Some mathematical results for Black-Scholes-type equations for financial derivatives

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- Introduction
- Analysis of incomplete market model
- Volatility identification from market data
- Credit risk modeling

Financial derivatives

Financial derivative is a financial instrument whose value depends on a underlying (stocks, commodities etc.)

Usage:

- Risk control (hedging)
- Speculation

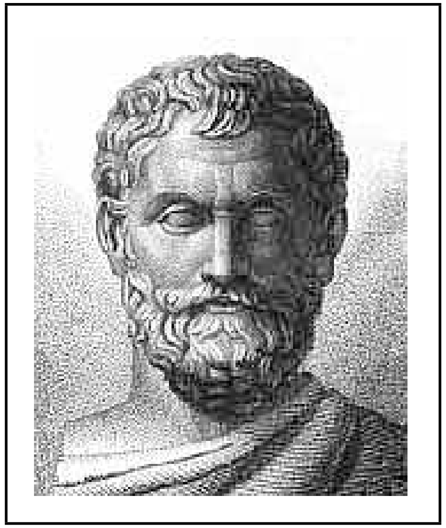
Example: European options

Holder has right but not the obligation to buy (call option) or to sell (put option) an asset at time T for fixed price K

Call option: if asset price $S_T \geq K$, buy asset for price K; if $S_T < K$, buy asset on the market

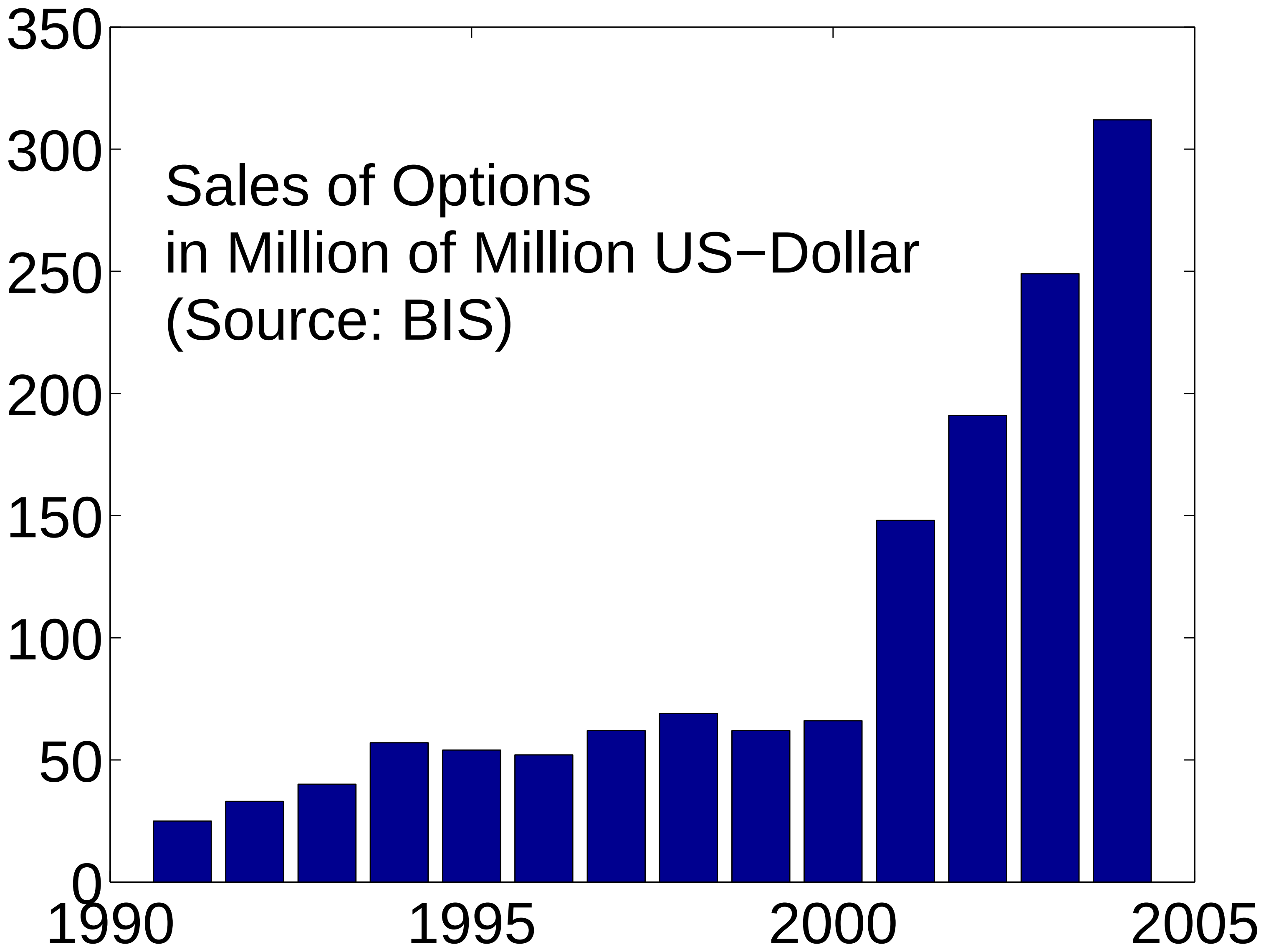
Main question: How much is the option worth?

Short history of financial derivatives



Thales

600 B.C.	Option on olive oil presses (Thales von Milet)
1630	Options on tulips (Holland) first market crash 1637
1650	Standardized futures on rice (Japan)
1728	Options of Royal Westindian Company (island St. Croix)
1848	Opening of Chicago Board of Trade
1973	Opening of Chicago Stock Options Exchange


Sales of Options
in Million of Million US−Dollar
(Source: BIS)
350
300
250
200
150
100
50
0
1990
1995
2000
2005

Question: What is the value V of an option?

Answer: Option price model of Black, Scholes, Merton (1973) (Nobel Prize in Economics 1997 for Merton and Scholes)

Black-Scholes world

Assumptions: Financial market is

- arbitrage-free (no instantaneous, riskfree gain)
- complete (every payoff is attainable)
- frictionless (no transaction costs, no taxes etc.)
- asset price S_t follows geometric Brownian motion
  $dS_t = \mu S_t dt + \sigma S_t dW_t$

Black-Scholes equation: σ: volatility, r: interest rate

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \quad S > 0,\ 0 < t < T$$

Beyond Black-Scholes

Relax Black-Scholes assumptions:

- Transaction costs allowed (Davis et al. 1993, Barles/Soner 1998)
- Feedback effects due to large traders (Frey 1998)
- Incomplete markets (Leitner 2001)

$\longrightarrow$ gives nonlinear Black-Scholes-type equations

$$V_t + \frac{1}{2} S^2 \sigma(t, S, V, V_S, V_{SS})^2 V_{SS} + rSV_S - rV = 0$$

$$V_t + \frac{1}{2} \sum_{i,j} \sigma_{ij}(V)^2 V_{S_i S_j} = f(t, S, V, |\nabla V|^2)$$

- Stochastic volatility (Scott '87, Hull/White '88, Heston '93)
- Other price processes (fractional Brownian, jump-diff...)

Overview

- Incomplete market model
- Volatility identification from market data
- Credit risk modeling

Objective: Find optimal trading strategy

(dates and number of shares in order to be traded to maximize profit)

Complete market: Solve Bellman equation, gives optimal value function e^u (Merton 1969)

Incomplete market: risky assets S_i and non-tradable variables S_i' (Leitner 2001)

$$\partial_t u - \frac{1}{2}\sum_{i,j} C_{ij}(u)\partial_{ij}u - \frac{1}{2}\sum_{i,j} C'_{ij}(u)\partial'_{ij}u$$
$$= \frac{1}{2}\nabla' u C'(u)\nabla' u - \frac{1}{2(p-1)}\nabla u C(u)\nabla u + f(S, S', u, \ldots)$$

in bounded domain or whole space + initial data

p: risk aversion parameter

Idea of derivation

- Utility: $U(x) = \mathsf{sign}(1-p)x^p/p$ (if $p = 0$: $U(x) = \ln x$)
- Value function: $v(x) = \sup_Y \mathsf{E}[U(Y(T))]$ taken over all self-financing portfolios $Y(t) \geq 0$ such that $Y(0) = x$
- Markovian market: W_i, W'_j correlated Wiener processes with covariance matrices (C_{ij}), (C'_{ij})

$$dS_i = \mu_i(S_i)dt + \sigma(S_i)dW_i,\ dS'_j = \mu'_j(S'_j)dt + \sigma(S'_j)dW_j$$

- Relation between optimal portfolio and optimal value function yields PDE for $u = \ln v$

$$\partial_t u - \frac{1}{2}\sum_{i,j} C_{ij}(u)\partial_{ij}u - \frac{1}{2}\sum_{i,j} C'_{ij}(u)\partial'_{ij}u$$
$$= \ \tfrac{1}{2}\nabla' u C'(u)\nabla' u - \tfrac{1}{2(p-1)}\nabla u C(u)\nabla u + f(S, S', u, \ldots)$$

Incomplete market model

$$\partial_t u - \frac{1}{2}\sum_{i,j} C_{ij}(u)\partial_{ij} u - \frac{1}{2}\sum_{i,j} C'_{ij}(u)\partial'_{ij} u$$
$$= \frac{1}{2}\nabla' u C'(u)\nabla' u - \frac{1}{2(p-1)}\nabla u C(u)\nabla u + f(S, S', u, \dots)$$

Mathematical features:

- quasilinear, quadratic gradients
- $p = 0$: exponential transformation removes quadratic gradients

Optimal trading strategy:

- Optimal portfolio strategy: $H(S, S') = (\lambda - \nabla u)/(p-1)$, $\lambda = C^{-1}(rS - \mu)$, r: interest rate, μ/S_i: rel. return, components of $H(S, S')$ = shares of the portfolio assets
- Optimal portfolio: $Y = H(S, S') \cdot S$

Goal: existence and uniqueness of solutions

Incomplete market model

$$\partial_t u - \frac{1}{2}\sum_{i,j} C_{ij}(u)\partial_{ij}u - \frac{1}{2}\sum_{i,j} C'_{ij}(u)\partial'_{ij}u$$
$$= \ \frac{1}{2}\nabla' u C'(u)\nabla' u - \frac{1}{2(p-1)}\nabla u C(u)\nabla u + f(S, S', u, \ldots)$$

Existence of solutions: (Düring/A.J., *Nonlin. Anal.* 2005)

$C(u)$, $C'(u)$ symmetric, positive definite. Then

$\exists$ weak solution u globally in time

Ideas of proof:

- Apply maximum principle to regularized problem
- Frehse's test function $\sinh(\lambda u_\varepsilon) \Rightarrow \|u_\varepsilon\|_{H^1} \leq K$
- Monotonicity method: test function $\sinh(\lambda(u_\varepsilon - u))$
  $\Rightarrow$ strong convergence of (u_ε) in H^1

Incomplete market model

$$\partial_t u - \frac{1}{2}\sum_{i,j} C_{ij}(u)\partial_{ij}u - \frac{1}{2}\sum_{i,j} C'_{ij}(u)\partial'_{ij}u$$
$$= \frac{1}{2}\nabla' u C'(u)\nabla' u - \frac{1}{2(p-1)}\nabla u C(u)\nabla u + f(S, S', u, \ldots)$$

Uniqueness of solutions: (Düring/A.J., *Nonlin. Anal.* 2005)

$C(u)$, $C'(u)$ symmetric, positive definite, $\partial C(u)/\partial u$ "small", $p > 1$. Then uniqueness of weak solution

Ideas of proof:

- Barles/Murat 1995: transformation of variables $u = \phi(v) = -A^{-1}\ln(e^{-KAv} + K^{-1})$
- Given two solutions v_1, v_2, use test function $\max\{0, v_1 - v_2\}^n$, $n \gg 1$

$\rightarrow$ Fully nonlinear parabolic eqs. (Papi 2002)

Incomplete market model

$$\partial_t u - \frac{1}{2}\sum_{i,j} C_{ij}(u)\partial_{ij}u - \frac{1}{2}\sum_{i,j} C'_{ij}(u)\partial'_{ij}u$$
$$= \frac{1}{2}\nabla' u C'(u)\nabla' u - \frac{1}{2(p-1)}\nabla u C(u)\nabla u + f(S, S', u, \ldots)$$

Numerical example

Parameters:

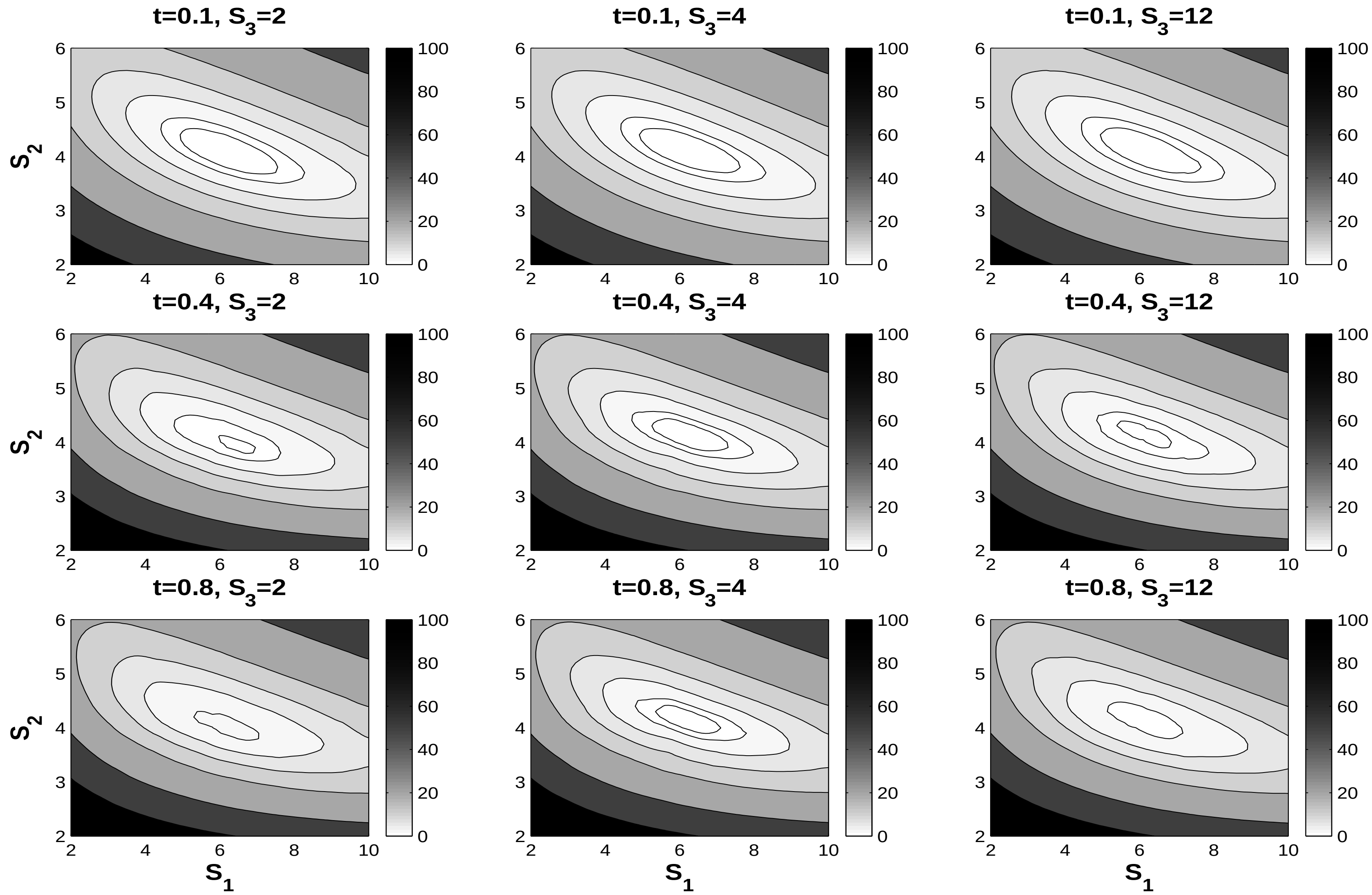
- Risk aversion parameter $p = \frac{1}{2}$
- Two risky assets S_1, S_2, one non-tradable variable $S' = S_3$
- Heuristic covariance matrices $C(u)$, $C'(u)$
- Returns: Ornstein-Uhlenbeck-type drift $\mu_i = (a_i - S_i)S_i$

Numerical method:

Quadratic finite elements, standard Runge-Kutta (FEMLAB package)

Incomplete market model

Numerical results: $u(S_1, S_2, S_3, t)$ versus S_1 and S_2



- Local minimum corresponds to expected return
- Maximal relative difference to minimum S_3^*: $2.9 \ldots 4.6$

Overview

- Incomplete market model
- Volatility identification from market data
- Credit risk modeling

Volatility identification

Black-Scholes equation: option price $V(S,t;K,T)$ solves

$$S_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \quad V(S,T) = V_0(S;K)$$

(σ: constant volatility, K: strike, T: expiration time)

- Market data shows: σ not constant
- Idea: replace σ by $\sigma(K,T)$

Dupire equation: option price $V(K,T;S_0,t_0)$ solves

$$V_T - \frac{1}{2}\sigma^2(K,T)K^2 V_{KK} + rKV_K = 0, \quad V(K,0) = V_0(S_0;K)$$

- Estimate $\sigma(K,T)$ from market data $\tilde{V}$
- Ill-posed problem (lack of continuous dependence of data)

Volatility identification

First idea: Dupire's formula

$$\sigma(K,T) = \left(\frac{2V_T + 2rKV_K}{K^2 V_{KK}}\right)^{1/2}$$

$+$ Computationally cheap

$-$ Possibly unstable, depends on interpolation method

(Dupire 1994, Bouchouev/Isakov 1999, Hanke/Rösler 2003,...)

Second idea: Regularization technique, e.g. minimize cost functional (Lagnado/Osher 1997)

$$J(\sigma, V) = \frac{1}{2}\int_\Omega |V - \tilde{V}|^2 dK + \frac{\beta}{2}\|\sigma\|^2$$

subject to constraint given by Dupire equation

(Jackson et al. '99, Achdou/Pironneau '02, Crépey '03, Egger/Engl '05)

$\longrightarrow$ Main focus on numerical results

$\longrightarrow$ Here: numerical analysis and stable algorithm

Volatility identification

Convention: write $V_T - \frac{1}{2}\sigma^2(K,T)K^2 V_{KK} + rKV_K = 0$

as $u_t - qu_{xx} = 0$

Optimal control problem

- Constraint:

  $e(u,q) = u_t - qu_{xx}$ plus boundary/initial conditions

- Cost functional: (X contains q_{xx}, $q_t \in L^2$)

  $$J(u,q) = \frac{1}{2}\|u(T) - \tilde{u}\|_{L^2}^2 + \frac{1}{2}\|q\|_X^2$$

- Problem: (C_{ad} contains $0 < q_{\min} \le q \le q_{\max}$)

  $\min J(u,q)$ subject to $e(u,q) = 0$ and $(u,q) \in C_{\mathrm{ad}}$

Theorem 1: (Düring/A.J./Volkwein 2006)

Problem has a solution in admissible set C_{ad}

Question: How to determine this solution?

Volatility identification

Lagrange functional:

$$L(\omega, p) = J(\omega) + e(\omega) \cdot p, \quad \omega = (u, q)$$

Theorem 2: (first-order necessary condition)
If $e'(\omega)$ surjective, $\omega^* = (u^*, p^*)$ solution of Problem then

$$\nabla L(\omega^*, p) = 0 \quad \forall p$$

Theorem 3: (second-order sufficient condition)
If $\|u^*(T) - \tilde{u}\|_{L^2}$ "small" then $\exists\ \kappa > 0$ such that

$$L''(\omega^*, p)(\omega, \omega) \geq \kappa \|\omega\|_X^2 \quad \text{for all } \omega \in C(\omega^*),$$

where $C(\omega^*)$ is the critical cone, i.e. the set of directions tangent to the feasible set

Lagrange-SQP (sequential quadratic programming) algorithm

① Choose $\omega^0 = (u^0, q^0)$, p^0

② Solve quadratic minimization problem for $\delta\omega^n$, δp^n

$$\min L'(\omega^n, p^n)\delta\omega^n + \frac{1}{2}L''(\omega^n, p^n)(\delta\omega^n, \delta\omega^n)$$

$$\text{subject to } e'(\omega^n)\delta\omega^n + e(\omega^n) = 0$$

③ Set $\omega^{n+1} = \omega^n + \delta\omega^n$, $p^{n+1} = p^n + \delta p^n$

- Discretization: linear finite elements, nonuniform grid
- Linear solver: GMRES with preconditioning
- Handling of L^∞ constraints for q:
  primal-dual active set strategy (Hintermüller 2003)
- Globalization strategy: line search with
  $\omega^{n+1} = \omega^n + \alpha_n\delta\omega^n$, $p^{n+1} = p^n + \alpha_n\delta p^n$

Numerical results: artificial data

Black-Scholes prices with $S_0 = 100$, $r = 0$, $\sigma = 0.15$, $T = 1$ month, 0.1 % noise

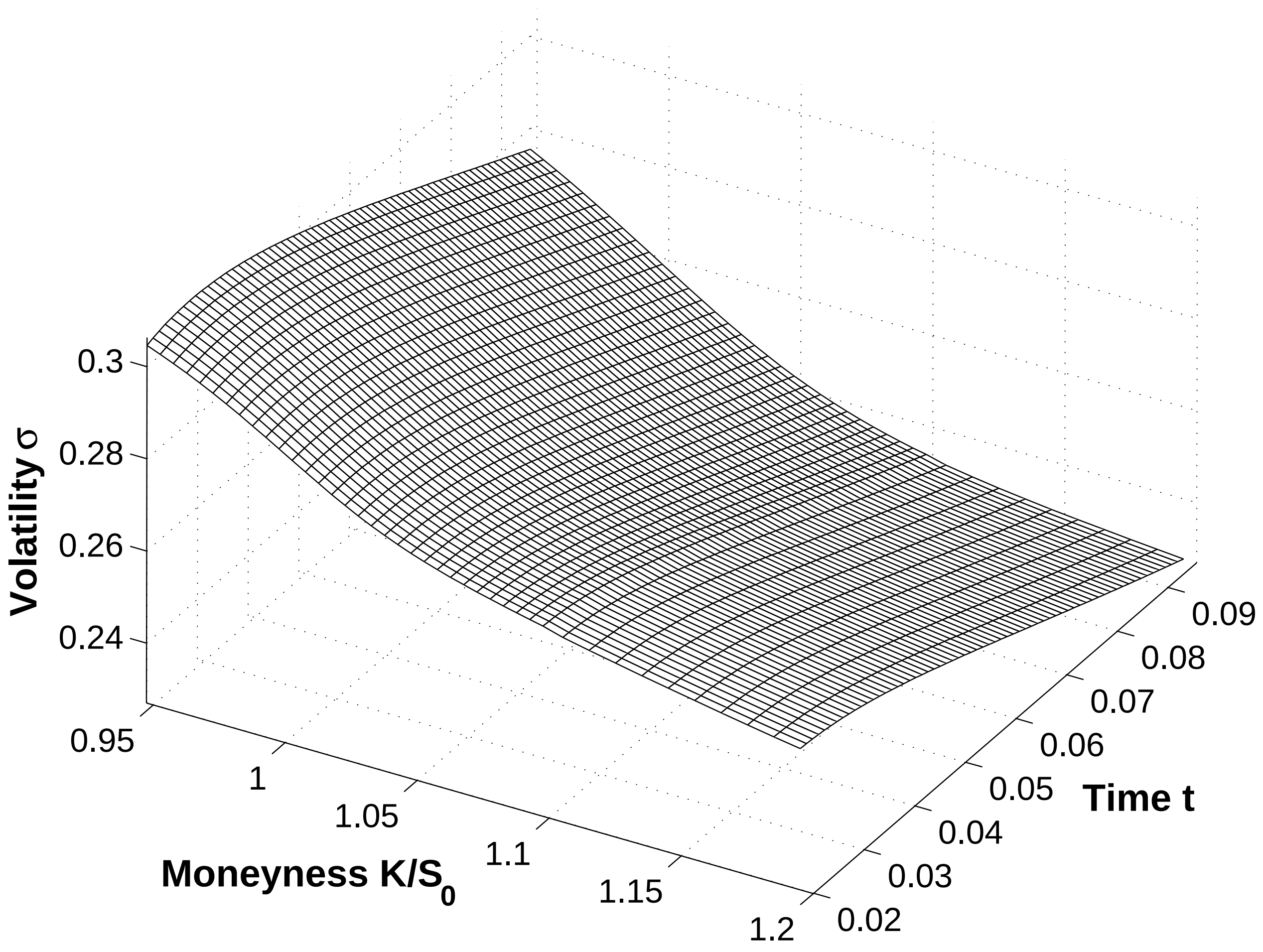
Strike	95	100	105
True value	5.24433	1.72734	0.28866
Good guess & noise	5.24429	1.72734	0.28861
Bad guess & noise	5.24435	1.72733	0.28866
Good guess & fine grid	5.24433	1.72734	0.28866

- dependence on a priori guess small
- robust regarding to small data noise (e.g. bid-ask spreads)
- option prices and volatilities very well recovered

Numerical results: market data

Data: FTSE 100 call option prices from Feb. 11, 2000

$\longrightarrow$ shows "volatility skew"



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Credit risk modeling

(Düring/A.J./Roth 2007)

Question: Does credit trade lead to higher credit allocation?

Modeling: bank portfolio consists of three securities

- Riskless bond $r = r_t$ (de Estrada 2005):

$$dr = \mu r dt + \sigma r dW, \quad t > 0$$

- Short-term credit r_1, long-term credit r_2, spread $h_i = r_i - r$:

$$dh_i = \mu_i h_i dt + \sigma_i \min\{h_i, H\} dW_i, \quad i = 1, 2$$

- Modeling of credit loss: $h_i \geq H$

① One representative bank: assume that r_i can not be sold
maximize utility $\Rightarrow$ trade strategy is maximizer of
Hamilton-Jacobi-Bellman equation

② Two banks: trade of long-term credit r_2 allowed
derive system of PDEs and solve numerically

Summary

Beyond Black-Scholes:

- Incomplete markets OK
- Non-constant volatility OK
- Credit risk modeling and feedback effects in progress
- Other price processes: modeling of pricing kernel (Düring/Lüders 2005)

Some research directions in mathematical finance:

- Stochastic volatility: numerics for stochastic diff. eqs.
- Multi-dimensional Back-Scholes models: sparse grid
- Energy derivatives: incomplete market
- Optimization of portfolios: efficient numerical algorithms