond and the option on the discount bond are contingent claims on the same factors and the same arbitrage arguments can apply to these securities, their PDEs are identical. They differ only by their boundary conditions. As a result the value of the option is a function of the market prices of risk and depend on the investor's preferences. Instead of solving a PDE for each kind of option, which is time consuming and requires tedious calculus, in the following sections, we use an alternative method, pioneered by Jamshidian (1989), to obtain preference-free pricing formulas for the interest-rate-sensitive options. We make use of the forward-neutral probability measure equivalent to the risk-neutral measure. Under this new measure, in absence of arbitrage opportunities, the forward price of any financial asset is a martingale having the same variance as under the historical probability. In this framework, interest rate options are valued by arbitrage independently of investors' preferences.

3. Options on discount bonds

Let $C_1(t)$ be the price at date t of a European call written on a discount bond price of maturity T , with strike price K and expiry date T_c , where $t \leq T_c \leq T$. The terminal value of the call at date T_c is: $C_1(T_c) = \max[B(T_c, T) - K, 0]$.

Under the risk-neutral probability Q , the current value of the option is given by:

$$C_1(t) = E^Q \left[\exp\left(-\int_t^{T_c} r(s) ds\right) \max[B(T_c, T) - K, 0] \middle| F_t \right] \quad (5)$$

One has to compute the expectation of a product of random variables, which is in general difficult. However, this expectation turns out to be equal to the expectation, under the forward-neutral probability measure Q^{T_c} , of its terminal payoff multiplied by the expectation, computed under the risk-neutral measure Q , of the discount factor. Then, the solution to equation (5), derived in the appendix, is:

$$C_1(t) = B(t, T)N(d_1) - KB(t, T_c)N(d_2) \quad (6)$$

$$\text{where: } d_1 = \frac{\text{Ln}\left(\frac{B(t, T)}{KB(t, T_c)}\right) + \frac{1}{2}\bar{\mathbf{s}}_1^2(t, T_c, T)}{\bar{\mathbf{s}}_1(t, T_c, T)},$$

$$d_2 = d_1 - \bar{\mathbf{s}}_1(t, T_c, T),$$

$$\begin{aligned} \mathbf{s}_1^2(t, T_c, T) = & \left(\mathbf{s}_r^2 + \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 \right) D_a^2(T_c, T) \left(D_a(t, T_c) - \frac{\mathbf{a}}{2} D_a^2(t, T_c) \right) \\ & + \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 D_m^2(T_c, T) \left(D_m(t, T_c) - \frac{\mathbf{a}}{2} D_m^2(t, T_c) \right) \\ & - 2 \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 D_a(T_c, T) D_m(T_c, T) D_{a+m}(t, T_c). \end{aligned}$$

This formula is similar to the Jamshidian's one-factor model (1989)³. The only difference between these two models is that, in our model, the bond prices depend on two factors⁴. $B(t, T)$ serves as the underlying asset and $B(t, T_c)$ is the numeraire associated with the forward-neutral probability measure.

4. Options on discount bond forward contracts

Forward and futures (see section 5) contracts themselves have no value. Thus, unlike pricing options, it is relevant to derive the forward or the futures prices but not to determine the values of these contracts. It follows from a simple arbitrage argument that, in

absence of arbitrage opportunities, the forward price, $G(t, T_G, T)$, at time t for a forward contract maturing at date T_G , written on a discount bond of maturity date T , where $t \leq T_G \leq T$, is given by: $G(t, T_G, T) = B(t, T)/B(t, T_G)$.

Let $C_2(t)$ be the value of a European call written on a bond forward contract of maturity T_G , with strike price K and expiry date T_c , where $t \leq T_c \leq T_G \leq T$. The terminal value of this call at date T_c is:

$$C_2(T_c) = \max[B(T_c, T_G)(G(T_c, T_G, T) - K), 0]$$

where the potential gain $(G(T_c, T_G, T) - K)$ must be discounted back from T_G to T_c since $G(T_c, T_G, T)$ is a forward price. Therefore at date t , we have :

$$C_2(t) = E^Q \left[\exp \left(- \int_t^{T_c} r(s) ds \right) \max[B(T_c, T_G)(G(T_c, T_G, T) - K), 0] \middle| F_t \right]$$

The solution to which is shown in the appendix to write :

$$C_2(t) = B(t, T_G) [G(t, T_G, T)N(d_1) - KN(d_2)] \quad (7)$$

$$\text{where: } d_1 = \frac{\text{Ln}\left(\frac{G(t, T_G, T)}{K}\right) + \frac{1}{2}\bar{\mathbf{s}}_2^2(t, T_c, T_G, T)}{\bar{\mathbf{s}}_2(t, T_c, T_G, T)},$$

$$d_2 = d_1 - \bar{\mathbf{s}}_2(t, T_c, T_G, T),$$

$$\begin{aligned} \bar{\mathbf{s}}_2^2(t, T_c, T_G, T) &= \left(\mathbf{s}_r^2 + \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 \right) e^{-2\mathbf{a}(T_G - T_c)} D_a^2(T_G, T) \\ &\quad \left(D_a(t, T_c) - \frac{\mathbf{a}}{2} D_a^2(t, T_c) \right) + \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 e^{-2\mathbf{m}(T_G - T_c)} \\ &\quad D_m^2(T_G, T) \left(D_m(t, T_c) - \frac{\mathbf{a}}{2} D_m^2(t, T_c) \right) - 2 \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 \\ &\quad e^{-\mathbf{a}(T_G - T_c)} e^{-\mathbf{m}(T_G - T_c)} D_a(T_G, T) D_m(T_G, T) D_{\mathbf{a}+\mathbf{m}}(t, T_c). \end{aligned}$$

The formula obtained is reminiscent of that of Black (1976) with stochastic interest rates. Note that when the maturity date of the forward contract coincides with that of the option then the option on the spot and the option on the forward have identical values. Indeed, at date $T_G = T_c$, the forward price is equal to the spot bond price.

5. Options on discount bond futures contract

As shown by Cox, Ingersoll and Ross (1981) the futures price may differ from the forward price due to the daily marking to market. The no-arbitrage futures price at time t , $H(t, T_H, T)$, of a

futures contract maturing at time T_H , written on a discount bond that matures at date T , where $t \leq T_H \leq T$, is equal to :

$$H(t, T_H, T) = E^Q [B(T_H, T) | F_t]$$

Since the futures contract is assumed to be marked to market continuously and then to have always zero-value, the futures price, under the risk-neutral probability measure Q , is a martingale. Because the futures price at maturity is equal to the spot price, it implies that the futures price at any date t is given by the risk-neutral expected value of the spot bond price at the futures contracts maturity.

Assuming that margins are called in a continuous rather than daily manner, the explicit solution to the above expectation is shown to the appendix to be :

$$\begin{aligned}
H(t, T_H, T) = G(t, T_H, T) \exp \left\{ \left[\mathbf{s}_r^2 + \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 \right] D_a(t, T_H) D_a(T_H, T) \right. \\
\left. \left(D_a(T_H, T) + \frac{1}{2} e^{-\mathbf{a}(T-T_H)} D_a(t, T_H) \right) - \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 \right. \\
\left. D_m(t, T_H) D_m(T_H, T) \left(D_m(T_H, T) + \frac{1}{2} e^{-\mu(T-T_H)} D_\mu(t, T_H) \right) + \right. \\
\left. \frac{1}{\mathbf{m}} \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 D_a(T_H, T) \left(D_a(t, T_H) - e^{-\mu(T-T_H)} D_{\mathbf{a}+\mathbf{m}}(t, T_H) \right) + \right. \\
\left. \frac{1}{\mathbf{a}} \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 D_m(T_H, T) \left(D_m(t, T_H) - e^{-\mathbf{a}(T-T_H)} D_{\mathbf{a}+\mathbf{m}}(t, T_H) \right) \right\}
\end{aligned}$$

Note that the futures price does not depend on the market prices of risk and therefore is preference-free. The futures price differs from the forward price by an adjustment factor that reflects the covariance between the bond price volatility and the forward price volatility. These prices are equal only when the interest rate and its short term mean are deterministic.

Let $C_3(t)$ be the price of a European call with maturity T_C and strike price K written on a bond futures contract with maturity T_H , where $t \leq T_C \leq T_H \leq T$. The value of this call at time t is given by :

$$C_3(t) = E^Q \left[\exp \left(- \int_t^{T_C} r(s) ds \right) \max [H(T_C, T_H, T) - K, 0] | F_t \right] \quad (8)$$

Since Futures price differ from forward price by an adjustment factor, we can use the same technique as in section 4 to evaluate an expression for the price of a European futures option. Using a suggestion from Jamshidian (1993) and computing first the forward price of the futures contract, the solution to equation (8), given in the appendix, is :

$$C_3(t) = B(t, T_c) [\mathbf{g}(t, T_H, T) H(t, T_H, T) N(d_1) - KN(d_2)] \quad (9)$$

$$\text{where: } d_1 = \frac{\text{Ln} \left(\frac{\mathbf{g}(t, T_c, T_H, T) H(t, T_H, T)}{K} \right) + \frac{1}{2} \bar{\mathbf{S}}_2^2(t, T_c, T_H, T)}{\bar{\mathbf{S}}_2(t, T_c, T_H, T)},$$

$$d_2 = d_1 - \bar{\mathbf{S}}_2(t, T_c, T_H, T),$$

$$\begin{aligned}
\mathbf{g}(t, T_c, T_H, T) &= \frac{1}{2} \left(\mathbf{s}_r^2 + \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 \right) e^{-\mathbf{a}(T_H - T_c)} D_a^2(t, T_c) D_a(T_H, T) \\
&\quad + \frac{1}{2} \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 e^{-\mathbf{m}(T_H - T_c)} D_m^2(t, T_c) D_m(T_H, T) - \\
&\quad \frac{1}{\mathbf{m}} \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 e^{-\mathbf{a}(T_H - T_c)} D_a(T_H, T) \\
&\quad \left(D_a^2(t, T_c) + D_{\mathbf{a}+\mathbf{m}}(t, T_c) \right) - \frac{1}{\mathbf{a}} \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 e^{-\mathbf{m}(T_H - T_c)} \\
&\quad D_m(T_H, T) \left(D_a^2(t, T_c) + D_{\mathbf{a}+\mathbf{m}}(t, T_c) \right).
\end{aligned}$$

$\bar{\mathbf{s}}_2^2(\cdot)$ is defined in equation (7) with $T_G = T_H$.

A straight comparison of the expressions $C_1(t)$ (equation (6)), $C_2(t)$ (equation (7)) and $C_3(t)$ (equation (9)) shows that all three values differ. If the options and both the forward and futures contracts have the same maturity (i.e., $T_c = T_G = T_H$), then the option on the bond and the option on the forward contract have identical values. Yet, the option on the futures has a different price because interest rates and the short term mean are stochastic. When the two factors (interest rate and its short run mean) are deterministic the prices of the three options are equal. Generally, the option maturity differs from that of the forward and futures contracts (i.e., $T_c < T_G = T_H = T^*$). If the two factors are deterministic then forward and futures prices are identical. However, the value of options on forward contracts is

different from that of options on futures contracts. This is due to the marking-to-market mechanism.

6. Conclusion

In this paper, we have derived formulas for European options on discount bonds and on discount bond forward and futures contracts. The bond price depends on two-state variables: the short rate and its short term mean. The choice of the state variables is based on empirical evidence and tries to remedy the theoretical weaknesses of existing models. The economic framework allows one to value any type of interest rate derivatives.

At least three possible directions for further research can be pursued along the lines of this model. First, a procedure can be found to fit the model to any given initial term structure. Second, the model can be extended to an international economy where both domestic and exchange rates are stochastic. Finally, an empirical work will be necessary to test the implications of the model in the valuation of interest rate contingent claims.

Mathematical Appendix

In order to derive the option prices (6), (7) and (9) in the main text, the forward-neutral probability measure, the expressions of $r(t)$ and $\theta(t)$ and the following result are used :

Result Let X is a Gaussian variable $N(\mu, \sigma^2)$ then :

$$E\left[e^X 1_{\{e^X > K\}}\right] = \exp\left\{\mathbf{m} + \frac{1}{2} \mathbf{s}^2\right\} N\left(\frac{\mathbf{m} + \mathbf{s}^2 - LnK}{\mathbf{s}}\right)$$

$$E\left[1_{\{e^X > K\}}\right] = N\left(\frac{\mathbf{m} - LnK}{\mathbf{s}}\right)$$

where: $1_{\{\cdot\}}$ the indicator function.

The solutions to (1') and (2') have the form :

$$\begin{aligned} r(T_u) = & e^{-a(T_u-t)} r(t) + \frac{\mathbf{a}}{\mathbf{a} - \mathbf{m}} \left(\mathbf{q}(T_u) - e^{-a(T_u-t)} \mathbf{q}(t) \right) - \mathbf{l}_r \mathbf{s}_r D_a(t, T_u) + \\ & \frac{ab^*}{\mathbf{a} - \mathbf{m}} D_a(t, T_u) + \mathbf{s}_r e^{-a(T_u-t)} \int_t^{T_u} e^{a(T_u-s)} d\tilde{W}_r(s) - \\ & \frac{\mathbf{a} \mathbf{s}_q}{\mathbf{a} - \mathbf{m}} e^{-a(T_u-t)} \int_t^{T_u} e^{a(T_u-s)} d\tilde{W}_q(s) \end{aligned} \quad (A1)$$

$$\mathbf{q}(T_u) = e^{-m(T_u-t)} \mathbf{q}(t) + b^* D_m(t, T_u) + \mathbf{s}_q e^{-m(T_u-t)} \int_t^{T_u} e^{m(T_u-s)} d\tilde{W}_q(s) \quad (A2)$$

where $T_u > t$.

The forward-neutral probability measure, denoted Q^{T_j} , is equivalent to Q , for $T_j > 0$. This change of measure makes numeraire the pure discount bond $B(t, T_j)$. Since $E\left[\exp\left(1/2 \int_0^{T_j} D_a(s, T_j)^2 ds\right)\right] < \infty$ and $E\left[\exp\left(1/2 \int_0^{T_j} A(s, T_j)^2 ds\right)\right] < \infty$

, then according to Girsanov's theorem, the processes $W_r^{T_j}(t) = \tilde{W}_r(t) + \mathbf{s}_r D_a(t, T_j)$ and $W_q^{T_j}(t) = \tilde{W}_q(t) - \mathbf{s}_q A(t, T_j)$ are Brownian motions under Q^{T_j} . In particular, the forward price $B(t, T)/B(t, T_j)$ is a martingale under this measure.

Derivation of equation (7) : calls on bond forward contracts

One wants to evaluate :

$$C_2(t) = E^Q \left[\exp \left(- \int_t^{T_c} r(s) ds \right) \max [B(T_c, T_G)(G(T_c, T_G, T) - K), 0] \middle| F_t \right]$$

Let Q^{T_G} the forward-neutral probability measure equivalent to Q . Under Q^{T_G} , the forward price $G(t, T_G, T)$ is a martingale such that⁵ :

$$\begin{aligned} \frac{dG(t, T_G, T)}{G(t, T_G, T)} = & -\mathbf{s}_r (D_a(t, T) - D_a(t, T_G)) dW_r^{T_G}(t) + \\ & \mathbf{s}_q (A(t, T) - A(t, T_G)) dW_q^{T_G}(t) \end{aligned} \quad (\text{A3})$$

Applying Itô's lemma to $G(t, T_G, T)$ yields :

$$\begin{aligned}
G(t, T_G, T) = \frac{B(0, T)}{B(0, T_G)} \exp \left\{ -\frac{1}{2} \mathbf{s}_r^2 \int_0^t (D_a(s, T) - D_a(s, T_G))^2 ds - \right. \\
\left. \frac{1}{2} \mathbf{s}_q^2 \int_0^t (A(s, T) - A(s, T_G))^2 ds - \right. \\
\left. \mathbf{s}_r \int_0^t (D_a(s, T) - D_a(s, T_G)) dW_r^{T_G}(s) + \right. \\
\left. \mathbf{s}_q \int_0^t (A(s, T) - A(s, T_G)) dW_q^{T_G}(s) \right\}
\end{aligned}$$

The forward price at date T_c of option expiry have lognormal distributions. Therefore :

$$\begin{aligned}
LnG(T_c, T_G, T) \sim N \left(LnG(t, T_G, T) - \frac{1}{2} \mathbf{s}_r^2 \int_t^{T_c} (D_a(s, T) - D_a(s, T_G))^2 ds \right. \\
\left. - \frac{1}{2} \mathbf{s}_q^2 \int_t^{T_c} (A(s, T) - A(s, T_G))^2 ds; \right. \\
\left. \mathbf{s}_r^2 \int_t^{T_c} (D_a(s, T) - D_a(s, T_G))^2 ds + \right. \\
\left. \mathbf{s}_q^2 \int_t^{T_c} (A(s, T) - A(s, T_G))^2 ds \right)
\end{aligned}$$

Under Q^{T_G} , the call price $C_2(t)$ is given by :

$$\begin{aligned}
C_2(t) &= B(t, T_G) E^{Q^{T_G}} \left[\max [(G(T_c, T_G, T) - K), 0] | F_t \right] \\
&= B(t, T_G) E^{Q^{T_G}} \left[G(T_c, T_G, T) 1_{G>K} | F_t \right] - K E^{Q^{T_G}} \left[1_{G>K} | F_t \right]
\end{aligned}$$

Using the result and rearranging terms yields equation (7).

Derivation of equation (6) : calls on discount bonds

Letting $T_C = T_G$ in equation (7) yields directly equation (6), a particular case of (7) when the option expiry date coincides with that of the forward contract : at $T_C = T_G$ the forward price is equal to the spot price.

Derivation of the equation of the price of the futures contract

The futures price $H(t, T_H, T)$ is given by :

$$H(t, T_H, T) = E^Q [B(T_H, T) | F_t] \quad (\text{A4})$$

To obtain the futures price as a function of the forward price, we first compute :

$$\begin{aligned}
\frac{B(T_H, T)}{G(t, T_H, T)} = & \exp\left\{-D_a(T_H, T)(r(T_H) - e^{-a(T_H-t)}r(t)) + \frac{a}{a-m}D_a(T_H, T) \right. \\
& \left. (\mathbf{q}(T_H) - e^{-a(T_H-t)}\mathbf{q}(t)) - \frac{a}{a-m}D_m(T_H, T) \right. \\
& \left. (\mathbf{q}(T_H) - e^{-m(T_H-t)}\mathbf{q}(t)) - \left(\frac{1}{2}\frac{\mathbf{s}_r^2}{a} + \frac{1}{2}\frac{a\mathbf{s}_q^2}{(a-m)^2} - \frac{a^2\mathbf{s}_q^2}{m(a-m)} \right. \right. \\
& \left. \left. + I_r\mathbf{s}_r - \frac{ab^*}{a-m}\right)D_a(t, T_H)D_a(T_H, T) - \frac{a}{a-m} \right. \\
& \left. \left(\frac{1}{2}\frac{a\mathbf{s}_q^2}{m(a-m)} - \frac{\mathbf{s}_q^2}{(a-m)} - b^*\right)D_\mu(t, T_H)D_\mu(T_H, T) - \right. \\
& \left. \frac{1}{4}\left(\frac{\mathbf{s}_r^2}{a} + \frac{a\mathbf{s}_q^2}{(a-m)^2}\right)\left(D_a^2(T_H, T) + D_a^2(t, T_H) - D_a^2(t, T)\right) \right. \\
& \left. - \frac{1}{4}\frac{1}{m}\left(\frac{a\mathbf{s}_q^2}{a-m}\right)^2\left(D_m^2(T_H, T) + D_m^2(t, T_H) - D_m^2(t, T)\right) - \right. \\
& \left. \frac{a(a+m)\mathbf{s}_q^2}{m(a-m)^2}D_{a+m}(t, T_H)D_{a+m}(T_H, T)\right\} \quad (A5)
\end{aligned}$$

Let

$$\begin{aligned}
Y(t, T_H) = & r(T_H) - e^{-a(T_H-t)}r(t) - \frac{a}{a-m}\left(\mathbf{q}(T_H) - e^{-a(T_H-t)}\mathbf{q}(t)\right) \\
= & -I_r\mathbf{s}_rD_a(t, T_H) + \frac{ab^*}{a-m}D_a(t, T_H) + \mathbf{s}_r e^{-a(T_H-t)} \\
& \int_t^{T_H} e^{a(T_H-s)}d\tilde{W}_r(s) - \frac{a\mathbf{s}_q^2}{a-m}e^{-a(T_H-t)}\int_t^{T_H} e^{a(T_H-s)}d\tilde{W}_q(s)
\end{aligned}$$

$$Z(t, T_H) = \mathbf{q}(T_H) - e^{-\mathbf{m}(T_H-t)} \mathbf{q}(t) = b^* D_{\mathbf{m}}(t, T_H) + \mathbf{s}_q e^{-\mathbf{m}(T_H-t)} \int_t^{T_H} e^{\mathbf{m}(T_H-s)} d\tilde{W}_q(s)$$

Given (A1) and (A2), $Y(t, T_H)$ and $Z(t, T_H)$ are normally distributed with conditional means and variances as follows :

$$E^Q[Y(t, T_H)|r(t), \mathbf{q}(t)] = -\mathbf{1}_r \mathbf{s}_r D_{\mathbf{a}}(t, T_H) + \frac{\mathbf{a}b^*}{\mathbf{a} - \mathbf{m}} D_{\mathbf{a}}(t, T_H) \quad (\text{A6a})$$

$$\text{Var}(Y(t, T_H)|r(t), \mathbf{q}(t)) = \left(\mathbf{s}_r^2 + \left(\frac{\mathbf{a}\mathbf{s}_q}{\mathbf{a} - \mathbf{m}} \right)^2 \right) \left(D_{\mathbf{a}}(t, T_H) - \frac{\mathbf{a}}{2} D_{\mathbf{a}}^2(t, T_H) \right) \quad (\text{A6b})$$

$$E^Q[Z(t, T_H)|\mathbf{q}(t)] = b^* D_{\mathbf{m}}(t, T_H) \quad (\text{A6c})$$

$$\text{Var}(Z(t, T_H)|\mathbf{q}(t)) = \mathbf{s}_q^2 \left(D_{\mathbf{m}}(t, T_H) - \frac{\mathbf{m}}{2} D_{\mathbf{m}}^2(t, T_H) \right) \quad (\text{A6d})$$

By plugging (A5) into (A4) and by using expressions (A6), one can compute the expectation under Q that gives the futures price.

Derivation of equation (9) : calls on bond futures contracts

Computing the forward price $\phi(t, T_C, T_H, T)$ of the futures contract, one can use the same technique as for calls on forward contracts. Following Jamshidian (1993), one obtains the following expressions for the forward price of the futures contract :

$$\begin{aligned}
f(t, T_c, T_H, T) &= \frac{E^Q \left[\exp \left(- \int_t^{T_c} r(s) ds \right) H(T_c, T_H, T) \middle| F_t \right]}{B(t, T_c)} \\
&= H(t, T_H, T) \exp \left\{ \int_t^{T_c} \text{cov} \left(\frac{dH(s, T_H, T)}{H(s, T_H, T)}, \frac{dB(s, T_c)}{B(s, T_c)} \right) \right\} \\
&= H(t, T_H, T) \exp \left\{ \mathbf{S}_r^2 \int_t^{T_c} D_a(s, T_c) (D_a(s, T) - D_a(s, T_H)) ds \right. \\
&\quad \left. \mathbf{S}_q^2 \int_t^{T_c} A(s, T_c) (A(s, T) - A(s, T_H)) ds \right\} \\
&\equiv H(t, T_H, T) \exp(\mathbf{g}(t, T_c, T_H, T))
\end{aligned}$$

Defining a new forward-neutral probability measure Q^{T_c} equivalent to Q , the futures contract price become a martingale. Then remark that $C_3(t)$ can be written :

$$C_3(t) = E^Q \left[\exp \left(- \int_t^{T_c} r(s) ds \right) \max [B(T_c, T_c) (F(T_c, T_c, T_H, T) - K), 0] \middle| F_t \right]$$

If this formula is compared with that giving $C_2(t)$, it is readily seen that equation (7) can be applied with T_c instead of T_G , $H(t, T_H, T) \exp(\mathbf{g}(t, T_c, T_H, T))$ instead of $G(t, T_H, T)$, which yields directly equation (9).

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¹ As Dybvig (1990) states "the second factor (if any) in a term structure model should be related to the distributional features of interest rates, not additive in levels of interest rates as is usually assumed".

² The main idea of the risk-neutral pricing method is analogous to that of the Black and Scholes (1973) model. In this model, the expectation of the instantaneous yield of the underlying asset does not affect the value of the option. The latter is a function of the riskless rate. Since the preference parameter (the drift) drops out of the valuation process, all assets should earn what a risk-neutral investor would expect, the risk-free rate.

³ Puts can be valued in the same manner or through the standard call-put parity.

⁴ It is obvious that the solution given by the formula (6) is a Black and Scholes (1973) type. The similarity between this solution and the Black-Scholes stock option model has been discussed by Jamshidian (1989).

⁵ The dynamics of the bond price is given by

$$\frac{dB(t,T)}{B(t,T)} = \mathbf{m}_b dt + \mathbf{s}_r \frac{B_r}{B} d\tilde{W}_r(t) + \mathbf{s}_q \frac{B_q}{B} d\tilde{W}_q(t)$$

where

$$\mathbf{m}_b = \frac{1}{B} \left(\frac{1}{2} B_{rr} \mathbf{s}_r^2 + \frac{1}{2} B_{qq} \mathbf{s}_q^2 + [\mathbf{a}(q(t) - r(t)) - \mathbf{l}_r \mathbf{s}_r] B_r + [\mathbf{m}(b - q(t)) - \mathbf{l}_q \mathbf{s}_q] B_q + B_t \right)$$

Using the solution (4), the instantaneous rate of return of the discount bond can be written as :

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt - \mathbf{s}_r D_a(t,T) d\tilde{W}_r(t) + \mathbf{s}_q A(t,T) d\tilde{W}_q(t)$$

Applying Itô's lemma to $G(t, T_G, T) = B(t, T)/B(t, T_G)$ and using the forward-neutral probability measure yields (A3).