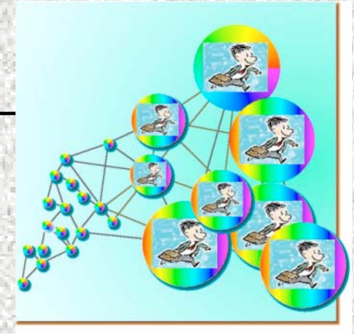
Workshop KTASEEM
Orléans, March 15-16, 2007

MAPMO



Fokker-Planck equations for opinion formation: modeling, analysis and asymptotics

Giacomo Aletti, Giovanni Naldi*
Università di Milano
Giuseppe Toscani,
Università di Pavia

• G. Aletti, G.N., G. Toscani, arXiv:cond-mat/0605092 May 2006, SIAM J. Appl.Math. (in press)

Ricerca Scientifica e Tecnologica
Ministero dell'Istruzione dell'Università e della Ricerca
PRIN2005

Opinion formation
(Marketing, Finance, Economic processes,…)

In recent years there has been interest in applications of physical paradigms to a quantitative description of of social and economical processes.

Some methods:
- stochastic processes
- lattice gas models
- deterministic chaos and discrete dynamical systems
- statistical mechanics
- Game Theory
- ...

Formation of opinion is among the challenging problems in social science due to complex dynamics which may depend on different internal and external influences.

Opinion formation
(A non so serious example)

We look for the dynamics of the opinion: quantitative approach is mainly based on the concept of *social impact*. The social impact describes the force on an individual *to keep* or *to change* its *current opinion*.

Opinion formation

(Sznajd Model - related to the Ising model)

•Social opinion is the outcome of by individual opinions

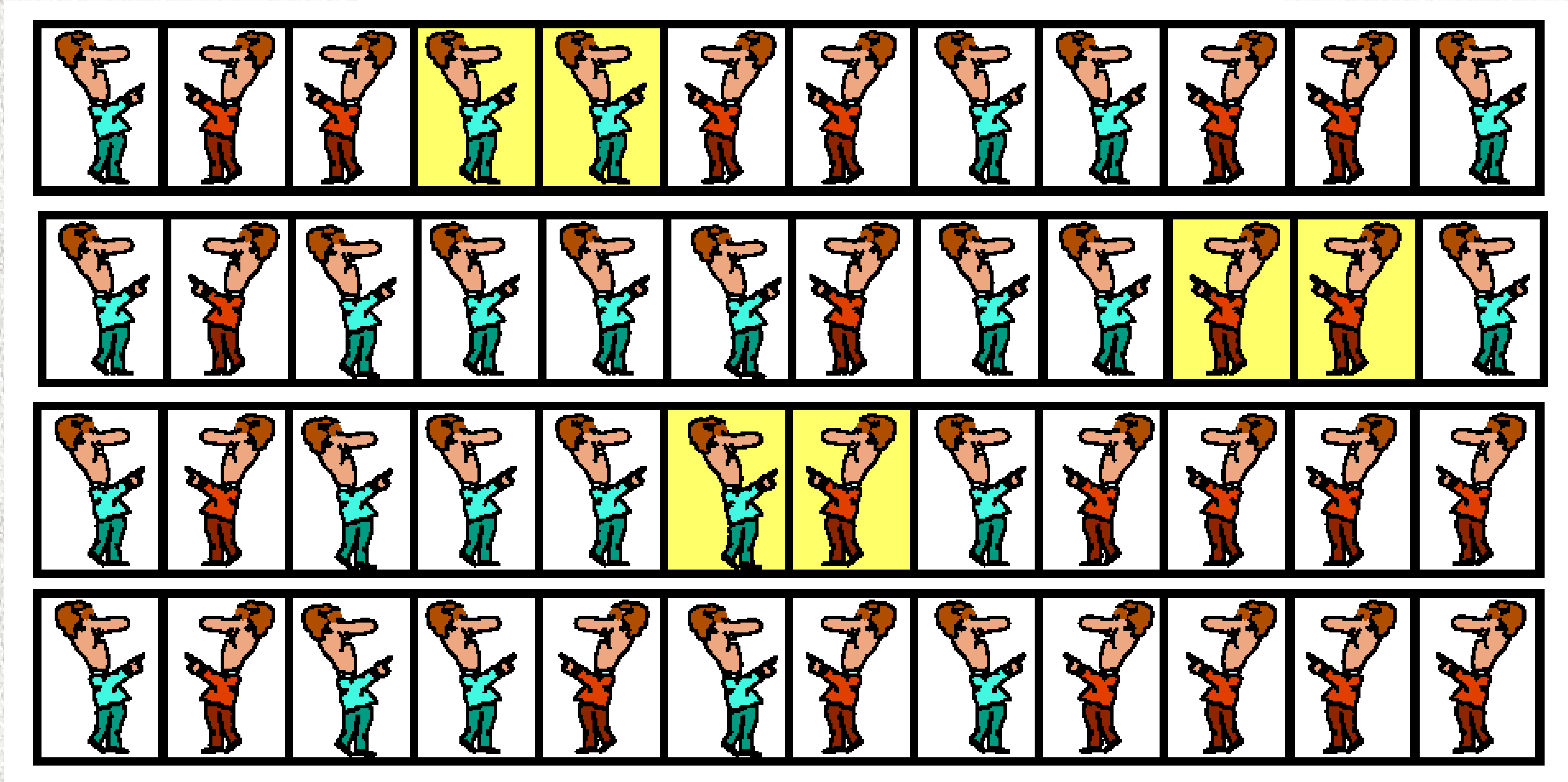
•Individual opinion is represented by Ising spins: *yes* or *no*

•rule: ``united we stand divided we fall''

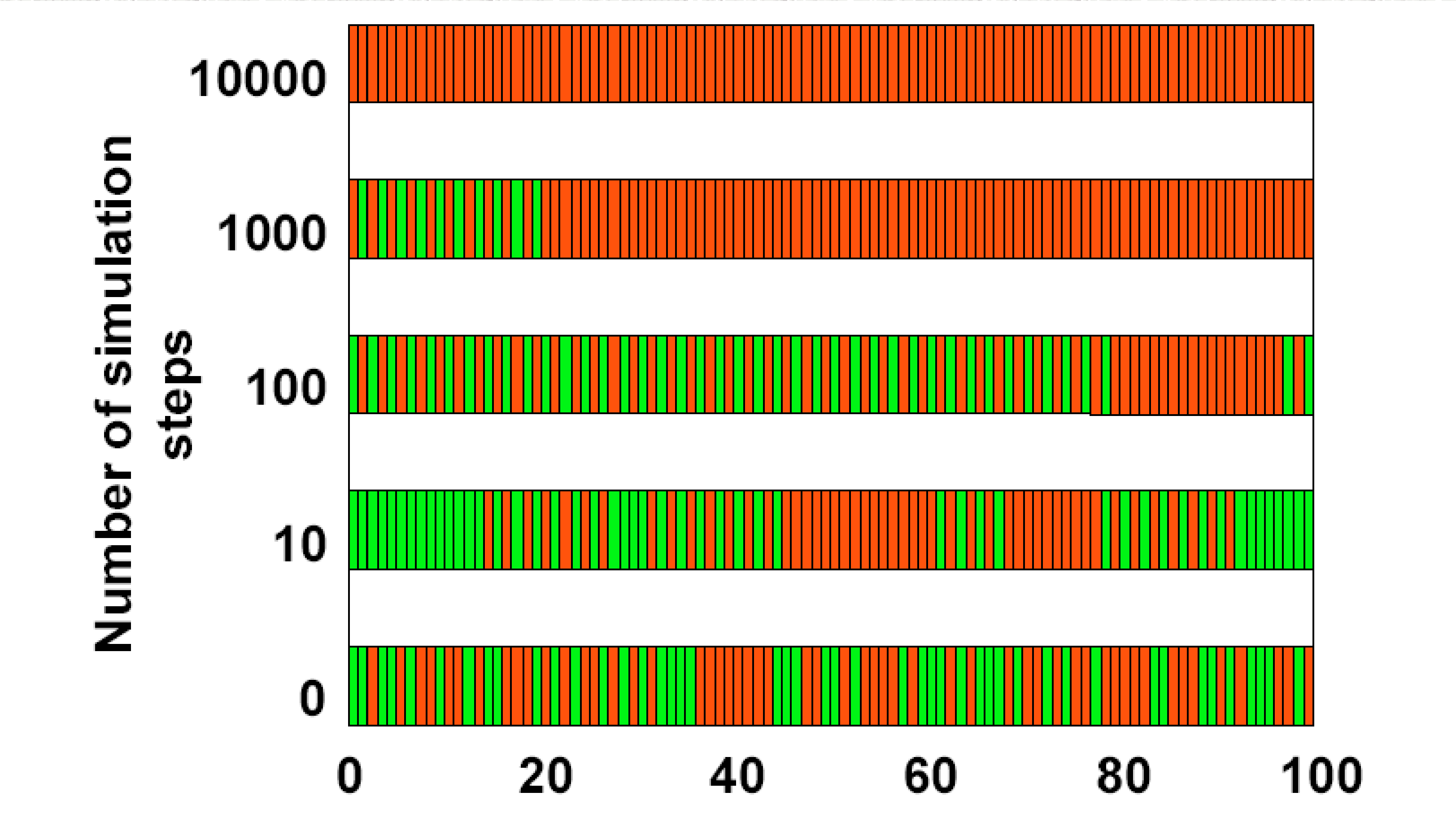
•chain (1D model) of opinions S_i, $i=1,\dots,N$, $S_i \in \{-1,1\}$

Dynamic rules:

- ♦ if $S_i S_{i+1}=1$, then S_{i-1} and S_{i+2} take the direction of the pair $(i,i+1)$
- ♦ if $S_i S_{i+1}=-1$, then S_{i-1} takes the direction of S_{i+1} and S_{i+2} the direction of S_i.



random sequential updating



Typical Monte Carlo simulation.

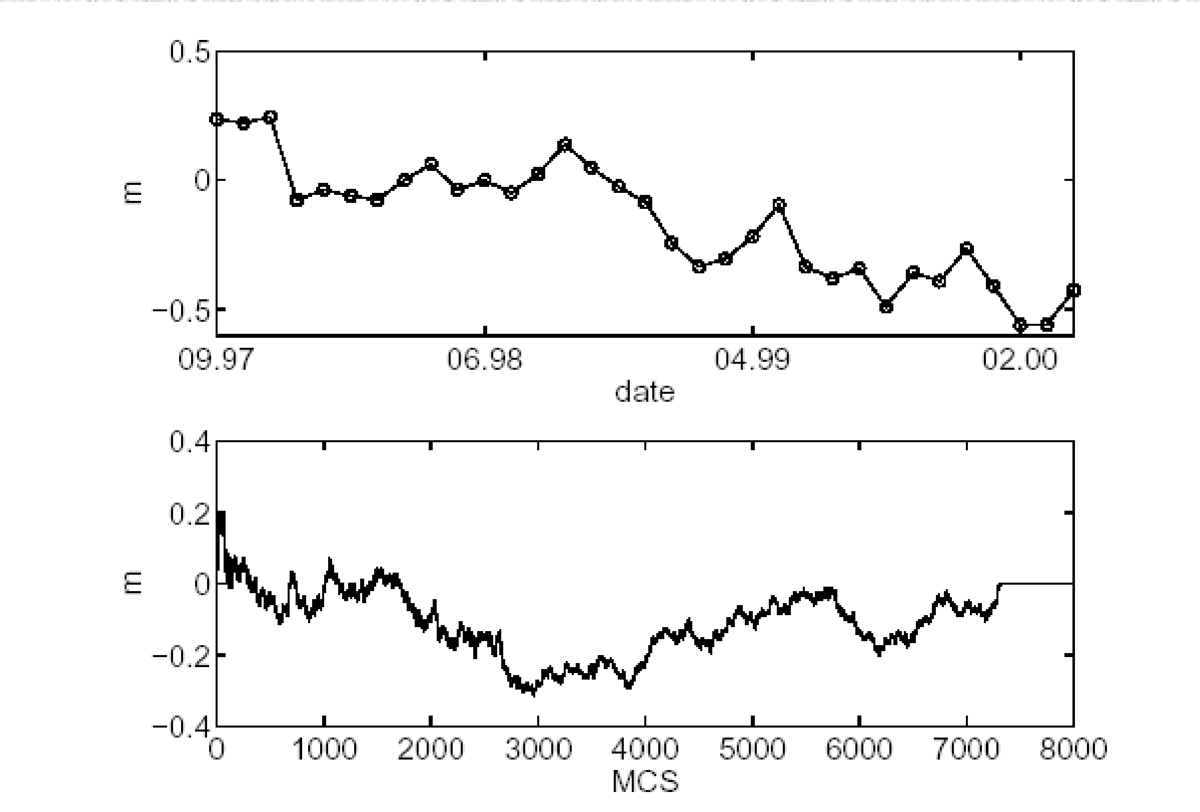
Three asymptotic cases:

- all S_i =1
- all S_i =-1
- 50% of S_i =1, 50% of S_i = -1

Opinion formation
(Sznajd Model - related to the Ising model)

``macroscopic'' variable, let us define the decision as a magnetization

$$m = \frac{1}{N}\sum_{i=1}^{N} S_i$$



(From OPINION EVOLUTION IN CLOSED COMMUNITY, K. SZNAJD-WERON, J. SZNAJD, International Journal of Modern Physics C, Vol. 11, No. 6 (2000) 1157-1165)

Opinion formation
(Sznajd Model - related to the Ising model)

Many possible variations, for example with some noise: probability p that an individual, instead of following the dynamics rules, will make a random decision (it is possible to have some kind of phase transitions).

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Note: mean value of opinion could be very difficult to fix

A huge number of opinion and of team coach: which kind of interaction?

Opinion formation
(Characteristics)

Each individual i (i = 1, 2, . . .N) has one opinion S_i (on one particular question).

- This opinion can be binary (0 or 1), multivalued integer ($S_i \in \{1,2,\ldots,Q\}$) or continuous real ($0 \leq S_i \leq 1$).
- The neighbours j of individual i may be those on a lattice (1D, 2D, …) or on network, or any other individual.
- Because of interactions between individuals i and j, one of them or both may change opinion from one time step t to the next t+1, according to rules to be specified.

Here we consider here a class of kinetic models of opinion formation, based on two-body interactions involving both compromise and diffusion properties in exchanges between individuals. Compromise and diffusion will be quantified by two parameters, which are mainly responsible of the behavior of the model, and allow for a rigorous asymptotic analysis (see G. Toscani, Kinetic models of opinion formation, 2006).

Kinetic Model (I)

The goal of the kinetic model of opinion formation, is to describe the evolution of the distribution of opinions in a society by means of microscopic interactions among agents or individuals which exchange information.

We associate the opinion with a variable *w* which varies continuously from -1 to 1, where -1 and 1 clearly enote two (extreme) opposite opinions: extreme opinions can not be crossed.
Let I = [-1,+1] denote the interval of possible opinions. From a microscopic view point, we describe the binary interaction between agents by the rules

$$w' = w - \gamma\, P(|w|)\, (w - w_*) + \eta\, D(|w|)$$

$$w_*' = w_* - \gamma\, P(|w_*|)\, (w_* - w) + \eta_*\, D(|w^*|)$$

where the pair (w, w_*), with $w, w_* \in I$, denotes the opinions of two arbitrary individuals before the interaction and (w', w_*'), their opinions after exchanging information between them and with the exterior.

Kinetic Model (II)

Let *f(w, t)* denote the distribution of opinion w, at time t. A direct application of standard methods of kinetic theory of binary interactions allows to recover the time evolution of f as a balance between bilinear gain and loss of opinion terms, described by the integro-differential equation of Boltzmann type

$$\frac{\partial f}{\partial t} = \int_{B^2} \int_I \left({}'\beta \frac{1}{J} f({}'w) f({}'w_*) - \beta f(w) f(w_*) \right) dw_* \, d\eta \, d\eta_* ,$$

where $({}'w, {}'w_*)$ are the pre-interaction opinions that generate the couple (w, w_*) of opinions after the interaction, J is the Jacobian of the transformation of (w, w_*) into (w', w_*'), while the kernels ${}'\beta$ and β are related to the details of the binary interaction.

Note. In the model $\gamma \in (0,1/2)$, η and η_* are random variables with the same distribution with variance σ^2 and zero mean, taking values on a set $B \subseteq \mathbb{R}$. The constant γ and the variance σ^2 measure respectively the ***compromise propensity*** and the modification of opinion due to ***diffusion***. Finally, the functions $P(\cdot)$ and $D(\cdot)$ describe the local relevance of the compromise and diffusion for a given opinion.

Kinetic Model (III)
(Fokker-Planck equations)

In general it is quite difficult both to study in detail the evolution of the opinion density, and to describe its asymptotic behavior. For a general kernel one has in addition to take into account that the mean opinion is varying in time. As is usual in kinetic theory, however, particular asymptotics of the equation result in simplified models (generally of Fokker-Planck type), for which it is relatively easier to find steady states, and to prove their stability.

These asymptotics are particularly relevant in case they are able to describe with a good approximation the stationary profiles of the kinetic equation.

Example.
Letting both $\gamma \rightarrow 0$ and $\sigma \rightarrow 0$ in such a way that $\sigma^2 / \gamma = \lambda$, the scaled density $g(v,\tau) = f(v,t)$ with $t = \tau/\gamma$, satisfies

$$\frac{\partial g}{\partial \tau} = \frac{\lambda}{2}\frac{\partial^2}{\partial w^2}\left(D(|w|)^2 g\right) + \frac{\partial}{\partial w}\left((w-m)g\right)$$

Kinetic Model (IV)
(Fokker-Planck equations)

Example. In the diffusion dominated case, $\sigma^2 / \gamma \to \infty$, we obtain

$$\frac{\partial g}{\partial \tau} = \frac{\lambda}{2} \frac{\partial^2}{\partial w^2} \left(D(|w|)^2 g \right)$$

Example (our case).
In the compromise dominated case, $\sigma^2 / \gamma \to 0$, we obtain the pure drift equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial w} \left(P(|w|)(w - m(t))g \right)$$

where m(t) is the mean opinion $m(\tau) = \int_{\mathcal{I}} w g(w,\tau)\, dw = \int_{\mathcal{I}} w f(w,t)\, dw.$

Pure drift equation

The choice (considered by F. Slanina and H. Lavička, 2003)

$$P(|w|) = 1 - w^2$$

gives the following equation

$$\frac{\partial g}{\partial \tau} = \frac{\partial}{\partial w}\left((1 - w^2)(w - m(t))g\right)$$

Remark. We note that $$\frac{dm(\tau)}{d\tau} = -m(\tau)\int_{\mathcal{I}} w^2 g(w,\tau)\,dw + \int_{\mathcal{I}} w^3 g(w,\tau)\,dw.$$

Thus, our equation differs from the pure drift in magnetization obtained by Slanina, Lavička, as the mean field limit of the Sznajd model in case of two opinions. There the first-order partial differential equation reads

$$\frac{\partial g}{\partial \tau} = -\frac{\partial}{\partial w}\left((1 - w^2)wg\right)$$

Drift equation
(The problem)

Analysis and large-time behavior of solutions of the equation (notation, now the unknown probability density is **f(x,t)**)

$$\frac{\partial f}{\partial t} = \gamma \frac{\partial}{\partial x}\left((1-x^2)(x-m(t))f\right)$$

with $x \in [-1,1]$ and $m(t)$ represents the mean value of $f(\cdot, t)$, the constant γ is linked to
- the spreading, $\gamma = -1$
- the concentration, $\gamma = +1$

of opinions.

Let $F(x)$ denote the probability distribution induced by the density $f(x)$,

$$F(x) = \int_{(-\infty,x]} f(y)\,dy$$

For $\rho \in (0,1)$, let

$$X^F(\rho) = \inf\{x : F(x) \geq \rho\}$$

We proved that the drift equation for f(x, t) takes a simple equivalent form if written in terms of its pseudo inverse $X(\rho, t)$:

$$\frac{\partial X(\rho,t)}{\partial t} = -\gamma\,(X(\rho,t) - m(t))\,(1 - X^2(\rho,t)),$$

$$m(t) = \int_0^1 X(\rho,t)\,d\rho.$$

Among the metrics which can be defined on the space of probability measures on I, which metricize the weak convergence of measures we consider the L^p-distance of the pseudo inverse functions (Wasserstein metric)

$$d_p(X,Y) = \Big(\int_0^1 |X(\rho) - Y(\rho)|^p\,d\rho\Big)^{1/p}.$$

THEOREM *For any probability density $f_0(x)$, there exists a unique function $\mu \in C^0(\mathbb{R}, \mathcal{K})$ such that, if $f(x,t)$ denotes the weak derivative of the probability distribution $\mu(t)$, $f(x,t)$ satisfies*

$$\frac{\partial f}{\partial t} = \gamma \frac{\partial}{\partial x}\left((1-x^2)(x-m(t))f\right)$$

with initial value $f_0(x)$ Moreover, for any $t \in \mathbb{R}$, the solution depends in a continuous way on the initial datum: the problem is well–posed in $C^0(\mathbb{R}, \mathcal{K})$.

Proof. [Existence] We prove the existence of a solution of the equivalent problem in a constructive way. More precisely,

A we construct a sequence $\{X_n, n \in \mathbb{N}\}$ which approximates a target solution;
B by compactness arguments, we find a convergent subsequence $X_{n_l} \to X$;
C the limit X satisfies

[Uniqueness and well-position] Let $Y(\rho,t), X(\rho,t)$ be two solutions of (1.1)–(1.2). Denote by $m_X(t)$ and $m_Y(t)$ the mean values of X and Y respectively at time t. Since $|m_Y(t) - m_X(t)| \le d_1(X,Y)$ we have (by Lemma 3.2 and (3.5))

$$\begin{aligned}
\frac{d}{dt} d_2(\mu_X(t), \mu_Y(t)) &= \frac{d}{dt}\int_0^1 \left(Y(\rho,t) - X(\rho,t)\right)^2 d\rho \\
&\le 2\int_0^1 \left|Y(\rho,t) - X(\rho,t)\right|\left(4\left|Y(\rho,t) - X(\rho,t)\right| + \left|m_Y(t) - m_X(t)\right|\right) d\rho \\
&\le 10 \cdot d_2(\mu_X(t), \mu_Y(t)).
\end{aligned} \tag{3.12}$$

Gronwall's lemma completes the proof. □

Large–time behavior of the solutions.

Main tools:

- Properties of Abel's equation (ODE equation)
- Conservation law

$$T(t) := \int_0^1 \log\Big(\frac{1+X(\rho,t)}{1-X(\rho,t)}\Big)\,d\rho\,.$$

- Lyapunov functional (variance)

$$V(t) := \int_0^1 (X(\rho,t))^2\,d\rho - \Big(\int_0^1 X(\rho,t)\,d\rho\Big)^2\,.$$

Spreading of opinions (γ = -1), ``no delta masses in the internal''.

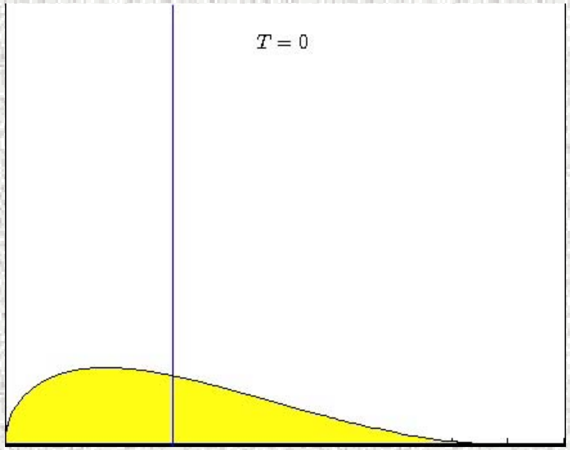
THEOREM *Assume* $X(\rho_1,0) = X(\rho_2,0) \iff \rho_1 = \rho_2$ *or* $(X(\rho_1,0))^2 = 1$. *Then, the limit distribution exists and it is given by two masses located in* -1 *and* $+1$.

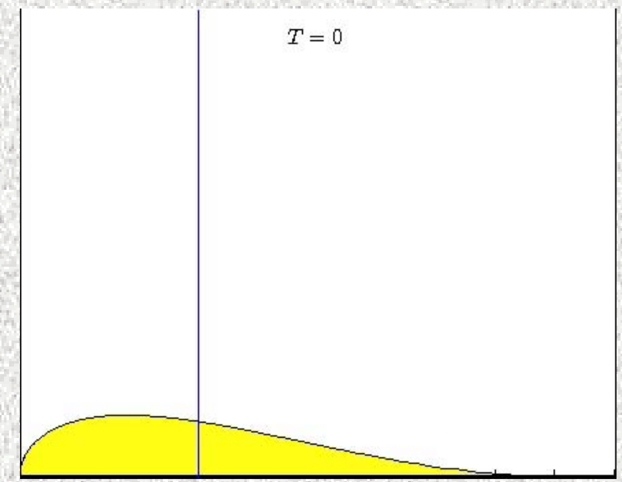
Concentration of opinions (γ = +1).

THEOREM *If* $(1-p_1)(1-p_{-1}) < 1$ *(i.e., if there are masses in* ± 1 *at time* $t = 0$*) then* $m_\infty = p_1 - p_{-1}$. *Otherwise, if* $\log\big((1+X(\rho,0))/(1-X(\rho,0))\big) \in L^1(0,1)$ *then*

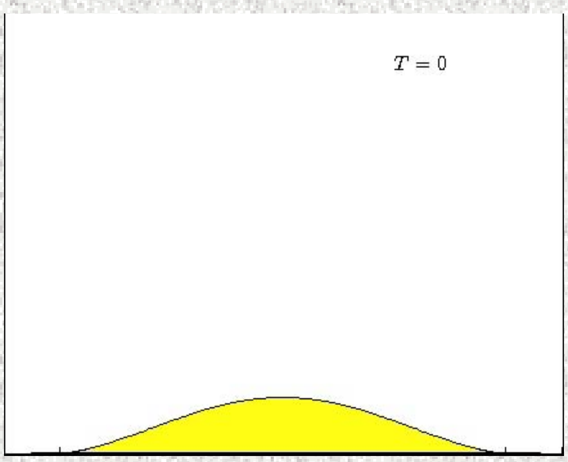
$$m_\infty = \frac{\exp\{T(0)\} - 1}{\exp\{T(0)\} + 1}$$

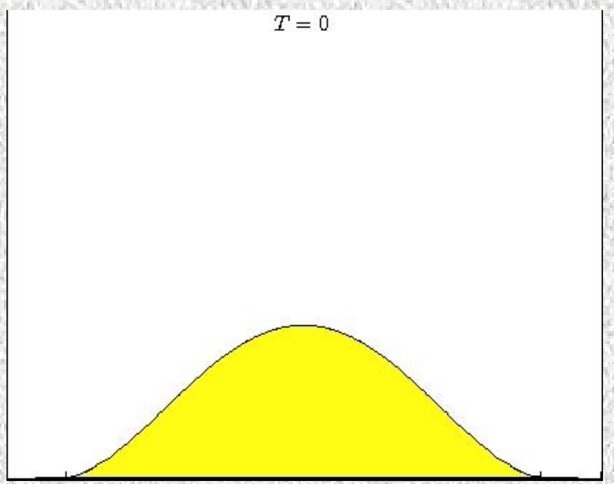
Numerical examples





Non symmetric data, concentration (left), spreading (right)

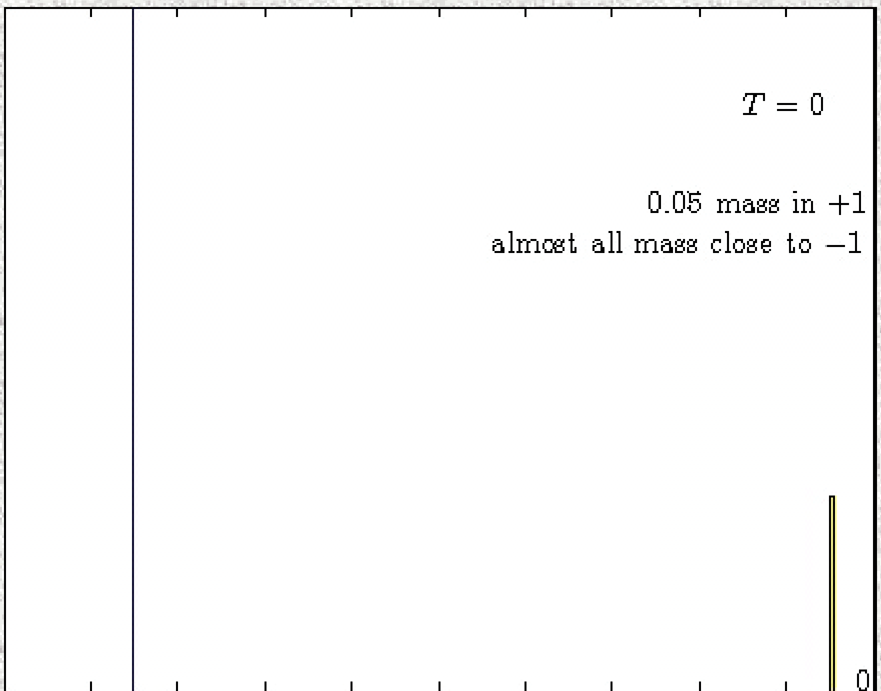


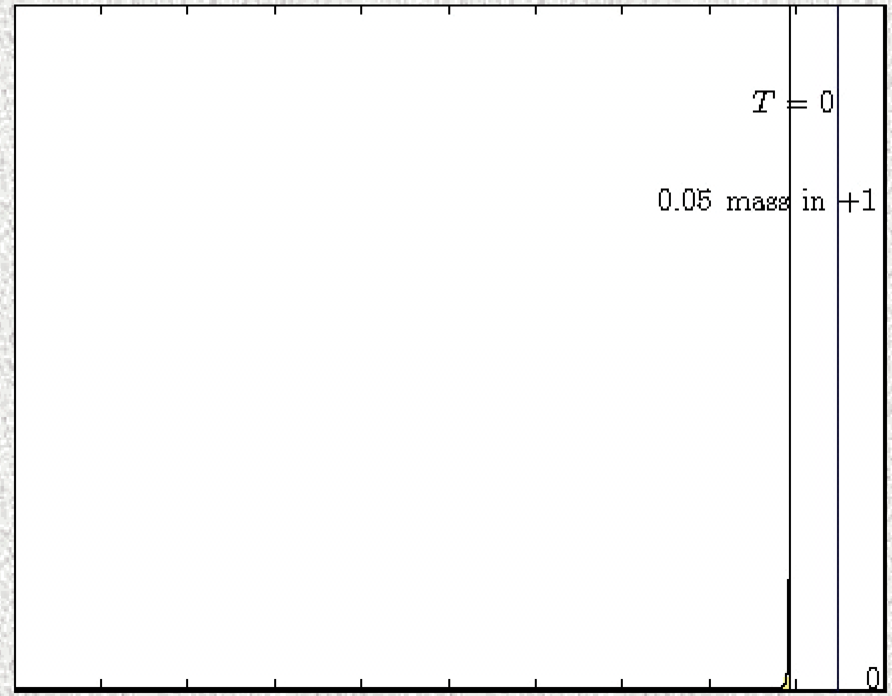


Symmetric data, concentration (left), spreading (right) [Slanina et al. Model]

Numerical examples

An interesting and “counterintuitive” example





Numerical examples (with linear diffusion)

Concentration Assym. →

Concentration Sym. →

Spreading Asymm. →

Spreding Sym. →

Conclusions

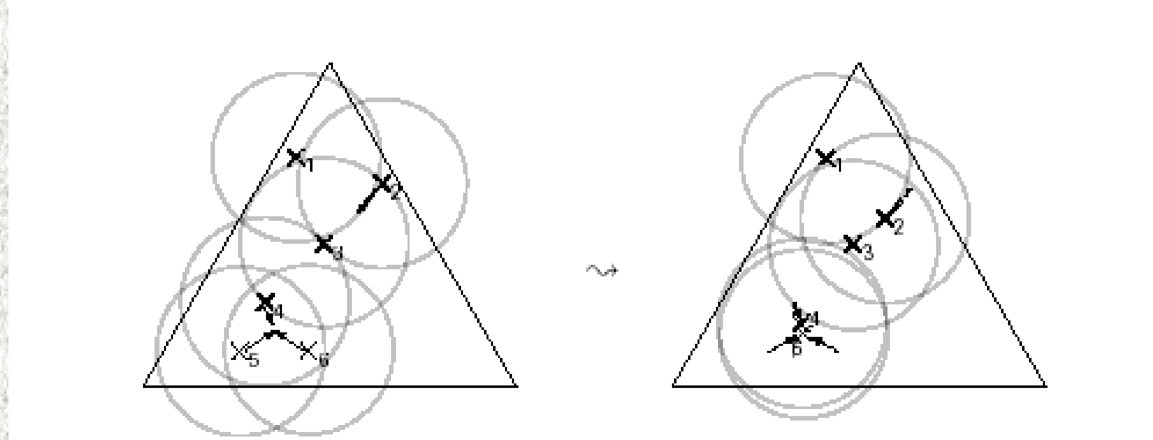
- A simple drift equation for concentration/spreading of opinions (possible also to link to micromagnetic phenomena).
- A theoretical and numerical analysis
- Asymptotic behavior (with some unexpected prediction?)

Next works

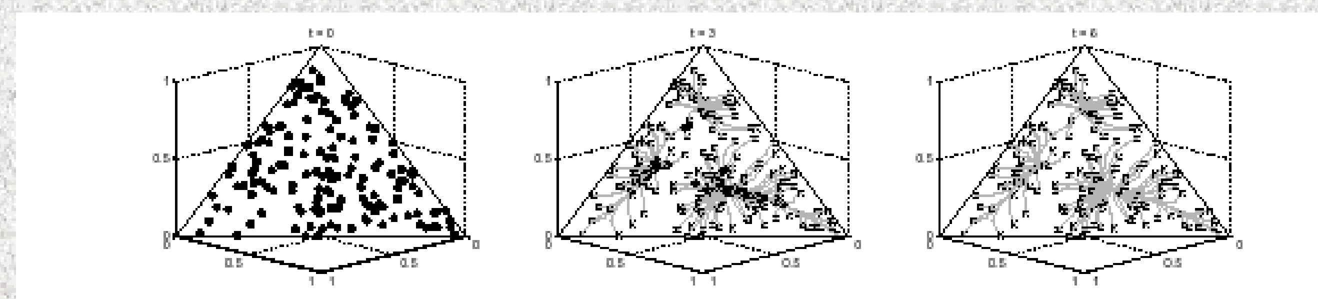
- More then two strong opinions (1-D or n-D models?)
- Other Fokker-Planck equations or other approaches
- Comaprison with some (real) data in Economical processes
- ...

… But this talk is only my opinion!

Example from J. Lorenz, 2003



An example with six agents with 3-dimensional opinions. A group of m agents is to find a common agreement about a certain issue. Consider this issue to be an n-dimensional vector of real numbers, for example the allocation of a fixed sum of money to n projects. Each agent has an opinion about the allocation, which he may revise (every agent averages all opinions which are closed to his own opinion).



Simulation by using discrete dynamical system (Lorenz, 2003)

Example: the **Opinion Changing Rate** (OCR) **model** of *Pluchino, Latora, Rapisarda,* Intern. J. Modern Physics (2004).

$$\dot{x_i}(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(x_j - x_i)e^{-\alpha|x_j - x_i|} \qquad i = 1, \ldots, N$$

Here $x_i(t)$ is the opinion (an *unlimited* real number) of the ith individual at time t, while ω_i represents the so called natural opinion changing rate, i.e. the intrinsic inclination, or natural tendency, of each individual to change his opinion (corresponding to the natural frequency of each oscillator in the Kuramoto model). The parameter K>0 measures the coupling strength in the global coupling term.

For example adding a diffusion term, as by E. Ben-Naim (2005) in the discrete case about the dynamics of parties.

The density $P_n(t)$ of agents with opinion n at time t obeys the master equation

$$\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2}) + D(P_{n-1} + P_{n+1} - 2P_n).$$

The total population and the total opinion are conserved: $\Sigma_n P = \text{const.}$ $\Sigma_n nP = \text{const.}$

Ben-Naim found that the level of diffusion (as a level of noise) determines the nature of the ``political system''. Strong diffusion leads to a uniform distribution of opinions. With weak diffusion the system organizes into clusters (parties) with large clusters continuously overtaking small ones. Without diffusion, the system evolves into a frozen pattern of clusters with equal weight and equal separations.