A kinetic model for particles deposit

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Plan

* RSOS model

* KPZ equation

* Kinetic model

* Model properties

* Numerical results

* Asymptotics $a \longrightarrow 0$

* Conclusions

RSOS model

$P(H,t)$ probability configuration $H = (h(j))_{j=1...N}$, at time $t > 0$

$$\frac{\partial P}{\partial t}(H,t) = \sum_{H'} [\sigma(H',H)P(H',t) - \sigma(H,H')P(H,t)] \tag{1}$$

$\sigma(H,H')$ probability $H \to H'$

$$\sigma(H,H') = \frac{1}{\tau} \sum_{j=1}^{N} \Big(\theta(h(j+1) - h(j))\, \theta(h(j-1) - h(j)) \times \\ \delta(h'(j), h(j) + a) \prod_{i \neq j} \delta(h'(i), h(i)) \Big) \tag{2}$$

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

KPZ equation

$$\partial_t h(x,t) = \nu \nabla^2 h(x,t) + \frac{\lambda}{2}[\nabla h(x,t)]^2 + \eta(x,t) \tag{4}$$

* $\eta(x,t)$ white noise
* existence and regularity of solutions: open problem

* mean height: $\overline{h}(t) = \frac{1}{L}\int_0^L h(x,t)\,dx$
* standard deviation: $W(t) = \frac{1}{L}\left(\int_0^L (h(x,t) - \overline{h}(t))^2\,dx\right)^{1/2}$

  $W(t)$ goes to ∞ as $t \to \infty$

M.Krandar-G.Parisi-Y.Zhang (Phys. Rev. Lett. - 1986)

K.Park-B.N.Kahng (Phys. Rev. E - 1995)

Simplifying assumptions

Heights distributions indep.: $P(H = (h(j))_{j=1..N}, t) = \prod_{j=1}^{N} p_j(h(j), t)$

$\exists\, p : \mathbb{N} \times \mathbb{R}^{+} \rightarrow [0,1]$ s.t. $p_j(h(j), t) = p(h(j), t) \quad \forall j = 1...N$

$$\Rightarrow P(H, t) = \prod_{j=1}^{N} p(h(j), t) \tag{5}$$

Goal: equation for $p(h = ma, t) = p_m(t) \quad m \in \mathbb{N}$

$$\overline{h} = \sum_{m=0}^{\infty} p_m(t) h = \sum_{m=0}^{\infty} p_m(t)\, ma,$$

$$W = \left(\sum_{m=0}^{\infty} p_m(t) (h - \overline{h})^2 \right)^{1/2}$$

Kinetic model

Lemma Fixed i, the probability to have $H_i = h(i)$, is given by:

$$p(h(i),t) = \int_{h(j)\geq 0,\, j\neq i} P(H,t) \prod_{j\neq i} dh(j). \tag{6}$$

Theorem Let $P(H)$ be a solution of (1)-(2) satisfying (5), then $p_m(t) = p_m$ is a solution of:

$$\partial_t p_m = q_{m-1}^2 p_{m-1} - q_m^2 p_m, \tag{7}$$

where $q_m(t) = q_m$ is given by:

$$q_m = \sum_{j\geq m} p_j. \tag{8}$$

Model properties

Let E be the space:

$$E = \{(p_m)_{m\in\mathbb{N}} \in \mathbb{R}^+ \ : \ \sum_{m\in\mathbb{N}} p_m < +\infty\} \tag{9}$$

with norm:

$$\|p\| = \sup_{m\in\mathbb{N}} |p_m| + \sum_{m\in\mathbb{N}} |p_m|. \tag{10}$$

$$(7) \quad \Rightarrow \quad \begin{cases} \dot{p}_m(t) = q_{m-1}^2 p_{m-1} - q_m^2 p_m \\ p_m(0) = \delta_0, \end{cases} \qquad \forall\, m \tag{11}$$

Theorem There exists a unique solution, in the space E (9)-(10), of problem (11) on a maximal interval $[0, T]$.

Lemma $p_m(t)$ and $q_m(t)$ satisfy $\forall\, t \geq 0$:

$$\sum_{m\in\mathbb{N}} p_m(t) = 1\ , \qquad q_m(t) \leq 1.$$

Numerical simulations

* $\exists\, T > 0$ s.t. $\forall\, m$: $p_m(t+T) = p_{m-1}(t)$, as t goes to ∞.
* for large time the mean height $\overline{h}(t) \propto \alpha t$ and that the standard deviation $W(t) \rightarrow \overline{W}$.

Asymptotics $a \to 0$

Consider the piecewise constant function:

$$p(h,t) = \frac{p_m(t)}{a} \quad \text{if } h \in [am, a(m+1)[$$

Then: $q(h,t) = \int_{h=am}^{\infty} p(h,t)\,dh \;\Rightarrow\; \partial_h q(h,t) = -p(h,t)$

Finally, rescaling time as t/a, we obtain:

$$\begin{cases} \partial_t p(h,t) = \partial_h (q^2(h,t) p(h,t)) \\ p(h,0) = \delta_0 \end{cases} \tag{12}$$

Concluding Remarks

* Kinetic model adapted to the ballistic deposit

* different from KPZ model

* Proof of unique profile