

Infinite-dimensional reaction  
diffusion equations for  
population dynamics in the context of  
an artificial selective pressure

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## The logistic model

Number Density

$$N = N(t).$$

$N(t)$  = number of individuals at time  $t$

Logistic equation:

$$N'(t) = \mu N(t) - \sigma N(t)^2.$$

## Typical Equation for Populations

### in the Context of a Continuous Space

Number Density

$$f \equiv f(t, x).$$

$f(t, x) dx =$  number of individuals at time  $t$   
between  $x$  and  $x + dx$ .

Fisher Equation:

$$\frac{\partial f}{\partial t}(t, x) - m \Delta_x f(t, x) = \mu f(t, x) - \sigma f(t, x)^2.$$

Homogeneous Neumann Boundary Condition :

$$\nabla_x f(t, x) \cdot n(x) = 0, \quad x \in \partial\Omega.$$

## Typical Equation for Populations

### with a Continuous Trait

Number Density

$$f \equiv f(t, v).$$

$f(t, v) dv =$  number of individuals at time  $t$   
having a trait between  $v$  and  $v + dv$ .

Equation for Selection/Mutations and Competition:

$$\frac{\partial f}{\partial t}(t, v) = \mu(v) f(t, v) + \varepsilon \int K(v, v') f(t, v') dv' - \left( \int L(v, v') f(t, v') dv' \right) f(t, v).$$

## PDE Combining the Continuous Dependence

### with Respect to Space and Trait

Number Density

$$f \equiv f(t, x, v).$$

Equation for Migration, Selection/Mutations

and Competition

(Reaction-Diffusion in Infinite Dimension, or Continuous Reaction-Diffusion)

$$\frac{\partial f}{\partial t}(t, x, v) - m \Delta_x f(t, x, v) = \mu f(t, x, v) + \int K(v, v') f(t, x, v') dv' - \left( \int L(v, v') f(t, x, v') dv' \right) f(t, x, v).$$

## Interaction between the Space and Trait Variables

Example : Trait = Migration Rate

$$\frac{\partial f}{\partial t}(t, x, v) - v \Delta_x f(t, x, v) = \mu f(t, x, v)$$

$$+ \varepsilon \int K(v, v') f(t, x, v') dv' - \left( \int f(t, x, v') dv' \right) f(t, x, v).$$

## Simulation through a (Monte-Carlo Type)

### Particle Method

Principle of the  $x, v$ -discretization

$$f(t, x, v) \sim \sum_{i=1}^N r_i(t) \delta_{x_i(t), v_i(t)}.$$

Migrations :  $x_i$  move according to a simulated Brownian motion

Birth/Mutations :  $v_i$  move according to a jump process

Not expensive (with respect to the dimension); not precise

## Typical Equation for Populations

### with an Age Dependence

Number Density

$$f \equiv f(t, a).$$

$f(t, a) da =$  number of individuals of age

between  $a$  and  $a + da$ .

Leslie Equation

$$\frac{\partial f}{\partial t}(t, a) + \frac{\partial f}{\partial a}(t, a) =$$
$$-\left[ l(a) + \int L(a, a') f(t, a') da' \right] f(t, a);$$

$$f(t, 0) = \int K(a') f(t, a') da'.$$

PDE Combining the Continuous Dependance  
with Respect to Space, Trait and Age

Number Density

$$f \equiv f(t, x, v, a).$$

Equation for Age, Migration, Selection/Mutations  
and Competition

(Convection-Reaction-Diffusion in Infinite Dimension,  
or Continuous Convection-Reaction-Diffusion)

$$\frac{\partial f}{\partial t}(t, x, v, a) + \frac{\partial f}{\partial a}(t, x, v, a) - m \Delta_x f(t, x, v, a) =$$

$$- \left[ l(a, v) + \int L(a, a', v, v') f(t, x, v', a') dv' da' \right] f(t, x, v, a);$$

$$f(t, x, v, 0) = \int \mu(v, a') f(t, x, v, a') da'$$

$$+ \varepsilon \int K(v, v', a') f(t, x, v', a') dv' da'.$$

## Sexual Reproduction

$$\frac{\partial f}{\partial t}(t, x, v) - m \Delta_x f(t, x, v) =$$

$$\left( \int f(t, x, v) dv \right)^{-1} \int \int K(v, v', v'_*) f(t, x, v') g(t, x, v'_*) dv' dv'_* \\ - f(t, x, v) \int \left( L_{11}(v, v') f(t, x, v') + L_{12}(v, v') g(t, x, v') \right) dv',$$

$$\frac{\partial g}{\partial t}(t, x, v) - m \Delta_x g(t, x, v) =$$

$$\left( \int f(t, x, v) dv \right)^{-1} \int \int K(v, v', v'_*) f(t, x, v') g(t, x, v'_*) dv' dv'_* \\ - g(t, x, v') \int \left( L_{21}(v, v') f(t, x, v') + L_{22}(v, v') g(t, x, v') \right) dv'.$$

## Coupling Ecology and Economy

Logistic equation with harvesting

$$N'(t) = \mu N(t) - \sigma N(t)^2 - q N(t) E$$

Without economy :

$$E = \alpha$$

Model of Idels and Wang :

$$E = \alpha - \beta N'(t)/N(t).$$

Possibility of thresholding.

## Possible improvements : ecology

1. Taking into account the space variable
2. Taking into account the size of adults
3. Taking into account the age

## Possible improvements : economy

1. Establishment of a price
2. Strategy of effort for a given price
3. Dependence of the effort/price with respect to distance, size of the preys

## Example of modelling

Harvested quantity :

$$H(t, v) = \int \int q(x, v) f(t, x, v, a) E(t, x) dx da$$

Price :

$$P(t, v) = F\left(\int H(t, v') k(v, v') dv'\right),$$

Effort :

$$E(t, x) = Cst \int_{t-t_1}^{t_2} \int \int P(\tau, v) q(x, v) f(t, x, v, a) da dv d\tau.$$

Evolution of the resource :

$$\begin{aligned} & \frac{\partial f}{\partial t}(t, x, v, a) + \frac{\partial f}{\partial a}(t, x, v, a) \\ & + Cst \mathbf{1}_{a \leq a_0(v)} \frac{\partial f}{\partial v}(t, x, v, a) - m \Delta_x f(t, x, v, a) = \\ & - \left[ l(a, x, v) + \int L(a, a', x, v, v') f(t, x, v', a') dv' da' \right. \\ & \quad \left. + q(x, v) E(t, x) \right] f(t, x, v, a); \end{aligned}$$

$$f(t, x, v, 0) = \int \mu(x, v, a') f(t, x, v, a') da'$$

$$+ \varepsilon \int K(x, v, v', a') f(t, x, v', a') dv' da'.$$